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A SYSTEM OF TWO CITIES MODELLED WITH OVERLAPPING GENERATIONS*

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1. Introduction

Unbalanced distribution of population between the metropolitan areas and peripheral regions has been the most serious issue in nationwide spatial planning, stimulating the development of the related theoretical models. Among the models proceeding this paper, UEDA (1993)[1], and MORISUGI et'al (1992)[2] showed the population unevenly distributed in a static system of two cities, which included economy and diseconomy of agglomeration. However, these model dealt with only one type of locaters, or in other words, ignored differences in locater's attributes. On the other hand, as shown in UEDA(1993), we have known it as a stylized fact that a majority of immigrants into Tokyo metropolitan area belong to ***the age group from 18 to 24 years***. The mobility of this group seems to be dominant in forming the unbalanced distribution of population, resulting in so called ***mono-polarization*** in a system of cities .

In fact, some people say, " The young is enjoying various kinds of opportunities for leisure and recreation activities in metropolitan areas, while he/she burdens less cost than the old. In particular, since that age group includes many students, the young are free from social duties and cost sharing. Then the young might be called " a free rider on agglomeration". On the contrary, although the old may take some benefits due to agglomeration of the young, he/she must pay much cost like high land prices, congestion cost, local tax, and so on, not only for him/herself also for the young instead. The old suffers from the agglomeration. ". The contrast between the old and the young suggests that easy aggregation of two group in to one in modelling may hide something important.

So, we are now facing a new question how these two group of population interacting with each other distribute among cities. This paper analyses population distribution in a system of two cities modelled with overlapping generation, so as to an answer to the question. In framework of the model, we discuss "why does unevenly distributed population exist ?" from the point different from the above proceeding models.

2. Model

(1) Sketch of the model

Major assumptions in the model are;1) A system consists of two cities, denoted by subscript i =1,2 .
2) Calender is denoted by a discrete period, the length of which is normalized to be 1. 3) A person can be alive during two periods, as the young in one and the old in the other, denoted by superscript $g=y,\ o$. 4) Any person chooses his/her location once in each period, and reproduces one person at the end of period when he/she is in old generation. 5) Location choice behaviour is formulated as binary logit model including no relocation cost. Although the model might be a kind of Cohort model, the population of both generations can be nomralized to be 1, becuase of 3) and 4).

(2) Formulation of the model

Dynamics of a system is formulated as,

$$\begin{bmatrix} N_1^y(t+1) \\ N_1^o(t+1) \\ N_2^y(t+1) \\ N_2^o(t+1) \end{bmatrix} = \begin{bmatrix} 0 & R_{11}^{oy}(t) & 0 & R_{21}^{oy}(t) \\ R_{11}^{yo}(t) & 0 & R_{21}^{yo}(t) & 0 \\ 0 & R_{12}^{oy}(t) & 0 & R_{22}^{oy}(t) \\ R_{12}^{yo}(t) & 0 & R_{22}^{yo}(t) & 0 \end{bmatrix} \begin{bmatrix} N_1^y(t) \\ N_1^o(t) \\ N_2^y(t) \\ N_2^o(t) \end{bmatrix} \tag{1}$$

where t denotes a period, and $N_i^g(t)$, population by city and by generation, $R_{ij}^{gg'}(t)$ is a probability that a person in generation g and residing in city i chooses location j when in generation g'. If $g=o$, then $R_{ij}^{oy}(t)$ means the choice probability of a reproduced person as explained in 4) . From constraint of total population in each generation, we have,

$$N_1^y(t)+N_2^y(t)=1, \quad and \quad N_1^o(t)+N_2^o(t)=1 \quad for \quad any \quad t \tag{2}$$

Then, we can rewrite the system of equations in (1) as,

$$\begin{bmatrix} N_1^y(t+1) \\ N_1^o(t+1) \end{bmatrix} = \begin{bmatrix} 0 & R_{11}^{oy}(t)-R_{21}^{oy}(t) \\ R_{11}^{yo}(t)-R_{21}^{yo}(t) & 0 \end{bmatrix} \begin{bmatrix} N_1^y(t) \\ N_1^o(t) \end{bmatrix} + \begin{bmatrix} R_{21}^{oy}(t) \\ R_{21}^{yo}(t) \end{bmatrix} \tag{3}$$

A person at the beginning of his/her young generation period chooses the location where the utility over two periods including both young and old periods is expected to be higher, while he/she recognizes the utility only over old period at the beginning of old period. With assumption of myopic insight and no relocation cost, choice probabilities in (3) are formulated as binary logit model as,

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$$R_{11}^{oy}(t) = \frac{\exp(\theta^y(V_1^y(t)+\rho V_1^o(t)))}{\exp(\theta^y(V_1^y(t)+\rho V_1^o(t)))+\exp(\theta^y(V_2^y(t)+\rho V_2^o(t)))} \quad R_{21}^{oy}(t) = \frac{\exp(\theta^y(V_1^y(t)+\rho V_1^o(t)))}{\exp(\theta^y(V_1^y(t)+\rho V_1^o(t)))+\exp(\theta^y(V_2^y(t)+\rho V_2^o(t)))}$$

$$R_{11}^{yo}(t) = \frac{\exp(\theta^o(V_1^o(t)))}{\exp(\theta^o(V_1^0(t)))+\exp(\theta^o(V_2^y(t)))} \quad R_{21}^{yo}(t) = \frac{\exp(\theta^o(V_1^o(t)))}{\exp(\theta^o(V_1^0(t)))+\exp(\theta^o(V_2^y(t)))}$$

…………(4.a),(4.b),(4.c),and (4.d)

where $V_i^g(t)$; the utility level in generation g and in city i , ρ ; discount coefficient of utility in old period, r_{ij}^g ; relocation cost paid by a person in generation g relocating from city i to j , and θ^g ; logit parameter.

The model is closed by specifying $V_i^g(t)$ and therefore, $R_{ij}^{gg'}(t)$ as a function of $N_i^g(t)$. We sepecify that,

$$V_i^y(t) = \alpha N_i^y(t) + \beta N_i^o(t) \qquad \textbf{(5.a)} \qquad\qquad V_i^o(t) = a N_i^y(t) + b N_i^o(t) \qquad \textbf{(5.b)}$$

where , , a, and b ; parameters. In this specification, we suppose that the young and the old interact with each other at their utility level. Such interactions are not symmetric in general, and the parameters in (5) represents magnitude and direction of externality in linear form. Of course, in other specification as non-linear functions, we may find some interesting results. However, the system already has non-linear structure in (4), for simplicity in analysis, we adopt linear formulation in (5).

3. Stationary state of the system

Although we can hardly examine dynamics of the system in an analytical way, we get a stationary state of the system from (3) by setting, $N_i^g(t) = N_i^g(t+1) = N_i^g(\infty)$. Then, we can rewrite (3) as,

$$\begin{bmatrix} N_1^y \\ N_1^o \end{bmatrix} = \frac{1}{|R|}\begin{bmatrix} (1+R_{11}^{oy}-R_{21}^{oy})R_{21}^{oy} \\ (1+R_{11}^{yo}-R_{21}^{yo})R_{21}^{yo} \end{bmatrix} \quad \text{where} \quad |R| = 1-(R_{11}^{oy}-R_{21}^{oy})(R_{11}^{yo}-R_{21}^{yo}) \quad \textbf{(6)}$$

Since we ignore relocation cost, binary logit model in (4) gives $R_{11}^{oy} = R_{21}^{oy} = R_1^y$ and $R_{11}^{yo} = R_{21}^{yo} = R_1^o$. By insetign them and rewriting remaining terms more specifically, we have the condition which the stationary state in (6) must satisfy as,

$$\begin{bmatrix} N_1^y \\ N_1^o \end{bmatrix} = \begin{bmatrix} R_1^y((V_1^y(N_1^y,N_1^o)+\rho V_1^o(N_1^y,N_1^o), V_2^y(N_1^y,N_1^o)+\rho V_2^o(N_1^y,N_1^o)) \\ R_1^o(V_1^o(N_1^y,N_1^o), V_2^o(N_1^y,N_1^o)) \end{bmatrix} \qquad \textbf{(7)}$$

For the convenience of later discussion, here let us rewrite (7) with some manupilations as,

$$\begin{bmatrix} N_1^y \\ N_1^o \end{bmatrix} = \begin{bmatrix} 1\Big/\left[1+\exp\{-\theta^y(V_1^{y+o}(N_1^y,N_1^o)-V_2^{y+o}(N_1^y,N_1^o))\}\right] \\ 1\Big/\left[1+\exp\{-\theta^o(V_1^o(N_1^y,N_1^o)-V_2^o(N_1^y,N_1^o))\}\right] \end{bmatrix} \qquad \textbf{(8.a)}$$

where we define,

$$\begin{aligned} V_1^{y+o}(N_1^y,N_1^o) &= V_1^y(N_1^y,N_1^o)+\rho V_1^o(N_1^y,N_1^o) = AN_1^y + BN_1^o \\ V_2^{y+o}(N_1^y,N_1^o) &= V_2^y(N_1^y,N_1^o)+\rho V_2^o(N_1^y,N_1^o) = A(1-N_1^y)+B(1-N_1^o) \qquad \textbf{(8.b)} \\ A &= \alpha+\rho a, \quad B = \beta+\rho b \end{aligned}$$

The system of equations in (7) or (8) equivalently, has structure of fixed point problem. We have defined that $[N_1^y, N_1^o]^t \in [0,1]\times[0,1] \subset R^2$, which is a bounded compact set. Then, so far as the function $R_{ij}^{gg'}(\cdot)$ is continuous, the existence of the solution satisfying (7) (or (8)) is to be proved from Brouwer's fixed point theorem. However, uniqueness and stability of the state are still other points of discussion.

Although it is not easy to approach to the solution analytically, we can examine uniqueness and stability in some manners. UEDA(1993) showed the stability condition in stochastic location equilibrium in the setting of one type locater and binary logit model, based on Samuelson's dynamic stability. Here we also apply this condition to the model. The adjustment process in the system is formulated as,

$$\begin{bmatrix} \frac{d}{dt}N_1^y(t) \\ \frac{d}{dt}N_1^o(t) \end{bmatrix} = K\begin{bmatrix} (V_1^{y+o}(N_1^y,N_1^o)-V_2^{y+o}(N_1^y,N_1^o)) - F_{1-2}^{y+o}(N_1^y(t)) \\ (V_1^o(N_1^y,N_1^o)-V_2^o(N_1^y,N_1^o)) - F_{1-2}^o(N_1^o(t)) \end{bmatrix} \qquad \textbf{(9)}$$

where

$F_{1-2}^{y+o}(\cdot)$ is an inverse function of $N_1^y(t) = \frac{1}{1+\exp(\theta^y(v_1^{y+o} - v_2^{y+o}))}$,

$F_{1-2}^{o}(\cdot)$ is also defined in the same manner, and K is a contant representing "adjustment speed".

The functions $F(N)$ represent utility level that give the numbers of locater N . The RHS of (9) means that the increment of population in infinitesimal period is proportional to the difference of the utility enough to attract population, $v_1^{y+o} - v_2^{y+o}$ in the above definition, and that dependent on population having located, $V_1^{y+o}(N_1^y, N_1^o) - V_2^{y+o}(N_1^y, N_1^o)$. The Jacobian matrix of the RHS of (9) and the necessary and sufficient condition for local stability of a stationary state are ,

$$J = K\begin{bmatrix} 2A - \frac{\partial F_{1-2}^{y+o}}{\partial N_1^y} & 2B \\ 2a & 2b - \frac{\partial F_{1-2}^{o}}{\partial N_1^o} \end{bmatrix} \qquad (12)$$

$$\begin{aligned} tr(J) &= 2A - \tfrac{\partial F_{1-2}^{y+o}}{\partial N_1^y} + 2b - \tfrac{\partial F_{1-2}^{o}}{\partial N_1^o} < 0 \\ \det(J) &= \left(2A - \tfrac{\partial F_{1-2}^{y+o}}{\partial N_1^y}\right)\left(2b - \tfrac{\partial F_{1-2}^{o}}{\partial N_1^o}\right) - 4aB > 0 \end{aligned} \qquad (13)$$

Examining this condition, we can verify the stability of a stationary state.

4. Population distribution

On the setting explained in Section 2. and 3., we analyze population distribution in several cases that we can examine analytically. Parameters in the utility function in (5), representing agglomeration merits interacting between old and young groups, characterizes each case to analyze.

The point $[N_1^y, N_1^o]^t = [1/2, 1/2]$, in the diagram, called the Centre in the rest of the paper is a trivial solution of the system of equations in (8) in any case. Since we have introduced no asymmetric factors a priori into the model, it is quite natural that the system has the solution of evenly distributed population of both age groups. However, the even distribution is not always stable, and we may face to multiple equilibria. Then, even in the setting of no asymmetry assumed a priori, we may find that the uneven distribution of population is attainable as a stable stationary state of the system.

The cases characterized by the parameters are $2^4 = 16$. Since we cannot discuss all of the cases because of limitation of space, let us focus on some particular cases. In the cases to analyze, we are to draw phase diagram of the system of differential equations in (10), with marking stationary states as intersections of curves, $\left[\frac{d}{dt}N_1^y = 0\right]$ and $\left[\frac{d}{dt}N_1^o = 0\right]$. The stability indicated in the diagram has been verified by using the condition (13).

Next, we explain the parameter setting and findings from phase diagram in selected cases. The drawing of it is to be explained in appendix.

Case 1) $A > 0,\ B > 0,\ a > 0,\ b > 0$

This case is that all externalities are positive, or that population agglomerating in a city, regardless of age groups, raises up utility level of any resident locating there. The phase diagram of this case is shown in Figure 1. Depending on the value, we can draw two patterns of diagram, the left one of which shows that there exist 5 stationary states at most. The point $[N_1^y, N_1^o]^t = [1/2, 1/2]$, the Centre, is a trivial solution of the system of equations in (8), as explained already. However, since this is not a stable state, judging from the condition (13), the unevenly distributed population is attainable. At the unevem distribution, more than half of both the young and the old groups are agglomerated into one of of the two cities.

Case 2) $A < 0,\ B < 0,\ a < 0,\ b < 0$

Completely reverse to case 1, all externalities are negative. Again, we can get two patterns of phase diagram, as shown in Figure 2 . In the left, there exist 3 stationary states includin the Centre and 2 uneven distribution of population. The Centre is unstable, and both in the rest are stable. Thus, the uneven population distribution is attainable. However, differing from Case 1 in the above, at the uneven distribution, les that half of the old age group concentrate in one city, while more than half of the young are agglomerated. In this case, the old and the young show a kind of segregation, as often observed in ecological world. On the contrary, in the right, there exists an unique stationary state at the Centre, and furthermore that is stable. Thus in this case, there are no uneven distribution.

Case 3) $A > 0,\ B < 0,\ a < 0,\ b < 0$

This case is that, agglomeration of the old decreases utility level of both the young and the old, while the young agglomerating in a city raises up the utility of the young, and reduces that of the old. Even though the agglomeration of the young is negative in the old period, a <0, the strong positive externality in the young period, >0 may result in A >0. As an example of such externality, we can suppose that urban-styled activities seems to be sustained by the agglomeration of the young, and to attract the young, in other words, ***snow-balling process*** . The diagram is in Figure 3. Although there assumed to exist the positive externality only within the young, the uneven distribution of population is *attainable*.

Case 4) $A > 0,\ B < 0,\ a > 0,\ b < 0$

This case is a modification of case 3, replacing a <0 with a >0. Then, the agglomeration of the young raises up the utility of the old. As an example of this type of externality, we suppose that the young makes a city more productive, and therefore, the old may be benefited. The diagram of this case including two patterns is in Figure 4. Although the diagram is a little different from case 3, the result in the left is very similar, and therefore we have the same interpretation as it. However, the right shows an interesting but difficult case to interpret. The Centre is an unique stationary state, but unstable. We cannot conclude dynamic behavior of the system from the diagram. This is still a point for us to investigate further.

As one of the remarks common among the cases, we can say that, it depends on an initial value of $[N_1^y(0), N_1^o(0)]^t$ into which stationary state the dynamic state of the system would converge. This is an indication that population distributed in a system of cities has the property, so called "***history dependence***".

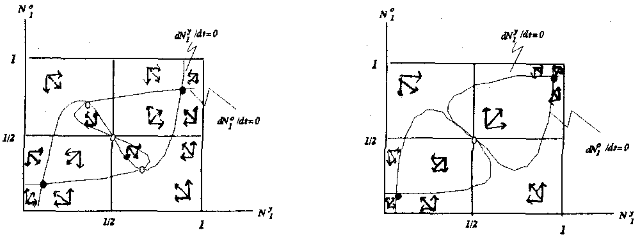

Fig.1 Phase diagram of Case 1

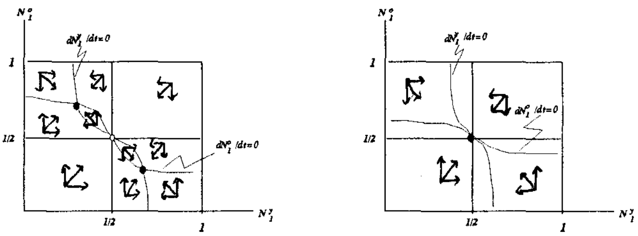

Fig.2 Phase diagram of Case 2

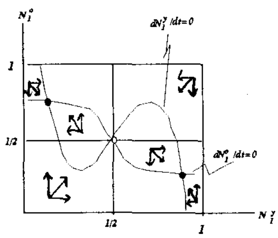

Fig.3 Phase diagram of Case 3

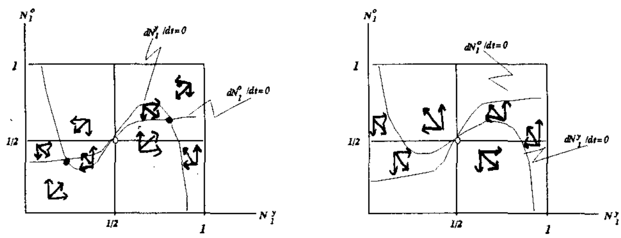

Fig.4 Phase diagram of Case 4

● Stable ○ Unstable

5. Concluding Remarks

Although we have shown only 4 selected cases, it is clearly shown that if there exists an positive externality within the group of young, then an uneven distribution of population is attainable, in spite of other coexisting negative externalities.

Many descriptive literatures on the uneven distribution of population, emphasized the role of asymmetrically distributed resource, or location specific factors. The model in this paper includes no asymmetric factors assumed a priori, but succeeded in reproducing the uneven distribution. This is nothing but the thought of " Second Nature" by Krugman (1991).

We have some tasks to tackle with in the next stage of research. The dynamics of the system shown in 1. was formulated originally with intending to introduce relocation cost into the system. Although the system would be more and more complicated, resulting in difficulties in handling it analytically, relocation cost might cause cohesion and hysteresis that can alter an unstable state in this paper to stable one. The model in this paper has not shown fully dynamic characteristics yet. The extension in this direction leads to the really dynamic analysis of a system of cities.

The other task is more fine microfundation for the relation between utility level and population. The utility level should be derived as a function of population, on a full setting of general equilibrium of an economy. UEDA and MORISUGI (1994)[3] provides one of such derivation. This work must be installed into framework of this paper.

Reference

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