A Derivation of Logit Model and Its Implications

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The Logit model is the well-known probabilistic choice model, which is derived in the random utility theory. While this model has some merits because of its simple formulation, there are problems of the independence from irrelevant alternatives (IIA) property and of dealing with generated demand. This paper indicates a derivation of the Logit type demand of model which is derived in the consumer behavior theory and the model enables us to explicitly deal with generated demand. Furthermore, some useful implications of this derivation are shown with respect to the setup of the indirect utility function from the demand function, the derivation of the log-sum function from the Logit model, and the benefit definition and estimation of the newly introduced transport facility.

I. *INTRODUCTION*

Economists have been paying increasing attention to the application of the Logit model in transport economic analysis, which is used as the simplest and convenient probabilisticbilistic choice model. Nevertheless, there are problems of the independence from irrelevant alternatives (IIA) property, and dealing with the generated transport demand by the use of the conventional Logit model. The former generated by the similarity of the choice axiom has been almost overcome by the development of the Nested-Logit model or the Probit model, but the latter due to the given total transport demand has not yet been overcome.

In the framework of the random utility theory, the Logit model is derived only by assuming that the random component (ε) varies according to the Gumbel distribution. On the other hand, this paper derives a Logit type of transport demand model in the consumer behavior theory, and shows that the model enables us to explicitly deal with the generated transport demand.

In this paper, furthermore, some useful implications of this derivation of the Logit model are indicated. First, even if the indirect utility function is unknown, the function form can be estimated by using the system of demand functions and the Roy's identity. Second, the log-sum function, that is, the satisfaction function is derived directly from the Logit model obtained here. Third, the benefit definition and estimation for the newly introduced transport facility are processed by this approach in the framework of Discrete-Continuous modeling.

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II. A DERIVATION OF LOGIT MODEL

We follow the micro-economic theory of consumer behavior in depicting the household decision problem which can be transformed into a demand function expressing the action of a consumer. It is assumed that the consumer maximizes its utility under some given budget constraints.

$$\max_{x}.\ U(x_1, x_2, \ldots x_n), \qquad (1.a)$$

$$\text{s.t. } \Sigma_j\ p_j x_j \le y,\ [j=1,2,\ldots,n]. \qquad (1.b)$$

where

- $U(.)$ denotes the direct utility function of individual,
- x_j is the demand of service j, generally assumed to be non-negative continuous variable,
- p_j is the price of service j,
- y is the income of individual.

The following is a solution which gives optimal amounts of x_j to above maximization problem (1):

$$x^{*}_{j} = x(p,y), \; [j=1,2,\ldots,n], \qquad (2)$$

Equation (2) is known as the demand function which expresses the choice of consumer for the given price vector p and income y.

The demand functions provide an expression for the "optimal" consumption of services that can now be substituted in the direct utility function. As the result, we obtain the maximum utility that is achievable under the given price vector and income, which is called the indirect utility function:

$$V(p,y) = \mathrm{Max.}\ \{U(x) \mid \Sigma_j\ p_j x_j \le y\}. \qquad (3)$$

The important property of the indirect utility function which we will use as the main tool in this paper is the Roy's identity which yields the observed market demand curves as a partial derivative of $V(p,y)$, that is,

$$x_i(p,y) = - \frac{\partial V(p,y)/\partial p_i}{\partial V(p,y)/\partial y}. \qquad (4)$$

To demonstrate this useful property, suppose that the indirect utility function has the form:

$$V(p,y) = \Sigma_j\ a_j\ \exp(b_j y - c_j p_j). \qquad (5)$$

From the Roy's identity as shown in equation (4), the demand function which corresponds to equation (5) can be found by differentiation:

$$x_i(p,y) = \frac{a_i c_i\ \exp[b_i y - c_i p_i)]}{\Sigma_j\ a_j b_j\ \exp[b_j y - c_j p_j]}. \qquad (6)$$

This demand function is not the Logit model exactly, but itself has the Logit type. However, we can easily show that in the case when $c_i = b_i$ for any i, this function becomes exactly the Logit model. Also, the share modeis the Logit form as:

$$\frac{x_i}{\Sigma_j x_j} = \frac{a_i c_i\ \exp[b_i y - c_i p_i]}{\Sigma_j\ a_j c_j\ \exp[b_j y - c_j p_j]}. \qquad (7)$$

As the result, shows that the share type of the Logit model can be derived directly from the specified indirect utility function without any assumption. Also, by using this demand function we can easily and explicitly deal with the generated demand. One of the typical implication of this can be used for explaining of the newly introduced transport facility. To emphasize our derivation some useful implications are demonstrated in the following section.

III. IMPLICATIONS

(1) DERIVATION OF INDIRECT UTILITY FUNCTION

As we know, the link between the indirect utility function and the demand functions through equation (4) shows that: if one has a specific functional form for the indirect utility function, the form of the ensuring demand functions follows from it through differentiation, using equation (4). If one has a specific form for a demand function, the indirect utility function which corresponds with it can be found by integration, also using equation (4). This property ensures that our derived demand functions which itself has the Logit type as shown in equation (6) can be use to derive the indirect utility function by using the Roy's identity. To demonstrate this useful argument, the basic idea used here is to start from a specified functional demand system:

$$x_i(p,y) = \frac{a_i c_i\ \exp[b_i y - c_i p_i)]}{\Sigma_j\ a_j b_j\ \exp[b_j y - c_j p_j)]}, \qquad (8.a)$$

$$= - \partial V(p,y)/\partial p_i \quad \partial V(p,y)/\partial y, \qquad (8.b)$$

where

- a_j, b_j, c_j are unknown paremeters.

For convenience, we consider a simple example with three services (j=1,2,3). Denote that the service with subscript j=1 is composite goods and the other with subscript j=2,3 are transport services, then we obtain the system of demand equations for x_1, x_2 and x_3 as follows:

$$x_1(p,y) = \frac{a_1 c_1\ \exp(b_1 y - c_1 p_1)}{\Sigma_j\ a_j b_j\ \exp(b_j y - c_j p_j)}, \qquad (9.a)$$

$$x_2(p,y) = \frac{a_2 c_2 \ \exp(b_2 y - c_2 p_2)}{\Sigma_j \ a_j b_j \ \exp(b_j y - c_j p_j)}, \quad (9.b)$$

$$x_3(p,y) = \frac{a_3 c_3 \ \exp(b_3 y - c_3 p_3)}{\Sigma_j \ a_j b_j \ \exp(b_j y - c_j p_j)}, \quad (9.c)$$

Note that the composite goods are taken as numeraire, that is, $p_1=1$.

Solve these linear partial differential equation systems by applying the method of completely integrable condition to derive the indirect utility function form as function:

$$V(p,y) = f[\Sigma_j \ a_j \ \exp(b_j y - c_j p_j)]. \quad (10)$$

Now we have a solution to the Roy's identity, but we need to check whether we have a valid indirect utility function which arises from consumer utility maximization. The derived function V(p,y) is monotonically increasing in income if $b_j>0$ and decreasing in price if $c_j>0$ and $a_j>0$ for any j=1,2,3 which satisfies the characteristicstic of the indirect utility function. Also, if we set $\exp[y]\equiv y$, $\exp[p_j]\equiv p_j$ and $b_j\equiv c_j$ for any j, then we can easily show that our derived indirect utility function has the form of the Indirect Addilog demand model. In addition, according to the Roy's identity which is an important property of the indirect utility function which we used, if one differentiates equation (10) with respect to p_j and y, the result obtained for demand systems must have the form as given in equation (9). It emphasizes that even if the indirect utility function is unknown, the function form is estimated by the system of demand function.

(2) DERIVATION OF LOG-SUM FUNCTION

The system of the demand function as given in equation (6) has the charateristic of which the function itself has Logit type. Its useful and convenience in which we can derive the indirect utility function already demonstrated in the first implication. However, it is interesting to show that if in the case when $b_1=0$ and $c_j=b_j$ for any $j\neq1$, the function given in equation (6) become exactly Logit model as follows:

$$x_2(p,y) = \frac{a_2 b_2 \ \exp[b_2(y-p_2)]}{\Sigma_j \ a_j b_j \ \exp[b_j(y-p_j)]}, \quad (11.a)$$

$$x_3(p,y) = \frac{a_3 b_3 \ \exp[b_3(y-p_3)]}{\Sigma_j \ a_j b_j \ \exp[b_j(y-p_j)]}. \quad (11.b)$$

Solving this partial differential equation system (11) by the same method those used before. The solution given is the indirect utility function:

$$V(p,y) = \ln \Sigma_j a_j \ \exp[b_j(y-p_j)] + C, \quad (12)$$

where
- Constant C takes any suitable value.

It is necessary to point out that the derived indirect utility in equation (12) has the log-sum function, that is satisfaction function. From this result, we come to the conclusion that the satisfaction function itself is the indirect utility function. In *this sense, it contributes one more piece of* evidence for the hypothesis that the inclusive value is superior to the other proposed definitions of benefit measurement. Obviously, if given the indirect utility function as (12), the Logit model will be derived.

(3) BENEFIT ESTIMATION OF NEW TRANSPORT MODE

The economic evaluation of the newly introduced transport facility, which had not yet appeared at the time of investigation, often plays an important role in the public transportation policy decision whether or not to introduce the new mode. In the application of the conventional benefit measurement framework, there is aproblem of imagining the situation of "before-introduction" for assigning the level of transport service of which it have not yet realized. In particular, there is the problem of estimating the cost of the new mode in the "before situation" which is often used as a basic tool for approximation in conventional approach. By application of this framework, the equivalent variation (EV) as an exact measure of welfare change is computed directly from an indirect utility function which is derived from the specified system of demand equations and the use of the Roy's identity as mentioned above.

Almost all of the conventional treatments of consumer behavior consider the maximization of the utility function defined over continuous variables. But for the new facility, the Discreate-Ccontinuous model which has both discrete and continuous variables, is more convenient because it can fully express both "existing" and "non-existing" situation of that new transport mode and take it as discrete variable accompanied with continuous consumption of the other goods. Let us consider a simple case in which there is existing only one transport service with its demand, say x_2. Now the situation includes one new transport mode that is introduced with its demand, say x_3. Recall the subscript i=1 which denotes composite goods as z. In the framework of Discrete-Continuous choice model the above maximization problem can be expressed as following:

$$V(p,y) = \underset{z,x}{\text{Max.}}\ U(z,x_2,x_3), \quad (13.a)$$

$$\text{s.t. } y = z + p_2x_2, \ (x_3=0), \quad (13.b)$$

$$y = z + p_2x_2 + p_3x_3, \ (x_3 \neq 0), \quad (13.c)$$

where
- (13.b) when the new mode doesn't exist,
- (13.c) when the new mode exists.

Within this framework, the above maximization problem which is explained by Discrete-Continuous choice model can be easily processed, not only in the case of three goods but also for the many goods case.

As the result, obtained by solving this maximization problem when dealing with the constraint (13.c) show that the derived indirect utility function given in the equation (10) is solution where the consumer maximizes his utility by controlling z, x_2 and x_3 under *his budget constraint* which corresponds to the "after situation" with the introduction of new transport mode. By using this framework the welfare change by the newly introduced transport facility can be expressed as follows:

a. "before situation"

$$V^0(a_3=0,y) = \underset{z,x2}{\text{Max}}\ U(z,x_2,x_3=0)$$

$$\text{s.t. } z + p_2x_2 = y$$

↓ Newly introduced Transport facility

b. "after situation"

$$V^1(a_3,y) = \underset{z,x2,x3}{\text{Max}}\ U(z,x_2,x_3)$$

$$\text{s.t. } z + p_2x_2 + p_3x_3 = y$$

For convenience and comparision, we set $x_3=0$ at the "before situation", where $x_3=0 \Leftrightarrow$ • $a_3=0$, and this notation expresses the situation when the new mode does not exist.

From the definition of the equivalent variation(EV) that is defined as the minimum amount of compensation which is needed for an individual in order to give up the project while sustaining his welfare level at the "after situation". For this case the welfare change, EV, can be defined formally as:

$$V(a_3=0,y+EV) = V(a_3,y), \quad (14)$$

that is,

$$\begin{aligned} &a_1 \exp[b_1(y+EV)-c_1p_1] \\ &\quad + a_2 \exp[b_2(y+EV)-c_2p_2] \\ &= a_1 \exp[b_1y-c_1p_1] \\ &\quad + a_2 \exp[b_2y-c_2p_2] \\ &\quad + a_3 \exp[b_3y-c_3p_3]. \end{aligned} \quad (15)$$

The value of EV which satisfies the above equations system is the benefit caused by the newly introduced transport facility that we are looking for. In term of the expenditure function:

$$EV = e(p^0,V^1) - y_0, \quad (16)$$

where
- $e(p^0,V^1)$ is the correspond expenditure function followed by equation (10).

Here the problem is the form of the function e(.) which we can not express mathematically in explicit form. However, by using the conventional technique and if given all the available market data then from the demand systems in equation (9) all parameters will be estimated by econometric procedure. Substituting these coefficients into equation (16) and using the computer program package, we can easily calculate the quantity EV as the exact measurement of the welfare change. Also, the quantity obtained for welfare change can be developed in this process by using the concept of the compensating variation CV.In the sense of the practical application, since the conventional methods can deal only with the improvement case, this approach seems preferable because only available market data is required. In *addition*, with the well-specified system of demand function we can calculate exact welfare change by the newly introduced transport service without any approximation.

We have derived the exact welfare change process from the specified demand systems which can be summarized as following:

. Step 1 : Solve the partial differential equations system obtained by using the specified *systems of demand* equations and Roy's identity equations. By doing so, we can derive unobserved indirect utility function, expenditure function and expression for the equivalent variation (EV) in term of the expenditure function.

. Step 2 : The equation obtained for (EV) has all unknown parameters of demand functions. Thus, we can estimate these parameters through the specification demand equations by econometric procedures using the available market data.

. Step 3 : By substituting all the estimated parameters into the equation obtained for EV, we can compute exact welfare change due to newly introduced transport facility.

IV. DISCUSSION

Let us consider some of the other well-known systems of demand equations.

a. The Translog functions

$$x_i(q) = \frac{q_i^{-1}(a_i + \sum_{j=1}^{n} b_{ij} \ln q_j)}{\sum_{k=1}^{n} a_k + \sum_{k=1}^{n}\sum_{m=1}^{n} b_{km} \ln q_m}$$

where

- b_{ij} are positive parameters for all i and j and $b_{ij}.b_{ji}$
- $q_i = p_i/y$,
- p_i is the price of service i,
- y is the income of individual,

and the share mode obtained:

$$\frac{x_i}{\sum_{k=1}^{n} x_k} = \frac{q_i^{-1}(a_i + \sum_{j=1}^{n} b_{ij} \ln q_j)}{\sum_{k=1}^{n} q_k^{-1}(a_k + \sum_{m=1}^{n} b_{km} \ln q_m)}$$

b. The Diewert functions

$$x_i(q = \frac{\sum_{j=1}^{n} b_{ij} q_i^{-1/2} q_j^{1/2} + b_{0i} q_i^{-1/2}}{\sum_{k=1}^{n}\sum_{m=1}^{n} b_{km} q_k^{1/2} q_m^{1/2} + \sum_{m=1}^{n} b_{0m} q_m^{1/2}}$$

and the share mode as follows:

$$\frac{x_i}{\sum_{k=1}^{n} x_k} = \frac{\sum_{j=1}^{n} b_{ij} q_i^{-1/2} q_j^{1/2} + b_{0i} q_i^{-1/2}}{\sum_{k=1}^{n}\sum_{m=1}^{n} b_{km} q_k^{-1/2} q_m^{1/2} + \sum_{k=1}^{n} b_{0k} q_k^{-1/2}}$$

These above systems of demand functions also have characteristic of which the transformation of the share has the Logit form just simple take exp[ln(.)]. However, these function itself do not have the Logit type. And also it is rather inconvenient to handle in this approach. In addition, from these system of demand function we can not derive the indirect utility function as the log-sum form. Thefore, the system of demand function we used here is significant and validity.

V. CONCLUSION

We have derived the Logit type of demand model within the framework of the consumer behavior theory, and also have demonstrated some of its useful implications. This approach seems to be worthwhile on three espects. First, we can use this demand model to explicitly treat with the generated transport demand, while the conventional Logit model does not have such ability. Also the system of demand functions used here is more convenient and easy to handle, especially, when setting the demand function of the newly introduced transport facility. Second, this approach is more generalized because we do not depend on the assumption of probabilistic distribution of the random component, but we still can derive the Logit model from the specified indirect utility function as the log-sum function form. Third, this method can be used to measure the welfare change both for the improvement and the newly introduced case. For the newly introduced case, this approach is very significant because it enables us to directly measure the exact welfare change without any approximation.

However, this approach has been developed only in the framework of the consumer behavior. But in the general equilibrium approach how can it be adopted ?. Also, based on individual choice mode which we have demonstrated in this framework how should it be developed generally in a joint choice framework such as route choice, destination choice and so on. We hope these questions can be brought out in further research.

VI. REFERENCES.

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