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Dynamic User Equilibrium in Networks with Queues*

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1. INTRODUCTION

The dynamic traffic assignment is to determine the time-varying flows on urban transportation networks, mainly at peak hours. There are two basic methods for dynamic traffic assignment problem: simulation methods which emphasize individual user behaviors, and analytical methods which mainly concern the average user behaviors. Only the analytical methods are of concern, and previous studies on analytical method can be grouped into three categories: mathematical programming approach, optimal control approach, and variational inequality approach. Merchant and Nemhauser [1), 2)] formulated the dynamic traffic assignment problem as a discrete time, nonlinear, and non-convex program based on system optimization for the first time, and Carey [3), 4)] proved that the Merchant-Nemhauser model satisfies the linear independence constraint qualification and modified the model as a convex program. The so-called many-to-one problems have been successfully solved by optimal control theory [5), 6), 7)]. Most of the mathematical programming models and discrete time optimal control models to date address system optimization rather than user equilibrium, thereby limiting their relevance to network prediction [8)]. Since the variational inequality problem is equivalent to the complementarity problem which is the native definition of user equilibrium, it has becomes the most powerful method to model the dynamic user optimal problem. Examples are departure time and route choice model by Friesz et al [9)] and link-based variational inequality model by Ran et al. [10)]

For short-run, users are intended to choose the route with the minimum estimated route travel cost and this leads to so-called user optimal flow pattern. Obviously, the user optimal principle does not require that equilibration of route travel costs actually experienced by drivers with same departure time and OD pair [6)]. Moreover, the estimated link travel time is also not necessarily the actual link travel time experienced by users. Such route choice model is especially useful to analyze the behaviors of users who receive traffic information from radios, navigators, etc. Reactive or instantaneous user optimal problem has been studied by numbers of researchers [11), 12), 13)].

On the other hand, the travel time estimated by users tends to the actual travel time experienced by users in the long-run if the traffic demands do not change. Since the users choose the route with real minimum travel cost, the resulting flow pattern is so-called user equilibrium. Akamatsu [14)] has formulated the dynamic user equilibrium as a nonlinear complementarity problem. Chen and Hsueh [15)] show that the dynamic user equilibrium problem can be formulated as a variational inequality problem under a certain flow propagation relationship. Their model does not impose link capacity constraints and no queue is simulated.

This paper deals with discrete time dynamic user equilibrium model incorporating point queue. Point queue is widely used in the dynamic traffic assignment since it satisfies the first-in-first-out (FIFO) principle [16)]. It is assumed that each link has fixed free travel time which includes the "normal" delay caused by the stops and red lights on the link [17)] and users have to spend extra time waiting in the queue if departure capacity is exceeded. To avoid inconsistency caused by the discretization, the travel time is defined as continuous variable in time so that FIFO principle can be preserved. Without imposing additional assumption, a solution method based on variational inequality is proposed which only requires solving the shortest route problem with the original size of network comparing with time-space method. Since the future travel time should be calculated in dynamic user equilibrium, the authors recognize that traffic information after studied period should be given to solve the user equilibrium.

2. NETWORK MODELING

Consider a many-to-many network represented by a directed graph $G(N, A)$, which includes a set of nodes, N, and a set of links, A. Let O denote the set of origin nodes, and D denote the set of destination nodes. The studied period is set as $[0,T]$, which is divided into small interval with identical length Δt. In this paper, the travel time is continuous in time although the model is discrete and so does the number of vehicles. On the other hand, the flows are average value during an interval and may not be continuous in time.

(1) Exit queue and departure rate

Let $u_a(t)$ and $v_a(t)$ be the average inflow and outflow during period $(t-\Delta t, t]$, respectively. The number of vehicles on link a, $x_a(t)$, and number of vehicles waiting in the queue, $x_a^q(t)$, are exactly defined at time t. For point queue model, users must spend fixed travel time τ_a before they arrive at the exit. Thus, the state equation for the queue is written as

$$\frac{x_a^q(t) - x_a^q(t-\Delta t)}{\Delta t} = u_a(t-\tau_a) - v_a(t) \tag{1}$$

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The number of vehicles on link at time t is

$$x_a^q(t) = x_a^q(t-\Delta t) + \left[u_a(t-\tau_a) - v_a(t)\right]\cdot \Delta t \tag{2}$$

Each link is assumed to have a maximum departure rate s_a. Noting that if $x_a^q(t) > 0$ then $v_a(t) = s_a$ and $v_a(t) \le s_a$, the departure rate from link a can be calculated as

$$v_a(t) = \min\left\{s_a, u_a(t-\tau_a) + \frac{x_a^q(t-\Delta t)}{\Delta t}\right\}. \tag{3}$$

(2) **Link travel time and queue delay**

Users who enter link at time t arrive at exit queue at time $t+\tau_a$ and face a queue with length of $x_a^q(t+\tau_a)$. Thus, the queue delay for those users is

$$w_a(t) = \frac{x_a^q(t+\tau_a)}{s_a} = w_a(t-\Delta t) + \frac{u_a(t) - v_a(t+\tau_a)}{s_a}\cdot \Delta t \tag{4}$$

The total link travel time for vehicles entering link at time t is the sum of fixed travel time and the queue delay, which is given by

$$c_a(t) = \tau_a + w_a(t) = c_a(t-\Delta t) + \frac{u_a(t) - v_a(t+\tau_a)}{s_a}\cdot \Delta t \tag{5}$$

By Eq. (3), the queue delay and link travel time become function of only the previous time and the inflow. The queue delay is

$$w_a(t) = \max\left\{0, w_a(t-\Delta t) + \frac{u_a(t) - s_a}{s_a}\cdot \Delta t\right\} \tag{6}$$

The link travel time is

$$c_a(t) = \max\left\{\tau_a, c_a(t-\Delta t) + \frac{u_a(t) - s_a}{s_a}\cdot \Delta t\right\} \tag{7}$$

(3) **Multi-destination outflows**

The FIFO principle can be used to calculate of multi-destination outflows. For the continuous time model, the following equation must hold [12), 13)]:

$$\frac{v_a^d(t+c_a(t))}{v_a(t+c_a(t))} = \frac{u_a^d(t)}{u_a(t)}. \tag{8}$$

Now considering discrete time model, users departing from link in one interval might not arrive at link in one interval. If users departing from link at time t arrive at link at time $t_a^c(t)$, then $t_a^c(t) + c_a(t_a^c(t)) = t$. Therefore, users departing from link a during $(t-\Delta t, t]$ will arrive at link a during period $(t_a^c(t-\Delta t), t_a^c(t)]$. Find two integers k_1 and k_2 such that $(k_1 - 1)\cdot\Delta t < t_a^c(t-\Delta t) \le k_1\cdot\Delta t$ and $(k_2-1)\cdot\Delta t \le t_a^c(t) < k_2\cdot\Delta t$, then the departure flow to destination d is given as

$$\frac{v^d(t)}{v_a(t)} = \frac{u_a^d(k_1\Delta t)\cdot\Delta t_1 + \sum_{k=k_1+1}^{k_2-1} u_a^d(k\Delta t)\cdot\Delta t + u_a^d(k_2\Delta t)\cdot\Delta t_2}{u_a(k_1\Delta t)\cdot\Delta t_1 + \sum_{k=k_1+1}^{k_2-1} u_a(k\Delta t)\cdot\Delta t + u_a(k_2\Delta t)\cdot\Delta t_2}, \tag{9}$$

where $\Delta t_1 = k_1\Delta t - t_a^c(t-\Delta t), \Delta t_2 = t_a^c(t) - (k_2-1)\Delta t$.

(4) **Flow conservation at node**

For any node, the outflow should be the sum of inflow and the demand generated at that node. Let $q_{jd}(t)$ be the demand generated at node j to destination d during period $(t-\Delta t, t]$, then the following equation holds

$$\sum_{a\in IN(j)} v_a^d(t) + q_{jd}(t) = \sum_{b\in OUT(j)} u_b^d(t) \tag{10}$$

where $IN(j)$ and $OUT(j)$ are the set of incoming links and the set of outgoing links, respectively.

3. DYNAMIC USER EQUILIBRIUM

There is a learning and forecasting process for the long-run users, and the travel time estimated by those users is expected to converge the actual travel time when the demands are relatively fixed. Users are intended to choose the routes minimum travel cost, leading to a dynamic user equilibrium flow pattern. The long-run users do not change their routes once they depart the origins. This is different from the reactive or predictive user optimal assignment, since users may seek better routes when they arrive at a new node if the original route is no long the one with minimum estimated cost.

(1) **Actual route travel time**

Consider route p connecting node j and destination d, which comprises m_p links, denoted by $a_1^p, a_2^p, \cdots, a_{m_p}^p$. Let $t_{a_i^p}$ be the time when users who depart node j at time t enter link a_1^p, then $t_{a_1^p} = t$ and $t_{a_i^p} = t_{a_{i-1}^p} + c_{a_{i-1}^p}(t_{a_{i-1}^p})$ for $i = 2, \cdots, m$. The actual route travel time on route p id defined as

$$\psi_p^{jd}(t) = \sum_{i=1}^{m_p} c_{a_i^p}(t_{a_i^p}) \tag{11}$$

Note that link travel time in Eq. (11) is the real link travel time experienced by users. This is different from predictive route travel time, which is given as $\hat{\psi}_p^{jd}(t) = \sum_{i=1}^{m_p} \hat{c}_{a_i^p}(t_{a_i^p})$, where $\hat{c}_{a_i^p}(t_{a_i^p})$ is *estimated* link travel time and not necessarily the actual link travel time.

(2) **Dynamic user equilibrium**

Generally, the definition of user equilibrium is based on route flows, and dynamic user equilibrium can also be defined on route flows. It however would be more convenient to define the dynamic user equilibrium on link flow, because demands are only assigned to the *first* link of a route and do not traverse the whole route immediately at in static assignment. On the other hand, the number of routes in transportation network is extremely large which is impossible to enumerate. The link-based definition may provide some new insights into the dynamic traffic assignment.

As discussed before, the link travel time and route travel time are actual travel time experienced by users rather than estimated by the users. This is the key difference between dynamic user equilibrium problem and predictive user optimal problem. Let i_a and j_a be start node and end node of link a, the definition of *dynamic user equilibrium* is given by [10)]

$$u_a^d(t)\cdot\left[c_a(t)+\pi_{j_a d}(t+c_a(t))-\pi_{i_a d}(t)\right]=0 \tag{12}$$

$$c_a(t)+\pi_{j_a d}(t+c_a(t))-\pi_{i_a d}(t)\geq 0 \tag{13}$$

$$u_a^d(t)\geq 0 \tag{14}$$

where $\pi_{jd}(t)$ is minimum actual travel time from node j to destination d at time t. It is obvious that the following equation holds if link a is utilized

$$c_a(t)+\pi_{j_a d}(t+c_a(t))-\pi_{i_a d}(t)=0 \tag{15}$$

Summing up those equations of all links on a used route, one can find that travel time on that path is exactly actual route travel time defined by Eq. (11).

4. SOLVING DYNAMIC USER EQUILIBRIUM PROBLEM

It is clear that the actual travel time depends on the traffic conditions in the future. Thus traffic assignment cannot be carried out without forecasting the future traffic conditions. For long-run transportation network, the traffic condition during one day can be considered to be stable. Generally, we are interested in network performance during the peak hours. The demands out of peak hours are relatively low and there is congestion. For example, the traffic flows before 4am and after 12pm are very low and drivers enjoy free travel time on all links. Traffic assignment during those periods can be easily solved by the shortest path method, or may be assumed to be empty network. It also assumes that the OD demands during the studied period are known so that the problem is a fixed demand problem. The studied period is from time 0 to T, which is divided into M identical interval with length of Δt.

Unlike the reactive user optimal assignment, the initial state of both before time 0 and time T should be given. Although any studied period is feasible, the solution method is easy to be implemented if we include the whole peak hours in the studied period. It is quite reasonable to assume that there is no queue delay when demands are very low, thus users simply choose the route with minimum free travel time.

(1) Diagonalization method

The dynamic user equilibrium problem indeed is a non-linear complementarity problem. Generally, dynamic user equilibrium problem is solved by converting the original network into time-space network (e.g., Drissi-Kaitouni and Hameda-Benchekroun 1992, Chen and Hseh 1998), but the number of links become extremely large. For example, the expanded time-space network has 2,500 link if the original network has 50 links and the studied period is divided into 50 intervals. A heuristic method is proposed to solve dynamic user equilibrium problem and keep the size of problem manageable.

The idea is to use backward method to update link flows of each interval $(t-\Delta t,t]$ one by one. The method is based on the diagonalization method. Assume that, at the kth iteration of the procedure, the values of all the link flow variables $\left\{u_a^{d(k-1)}(t)\right\}$ and $\left\{v_a^{d(k-1)}(t)\right\}$ are known. At this point, the following problem can be formulated for a given interval $(t-\Delta t,t]$:

$$u_a^{d(k)}(t)\cdot\left[c_a^{(k)}(t)+\pi_{j_a d}(t+c_a^{(k-1)}(t))-\pi_{i_a d}(t)\right]=0 \tag{16}$$

$$c_a^{(k)}(t)+\pi_{j_a d}(t+c_a^{(k-1)}(t))-\pi_{i_a d}(t)\geq 0 \tag{17}$$

$$u_a^{d(k)}(t)\geq 0 \tag{18}$$

where $c_a^{(k)}(t)=\tilde{c}_a^{(k)}(u_a)\Big|_{u_a=u_a^{(k)}(t)}$ and $\tilde{c}_a^{(k)}(u_a)$ is defined as

$$\tilde{c}_a^{(k)}(u_a)=\min\left\{\tau_a, c_a^{(k-1)}(t-\Delta t)+\frac{u_a-1}{s_a}\cdot\Delta t\right\} \tag{19}$$

Eq. (19) is the same as link travel time except that the previous travel time is given by $c_a^{(k-1)}(t-\Delta t)$ instead of $c_a(t-\Delta t)$. Moreover, $\tilde{c}_a^{(k)}(u_a)$ can be considered as a separable link cost function as it is function of only u_a. With no difficulty, we can show that Eqs. (16)–(18) are equivalent to the following variational inequality problem:

Find $\left\{u_a^{d(k)}(t)\right\}$ such that for all $\left\{u_a^d\right\}$ the following inequality holds

$$\sum_a\sum_d\left[c_a^{(k)}(t)+\pi_{j_a d}(t+c_a^{(k-1)}(t))\right]\cdot\left(u_a^d-u_a^{d(k)}\right)\geq 0 \tag{20}$$

where $\left\{u_a^d\right\}$ satisfies

$$\sum_{a\in OUT(j)}u_a^d=q_{jd}(t)+\sum_{a\in IN(j)}v_a^{d(k-1)}(t) \tag{21}$$

Problem (20) is equivalent to solve the following mathematical program

$$\min \quad \sum_a\int_0^{u_a}\tilde{c}_a^{(k)}(\omega)\mathrm{d}\omega+\sum_{jd}\int_0^{y_{jd}}\pi_{j_a d}(t+c_a^{(k-1)}(t))\mathrm{d}\omega$$

subject t o

$$\sum_{a\in OUT(j)}u_a^d=q_{jd}(t)+\sum_{a\in IN(j)}v_a^{d(k-1)}(t) \tag{22}$$

$$u_a=\sum_d u_a^d$$

$$y_{jd}=\sum_{a\in IN(j)}u_a^d$$

which is a standard fixed demand network equilibrium problem and can be solved by Frank-Wolfe Method.

(2) Iterative scheme

The general process of finding the predictive equilibrium flows is as follows:

Step 0. Initialization. Set $k=0$, use the static user equilibrium model to initialize the network for $t>T$. Calculate the shortest travel time from every node j to destination d

by the shortest path method, $\pi_{jd}(t)$ for $t > T$. Find feasible flow pattern $\left\{u_a^{d(k)}(t)\right\}$ for $t \in [0,T]$, which can be carried out by all-or-nothing assignment.

Step 1. Calculate the departure rates $\left\{v_a^{d(k)}(t)\right\}$ by Eqs. (9). Set $k = k + 1$.

Step 2. Assignment. Let $t = T$, calculate the flows during interval $(t - \Delta t, t]$ as follows

Step 2.1. Solve problem (22). This leads to $\left\{u_a^{d(k)}(t)\right\}$, calculate the shortest travel time $\pi_{jd}(t)$ by the shortest path method.

Step 2.2. Set $t = t - \Delta t$. If $t \leq 0$, stop and go to the main Step 3. Otherwise, go to Step 2.1.

Step 3. Convergence test. If $\sum_a \left|u_a^{(k)}(t) - u_a^{(k-1)}(t)\right|^2 \leq \varepsilon$, stop; otherwise go to Step 1.

It is easy to prove this algorithm produces the equilibrium flow pattern if the algorithm converges. However, there is no proof of the convergence of the algorithm and whether the algorithm converges is open for the future study.

5. CONCLUSIONS

A dynamic user equilibrium model in consideration of point queue is proposed in this paper. The differences between the reactive/predictive user optimal assignment and dynamic user equilibrium are examined. The necessity of using actual travel time for dynamic user equilibrium problem is discussed. The proposed solution method is based on variational inequality problem, and only solving shortest path problem of the original network is required to prevent the size of expanded time-space network become too large. The resulting sub-problem is shown as a standard fixed demand problem, on which Frank-Wolfe method can be employed.

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