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Sensitivity analysis for stochastic equilibrium network flows – A dual approach *

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1 Introduction

Recently much efforts have been made on the sensitivity analysis for the Wardropian equilibrium of traffic networks [3, 4, 5]. Here we present a method for the sensitivity analysis for the stochastic equilibrium. Our method is based on the dual formulation of the stochastic equilibrium analysis. By adopting Dial's algorithm for stochastic traffic assignment and a "path choice entropy decomposition" technique recently developed by Akamatsu [1], we are able to formulate a computationally efficient link cost based algorithm for the sensitivity analysis.

2 Sensitivity Analysis of Stochastic Equilibrium of Traffic Networks

notations:

- $N = \{i, j, \cdots\}$: set of nodes
- $A = \{ij, \cdots\}$: set of links
- $W = \{rs, \cdots\}$: set of OD pairs
- $R_{rs} = \{k, \cdots\}$: set of paths connecting rs
- x_{ij}: link flow, for $ij \in A$
- q_{rs}: OD demand, $rs \in W$
- $t_{ij}(x_{ij}, \epsilon_{ij})$: differentiable cost function of link ij with respect to flow x_{ij}, and parameter ϵ_{ij}. It is assumed that t_{ij} is strictly monotone with respect to x_{ij}. For fixed ϵ_{ij}, the inverse of the cost function is denoted as $x_{ij}(t_{ij}, \epsilon_{ij})$.

Dual Mathematical Program Formulation

It can be shown that $x_{ij}(t_{ij}, \epsilon_{ij})$, $ij \in A$, achieves the stochastic equilibrium ( ) if and only if t_{ij}, $ij \in A$, is a minimizing point of the function Z,

$$Z(\boldsymbol{t}, \epsilon) = \sum_{ij} \int_{t_{ij}(0,\epsilon_{ij})}^{t_{ij}} x_{ij}(\nu, \epsilon_{ij}) d\nu - \sum_{rs} q_{rs} S_{rs}(\boldsymbol{c}^{rs}(\boldsymbol{t})), \quad (1)$$

where

$$S_{rs}(\boldsymbol{c}^{rs}(\boldsymbol{t})) = -\frac{1}{\theta} \ln \sum_k \exp(-\theta c_k^{rs})$$

is the expected minimum cost for OD rs.

In fact, the minimizing condition for this unconstrained program is as follows

$$\frac{\partial Z}{\partial t_{ij}} = x_{ij}(t_{ij}, \epsilon_{ij}) - \sum_{rs} q_{rs} \sum_k \frac{\partial S_{rs}}{\partial c_k^{rs}} \frac{\partial c_k^{rs}}{\partial t_{ij}}$$

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$$
\begin{aligned}
&= x_{ij}(t_{ij},\epsilon_{ij}) - \sum_{rs} q_{rs} \sum_{k} \frac{\partial S_{rs}}{\partial c_k^{rs}} \delta_{ij,k}^{rs} \\
&= x_{ij}(t_{ij},\epsilon_{ij}) - \sum_{rs} q_{rs} \frac{\sum_k \exp(-\theta c_k^{rs})\delta_{ij,k}^{rs}}{\sum_p \exp(-\theta c_p^{rs})} \\
&= 0, \quad ij \in A.
\end{aligned} \tag{2}
$$

It is clear that

$$
x_{ij}(t_{ij},\epsilon_{ij}) = \sum_{rs} q_{rs} \frac{\sum_k \exp(-\theta c_k^{rs})\delta_{ij,k}^{rs}}{\sum_p \exp(-\theta c_p^{rs})}
$$

are the link flows in a stochastic user equilibrium state.

In a compact vector expression, (2) is rewritten as

$$
\nabla_{\boldsymbol{t}} Z = \mathbf{0}. \tag{3}
$$

It can be shown that the function $-\sum_{rs} q_{rs} S_{rs}(\boldsymbol{c}^{rs}(\boldsymbol{t}))$ is convex with respect to t_{ij}, or equivalently, its Hessian with respect to t_{ij} is semi-positive definite.

It is also trivial to show that the Hessian of the first term of Z

$$
\nabla_{\boldsymbol{t}}^2 (\sum_{ij} \int_{t_{ij}(0,\epsilon_{ij})}^{t_{ij}} x_{ij}(\nu,\epsilon_{ij}) d\nu) = \mathrm{diag}(\frac{\partial x_{ij}}{\partial t_{ij}})_{ij} \tag{4}
$$

is positive definite, since each diagonal entry is positive from the assumption that x_{ij} is strictly monotone in t_{ij}. Where diag( ) denotes a diagonal matrix with corresponding diagonal entries.

It then follows that the Hessian of Z

$$
\nabla_{\boldsymbol{t}}^2 Z = \left(\frac{\partial^2 Z}{\partial t_{ij} \partial t_{gh}} \right)_{ij,gh} \tag{5}
$$

is a positive definite matrix. This implies that Z is convex and the minimum point is unique.

Sensitivity Analysis

The sensitivity analysis is conducted by computing the changes of flows x_{ij} and costs t_{ij} according to small perturbation of ϵ_{ij}.

$\frac{\partial Z}{\partial t_{ij}}$ are functions in free variables t_{ij} and ϵ_{ij}. The equations $\frac{\partial Z}{\partial t_{ij}} = 0$ define t_{ij} as implicit functions in variables ϵ_{ij}. The derivatives of t_{ij} and x_{ij} with respect to ϵ_{ij} are computed as follows.

$$
\begin{aligned}
(\tfrac{\partial \boldsymbol{t}}{\partial \boldsymbol{\epsilon}}) = \left(\tfrac{\partial t_{ij}}{\partial \epsilon_{gh}}\right)_{ij,gh} &= -(\nabla_{\boldsymbol{t}}^2 Z)^{-1} \nabla_{\boldsymbol{\epsilon}} (\nabla_{\boldsymbol{t}} Z) \\
&= \left(\tfrac{\partial^2 Z}{\partial t_{ij} \partial t_{gh}}\right)_{ij,gh}^{-1} \mathrm{diag}(\tfrac{\partial x_{ij}}{\partial \epsilon_{ij}})_{ij}.
\end{aligned} \tag{6}
$$

$$
\frac{\partial x_{ij}}{\partial \epsilon_{gh}} = \frac{\partial x_{ij}}{\partial \epsilon_{ij}} \delta_{ij,gh} + \frac{\partial x_{ij}}{\partial t_{ij}} \frac{\partial t_{ij}}{\partial \epsilon_{gh}}. \tag{7}
$$

The part in the computation that appears difficult is that for computing $\frac{\partial^2 Z}{\partial t_{ij} \partial t_{gh}}$, which however can be efficiently computed by the following method based on Dial's algorithm.

$$
\begin{aligned}
&\frac{\partial^2 Z}{\partial t_{ij} \partial t_{gh}} \\
&= \frac{\partial x_{ij}}{\partial t_{gh}} - \sum_{rs} q_{rs} \left[\frac{-\theta \sum_k \exp(-\theta c_k^{rs}) \delta_{ij,k}^{rs} \delta_{gh,k}^{rs}}{\sum_p \exp(-\theta c_p^{rs})} \right. \\
&\qquad \left. - \frac{-\theta (\sum_k \exp(-\theta c_k^{rs}) \delta_{ij,k}^{rs})(\sum_l \exp(-\theta c_l^{rs}) \delta_{gh,l}^{rs})}{(\sum_p \exp(-\theta c_p^{rs}))^2} \right]
\end{aligned} \tag{8}
$$

By applying Dial's algorithm, Akamatsu [1] showed that

$$
\sum_p \exp(-\theta c_p^{rs}) = \exp(-\theta S_{rs})
$$

can be computed without enumerating all the paths for an OD pair rs. Here we show that

$$\sum_k \exp(-\theta c_k^{rs})\delta_{gh,k}^{rs} \quad \text{or} \quad \frac{\sum_k \exp(-\theta c_k^{rs})\delta_{gh,k}^{rs}}{\sum_k \exp(-\theta c_k^{rs})}$$

and

$$\sum_k \exp(-\theta c_k^{rs})\delta_{ij,k}^{rs}\delta_{gh,k}^{rs} \quad \text{or} \quad \frac{\sum_k \exp(-\theta c_k^{rs})\delta_{ij,k}^{rs}\delta_{gh,k}^{rs}}{\sum_k \exp(-\theta c_k^{rs})}$$

can also be computed efficiently based on Dial's algorithm.

It is clear that in a stochastic equilibrium state the fraction of the travelers from r to s who use link ij is

$$\frac{x_{ij}^{rs}}{q_{rs}} = \frac{\sum_k \exp(-\theta c_k^{rs})\delta_{ij,k}^{rs}}{\sum_k \exp(-\theta c_k^{rs})}. \tag{9}$$

Assuming that all of the link costs are fixed, x_{ij}^{rs} can be directly computed by Dial's algorithm. This implies that

$$\sum_k \exp(-\theta c_k^{rs})\delta_{gh,k}^{rs} = \left(\frac{\sum_k \exp(-\theta c_k^{rs})\delta_{gh,k}^{rs}}{\sum_k \exp(-\theta c_k^{rs})}\right)\left(\frac{x_{ij}^{rs}}{q_{rs}}\right)$$

can be efficiently computed.

Let x_{ij-gh}^{rs} denote the number of travelers from r to s who traces link ij and link gh later, and x_{gh-ij}^{rs} denote the number of travelers from r to s who traces link gh and link ij later. Note that either x_{ij-gh}^{rs} or x_{gh-ij}^{rs} is zero if only efficient paths are accounted for in the Dial's algorithm.

By viewing x_{ij}^{rs} as an OD demand from j to s, we have

$$\frac{x_{ij-gh}^{rs}}{x_{ij}^{rs}} = \frac{\sum_k \exp(-\theta c_k^{js})\delta_{gh,k}^{js}}{\sum_k \exp(-\theta c_k^{js})}. \tag{10}$$

It is clear that this can be computed efficiently by Dial's algorithm, in the same manner as for computing (9). Similarly we can efficiently compute

$$\frac{x_{gh-ij}^{rs}}{x_{gh}^{rs}} = \frac{\sum_k \exp(-\theta c_k^{hs})\delta_{ij,k}^{hs}}{\sum_k \exp(-\theta c_k^{hs})}. \tag{11}$$

The above equations can be also derived from the logit model in a more formal manner.

Now let $x_{ij,gh}^{rs} = x_{gh,ij}^{rs}$ denote the number of travelers who use both link ij and link gh, then we have

$$x_{ij,gh}^{rs} = x_{gh,ij}^{rs} = \begin{cases} x_{ij-gh}^{rs} + x_{gh-ij}^{rs} & ij \neq gh, \\ x_{ij}^{rs} & ij = gh. \end{cases} \tag{12}$$

We have

$$\frac{\sum_k \exp(-\theta c_k^{rs})\delta_{ij,k}^{rs}\delta_{gh,k}^{rs}}{\sum_k \exp(-\theta c_k^{rs})} = \frac{x_{ij,gh}^{rs}}{q_{rs}}. \tag{13}$$

To summarize, $\frac{\partial^2 Z}{\partial t_{ij}\partial t_{gh}}$ can be computed based on Dial's algorithm and a technique provided by Akamatsu [1] for computing $\sum_p \exp(-\theta c_p^{rs})$.

3 Extension and Discussion

Since the Wardropian equilibrium in a traffic network is an extreme case of stochastic user equilibrium with $\theta \to \infty$, the method presented here may also be used for the sensitivity analysis for the Wardropian equilibrium by taking θ large enough.

Recently Yang [6] provided a framework for analyzing the user equilibrium in a traffic network with two classes of drivers, one class equipped with advanced traveler information system (ATIS) thus enjoying a higher accuracy in choosing least costly routes, the other class being unequipped and having poorer accuracy.

Let q_{rs} be the number of ATIS-equipped drivers for OD rs and f_p^{rs} be the flow of equipped drivers on route p; let $\hat{q}_{rs}$ be the number of ATIS-equipped drivers for OD rs and $\hat{f}_p^{rs}$ be the flow of unequipped drivers on route p. The route choice behavior of the two classes are described by the following logit models.

$$f_p^{rs} = q_{rs}\frac{\exp(-\theta_1 c_p^{rs})}{\sum_{k\in R_{rs}} \exp(-\theta_1 c_k^{rs})}, \tag{14}$$

$$\hat{f}_p^{rs} = \hat{q}_{rs} \frac{\exp(-\theta_2 c_p^{rs})}{\sum_{k \in R_{rs}} \exp(-\theta_2 c_k^{rs})}. \tag{15}$$

It is understood that

$$\theta_1 > \theta_2, \tag{16}$$

standing for the assumption that the equipped car drivers enjoy a higher accuracy than the unequipped in choosing least costly routes, by paying an extra cost for the equipment.

This model also has the following dual formulation

$$\min_{\boldsymbol{t}} Z(\boldsymbol{t}, \epsilon) = \sum_{ij} \int_{t_{ij}(0,\epsilon_{ij})}^{t_{ij}} x_{ij}(\nu, \epsilon_{ij}) d\nu - \sum_{rs} q_{rs} S_{rs}(\boldsymbol{c}^{rs}(\boldsymbol{t})) - \sum_{rs} \hat{q}_{rs} \hat{S}_{rs}(\boldsymbol{c}^{rs}(\boldsymbol{t})), \tag{17}$$

where

$$S_{rs}(\boldsymbol{c}^{rs}(\boldsymbol{t})) = -\frac{1}{\theta_1} \ln \sum_k \exp(-\theta_1 c_k^{rs}),$$

$$\hat{S}_{rs}(\boldsymbol{c}^{rs}(\boldsymbol{t})) = -\frac{1}{\theta_2} \ln \sum_k \exp(-\theta_2 c_k^{rs}).$$

The minimizing condition for this unconstrained program is as follows

$$\begin{aligned} & \frac{\partial Z}{\partial t_{ij}} \\ = \; & x_{ij}(t_{ij}, \epsilon_{ij}) - \sum_{rs} q_{rs} \frac{\sum_k \exp(-\theta_1 c_k^{rs}) \delta_{ij,k}^{rs}}{\sum_p \exp(-\theta_1 c_p^{rs})} - \sum_{rs} \hat{q}_{rs} \frac{\sum_k \exp(-\theta_2 c_k^{rs}) \delta_{ij,k}^{rs}}{\sum_p \exp(-\theta_2 c_p^{rs})} \\ = \; & 0, \quad ij \in A. \end{aligned} \tag{18}$$

This is exactly the condition that the link flows should satisfy in the stochastic user equilibrium.

As we have seen in the previous section, the main obstacle in sensitivity analysis is the computation of the partial derivatives $\frac{\partial^2 Z}{\partial t_{ij} \partial t_{gh}}$, which however can be computed efficiently by the method developed in the previous section.

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