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A New Approach for Dynamic Traffic Assignment

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Abstract. This paper considers a general problem of dynamic system optimal (DSO) traffic assignment. Distinguished from the previous approaches, the many-to-many DSO traffic assignment model is established by using the constrained optimal control theory. According to the characteristics of cost functional, the DSO traffic assignment problems are divided into two groups, the normal DSO problems and singular DSO problems. The necessary optimality conditions are given for both kinds of problems. It is also shown that the DSO traffic assignment problems can be solved by using the general constrained optimal control approaches.

1. Introduction

In most of the dynamic traffic assignment models proposed to date, only very simple cases are investigated. Merchant and Nemhauser [4],[5] consider the dynamic system optimal (DSO) traffic assignment problem for a many-to-one case. Quite recently, Matsui [3] suggests a dynamic traffic assignment model applying to a simple one OD network by using the maximum principle. In order to study the general dynamic traffic assignment problems, the general DSO traffic assignment problems are investigated at first.

In this paper, our objective is to provide a macro-model for minimizing total system cost in dynamic traffic assignment. The model of many-to-many network DSO traffic assignment is established in the first part. This model is an advance over Merchant's many-to-one model. It differs from Merchant's method by using the constrained optimal control theory. The model is described as following.

The model is a nonlinear constrained optimal control problem. The state equation of every link is presented in a general nonlinear form. Suitable assumptions about cost function and exit function on one link are made. The state equations, together with other constraints of node conservation equations, the given O/D flow rate Q(t), and nonnegative conditions are given in order to minimize the total system cost.

Necessary optimality conditions are conducted as an application of the general theory. According to the different optimality conditions caused by different cost functional assumption, the DSO traffic assignment problems are classified into normal DSO and singular DSO problems.

In the applications of optimal control theory, the assumptions of cost function and exit function decide the solvablility of the problem and the characteristics of solutions. Generally in normal DSO traffic assignment problems, if the cost function is assumed to be convex with respect to in-flow u(t), the uniqueness of solution can be obtained. In the cases of other cost functional assumptions, the problem usually has no unique solution due to the singular solutions.

The next section presents the models of dynamic system optimal (DSO) traffic assignment. The transformation of DSO traffic assignment problems to a canonical optimal control formulation is described in section 3. This section

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Key Words. Dynamic System Optimal (DSO) Traffic Assignment, Optimal Control Theory.

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also highlights the principal first-order necessary conditions of two kinds of DSO traffic assignment programs, but only to the extent necessary for understanding the work herein. The general solution approaches of DSO traffic assignment are introduced in Sec. 4. In order to make some comparisons with previous works in this field, the analysis of dynamic traffic assignment models is reported in Sec. 5. Concluding comments are presented in the last section.

2. Formulation of Dynamic System Optimal Traffic Assignment

2.1 Basic Network Notation

N: node set, (n: total nodes number);

L: link set, (m: total links number);

L_{ok}: set of links starting from origin k, k=1,...,r, (r: total origins number);

L_{dk}: set of links directing to destination k, k=1,...,s, (s: total destinations number);

N_1: set of nodes which are not origins nor destinations, (nodes number: $n_1 = n - r - s$);

$x_i(t)$: number of vehicles on link i at time t;

$u_i(t)$: admitted flow into link i at time t;

$g_i(t)$: exit flow out of link i at time t;

A(q): $\{j \in L \mid j$ points out of node $q\}$;

B(q): $\{j \in L \mid j$ points into node $q\}$;

$Q_{ok}(t)$: generation in-flow from origin k at time t;

$Q_{dk}(t)$: attraction flow to destination k at time t;

$C_i[x(t),u(t),t]$: travel cost on link i for one vehicle entering at time t.

In the following we give the mathematical formulation of DSO traffic assignment programs.

2.2 Dynamic System Optimal (DSO) Traffic Assignment Problem

Consider a performance index of the form

$$I = F[t_o,x(t_o),t_1,x(t_1)] + \int_{t_o}^{t_1} f_o[x(t),u(t),t]\,dt \quad (2\text{-}1)$$

Here $F[t_o,x(t_o),t_1,x(t_1)]$ is the total cost of the system relating to initial and final state variables $x(t_o),x(t_1)$ and the second term of (2-1) is the total cost of the system during time interval $[t_o,t_1]$. The DSO traffic assignment problem is to find the functions u(t) that minimize (or maximize) I and at the same time to satisfy the following constraints:

The state function on link i is:

$$dx_i(t)/dt = u_i(t) - g_i(t), \quad i=1,\dots,m. \quad (2\text{-}2)$$

The in-flow into origin k during interval $[t_o,t_1]$

$$\sum_{j \in L_{ok}} u_j(t) = Q_{ok}(t) \text{ is given}, \quad k=1,\dots,r, \quad (2\text{-}3)$$

and out-flow from destination k during time interval $[t_o,t_1]$

$$\sum_{j \in L_{dk}} g_j[x_j(t)] = Q_{dk}(t) \text{ is also given}, \ k=1,\dots,s. \quad (2\text{-}4)$$

For each node $q \in N_1$, we have the conservation equation

$$\sum_{j \in A(q)} u_j(t) = \sum_{j \in B(q)} g_j[x_j(t)], \quad q=1,\dots,n_1. \quad (2\text{-}5)$$

The non-negative conditions are

$$x_i(t) \geq 0, \quad i=1,\dots,m, \quad (2\text{-}6a)$$

$$u_i(t) \geq 0, \quad i=1,\dots,m. \quad (2\text{-}6b)$$

The boundary conditions are

$$x_i(t_o)=x_{io}, \quad i=1,\dots,m. \quad (2\text{-}7)$$

We call (2-1)-(2-7) program (A). The OD conditions of (2-3) and (2-4) can also be given in an integral form, which means the total numbers of in-flow and out-flow vehicles during interval $[t_o,t_1]$ are given. The different types of OD conditions are preferred for different control purposes and situations.

Exit Function. The exit function $g_i(t)$ represents a physical phenomenon and is mainly dependent on existing vehicles x(t) and in-flow u(t). In a highly congested situation, density distribution along link i can be assumed to be constant and $g_i(t) = g_i[x_i(t)]$ can be thought as a suitable assumption. Therefore in this paper we use the following exit function assumption. The other assumptions of exit function will be discussed in other papers.

Exit Function Assumption (EFA): We say that $\{g_i[x_i(t)]\}$ satisfies EFA when

a) Exit function $g_i=g_i[x_i(t)]$ is a nondecreasing, differentiable, upper bounded, and concave function with respect to $x_i(t)$ for each link i;

b) $g_i[x_i(t),t]$ is continuous on $R^m \times R^1$.

Cost Functional. By changing the performance index, a wide variety of applications of DSO traffic assignment models can be found. The following types of performance index that may be of practical interest have been discussed at length:

1) The total travel cost of in-flow vehicles is minimized in interval $[t_o,t_1]$. The cost functional is

$$I = \int_{t_o}^{t_1} \sum_{i=1}^{m} u_i\, C_i[x_i(t),u_i(t),t]dt \qquad (2\text{-}8)$$

Here the cost function $C_i(t)$ on link i is assumed to be dependent on existing vehicles $x_i(t)$ and in-flow $u_i(t)$ on link i. Other cost function assumptions, such as cost function with link interactions, will be investigated in other papers. For convenience of solution, we introduce the cost function assumption of the present problem as following.

Cost Function Assumption (CFA): We say that $\{C_i[x_i,u_i,t]\}$ satisfies CFA when

a. $C_i[x_i,u_i,t]$ is nonnegative, nondecreasing, and convex with in-flow $u_i(t)$ for all link i;

b. $C_i[x_i,u_i,t]$ is differentiable with x_i and is continuous on $R^m \times R^m \times R^1$.

2) At the end of control($t=t_1$), the number of vehicles existing on the network is minimized. The cost functional is given as following:

$$I = \sum_{i=1}^{m} x_i(t_1) \qquad (2\text{-}9)$$

3. Necessary Conditions of DSO Traffic Assignment

The cost functional, state equations, state and control variables constraints, and boundary conditions determine the solvability of DSO traffic assignment problems. The different necessary conditions and the existence theorems may make it preferable to deal with one particular problem or the other. Herein for two types of DSO traffic assignment problems, or problems with normal solutions and singular solutions, we present the necessary optimality conditions, but only to the extent necessary for understanding the works here. In order to keep consistency with the general optimal control theory, the following canonical form of optimal control problem with general constraints is adopted:

$$\min I = F[t_o,x(t_o),t_1,x(t_1)] + \int_{t_o}^{t_1} f_o(x,u,t)\, dt \quad (3\text{-}1a)$$

with respect to the state $x(t)$ and control $u(t)$ which satisfy the constraints

$$dx(t)/dt - f(x,u,t) = 0, \quad t_o \leqq t \leqq t_1, \qquad (3\text{-}1b)$$

$$S(x,u,t) = 0, \quad t_o \leqq t \leqq t_1, \qquad (3\text{-}1c)$$

$$x(t_o) = x_o \quad \text{given}, \qquad (3\text{-}1d)$$

$$\phi[x(t_1),t_1] = 0. \qquad (3\text{-}1e)$$

For convenience, define the Hamiltonian function H as following:

$$H(x,u,t,\lambda,\rho) = \lambda^T f + f_o + \rho^T S. \qquad (3\text{-}2)$$

Since the constraints $S(x,u,t)$ in DSO traffic assignment problems are linear in the elements of control vector, the singular DSO traffic assignment problems can be defined as following: an extremal arc of the DSO traffic assignment problem is said to be singular if the m x m determinant

$$|H_{uu}|$$

vanishes at any point along it. H_{uu} is the m x m matrix with elements

$$\frac{\partial^2 H}{\partial u_i \partial u_j}, \quad i,j=1,\ldots,m.$$

If the Hamiltonian H is linear in one or more elements of the control vector then the extremal is singular. This is the case considered in the second part of this section.

3.1 Normal DSO Traffic Assignment Problem

First of all, we consider the case of the first cost functional assumption (2-8) in section 2. Since the Hamiltonian H is nonlinear with control variables $\{u_i\}$, this problem belongs to normal DSO traffic assignment problem. As the problem here has a general form of Bolza optimal control problem with equality and inequality constraints of state and control, we use the extended form of the maximum principle. For the sake of convenience, the bounded state variables and control variables are transformed into unbounded forms.

Introduce a set of auxiliary state variables $\{y_i: i=1,..,m\}$ defined by

$$x_i - y_i^2 = 0, \quad i=1,\dots,m, \quad (3\text{-}3)$$

and replace state inequality constraints (2-6a) with Eq.(3-3). Next, compute the time derivative of Eq.(3-3), discard Eq.(2-6a), and replace it with the nondifferential constraints

$$u_i - g_i(x_i) - 2y_iv_i = 0, \quad i=1,\dots,m, \quad (3\text{-}4)$$

the differential constraints

$$dy_i/dt = v_i, \quad i=1,\dots,m, \quad (3\text{-}5)$$

and the initial conditions

$$y_i(0) = \sqrt{(x_{io})}, \quad i=1,\dots,m. \quad (3\text{-}6)$$

Here, the symbol v_i denotes an auxiliary control variable.

As the control solution of the present problem is not singular, we can transform the bounded control into an unbounded form.

Introduce the set of auxiliary control variables $\{w_i: i=1,\dots,m\}$ and rewrite control inequality constraints (2-6b) in the form

$$u_i - w_i^2 = 0, \quad i=1,\dots,m. \quad (3\text{-}7)$$

In this transformation technique, the control vector u(t) is redefined so as to include w(t) as an additional component. However, it should be understood that the components of this augmented control are not free, but must be chosen consistently with (3-7), which becomes a nondifferential constraint to be satisfied everywhere along the interval of integration.

The out-flow constraints (2-4) from destinations are equality constraints which have no explicit dependence on the control variables.

$$G_i = \sum_{j\in L_{di}} g_j[x_j(t)] - Q_{di}(t) = 0, \quad i=1,\dots,s. \quad (3\text{-}8)$$

As the constraints are to apply for all $t_o \leq t \leq t_1$, their time derivatives along the time path must vanish. We must have

$$\frac{dG_i}{dt} = \frac{\partial G_i}{\partial t} + \sum_{j\in L_{di}} \frac{\partial G_i}{\partial x_j}\frac{dx_j}{dt} = 0, \quad (3\text{-}9) \text{ or}$$

$$\frac{dG_i}{dt} = -\frac{dQ_{di}(t)}{dt} + \sum_{j\in L_{di}} \frac{dg_j(x_j)}{dx_j}[u_j(t) - g_j(x_j)] = 0, \quad i=1,\dots,s. \quad (3\text{-}10)$$

Now (3-10) has explicit dependence on u. The total time derivatives of (3-8) play the role of control variables constraints of the type (3-1c). In addition, we add a set of boundary conditions (3-15) at $t=t_1$.

In the light of the previous discussion, we rewrite our DSO traffic assignment program (A) in the following form (B) which is similar to the canonical optimal control program (3-1).

$$\min \ I = \int_{t_o}^{t_1} f_o(x,u,t)dt = \int_{t_o}^{t_1} \sum_{i=1}^{m} u_i C_i[x_i(t),u_i(t),t], \quad (3\text{-}11)$$

with respect to the state x(t) and y(t), the control u(t), v(t) and w(t) which satisfy the following constraints:

differential constraints (2m equations)

$$dx_i/dt - f_i(x_i,u_i,t) = dx_i/dt - u_i(t) + g_i[x_i(t)] = 0, \quad i=1,\dots,m, \quad (3\text{-}12a)$$

$$dy_i/dt - f_{i+m}(y_i,v_i,t) = dy_i/dt - v_i(t) = 0, \quad i=1,\dots,m; \quad (3\text{-}12b)$$

equality constraints ((2m+n) equations)

$$S_i = \sum_{j\in L_{oi}} u_j(t) - Q_{oi}(t) = 0, \quad i=1,\dots,r, \quad (3\text{-}13a)$$

$$S_{i+r} = -\frac{dQ_{di}(t)}{dt} + \sum_{j\in L_{di}} \frac{dg_j(x_j)}{dx_j}[u_j(t) - g_j(x_j)] = 0, \quad i=1,\dots,s, \quad (3\text{-}13b)$$

$$S_{i+r+s} = \sum_{j\in A(i)} u_j(t) - \sum_{j\in B(i)} g_j[x_j(t)] = 0, \quad i=1,\dots,n_1, \quad (3\text{-}13c)$$

$$S_{i+n} = u_i - g_i(x_i) - 2y_iv_i = 0, \quad i=1,\dots,m, \quad (3\text{-}13d)$$

$$S_{i+n+m} = u_i - w_i^2 = 0, \quad i=1,\dots,m; \quad (3\text{-}13e)$$

boundary conditions (2m initial conditions)

$$x_i(t_o) = x_{io}, \quad i=1,\dots,m, \quad (3\text{-}14a)$$

$$y_i(t_o) = \sqrt{(x_{io})}, \quad i=1,\dots,m; \quad (3\text{-}14b)$$

side conditions

$$\phi_i(t_1) = \sum_{j\in L_{di}} g_j[x_j(t_1)] - Q_{di}(t_1) = 0, \quad i=1,\dots,s. \quad (3\text{-}15)$$

The above program (B) is one of the Bolza type and is characterized by the augmented functional

$$J = \sum_{i=1}^{s} \mu_i\phi_i + \int_{t_o}^{t_1} [f_o + \sum_{i=1}^{2m} \lambda_i(x_i - f_i) + \sum_{i=1}^{2m+n} \rho_i S_i]\, dt, \quad (3\text{-}16)$$

where μ, λ, ρ are vector Lagrange multipliers having dimensions s, 2m, (2m+n), respectively. It is known that the minimization of (3-16), subject to (3-12)-(3-15), is identical with the minimization of (3-11), subject to (3-12)-(3-15), regardless of the choice of the multipliers μ, λ, ρ.

The Hamiltonian H is:

$$H(x,y,u,v,w,t,\lambda,\rho) = \sum_{i=1}^{2m} \lambda_i f_i + f_0 + \sum_{i=1}^{2m+n} \rho_i S_i. \quad (3\text{-}17)$$

The first-order optimality conditions are given as following:

$$\partial H/\partial u = \lambda^T f_u + f_{0u} + \rho^T S_u = 0, \quad (3\text{-}18a)$$

$$\partial H/\partial v = \lambda^T f_v + f_{0v} + \rho^T S_v = 0, \quad (3\text{-}18b)$$

$$\partial H/\partial w = \lambda^T f_w + f_{0w} + \rho^T S_w = 0; \quad (3\text{-}18c)$$

(3m algebraic equations)

$$\dot{\lambda}^T = -H_x = -\lambda^T f_x - f_{0x} - \rho^T S_x, \quad (3\text{-}19a)$$

$$\dot{\lambda}^T = -H_y = -\lambda^T f_y - f_{0y} - \rho^T S_y; \quad (3\text{-}19b)$$

(2m differential equations)

$$\lambda^T(t_1) = (\mu^T \phi_x)_{t=t1}, \quad (3\text{-}20a)$$

$$\lambda^T(t_1) = (\mu^T \phi_y)_{t=t1}. \quad (3\text{-}20b)$$

(2m boundary conditions)

In summary, necessary conditions for program (B) to have an optimal value are: 2m differential state equations (3-12); 2m differential optimal condition equations (3-19); 3m algebraic optimal condition equations (3-18); (2m+n) equality constraint algebraic equations (3-13); 4m boundary conditions (3-14),(3-20); and s side conditions (3-15). The system composed of the feasibility equations (3-12)-(3-15) and the optimality conditions (3-18)-(3-20) constitutes a nonlinear, two-point boundary-value problem in which the unknowns are the functions x(t), y(t), u(t), v(t), w(t) and the multipliers $\lambda(t)$, $\rho(t)$.

3.2 Singular DSO Traffic Assignment Problem

Consider the case of the second cost functional assumption in section 2. The canonical DSO traffic assignment program (A) can be rewritten as program (C):

$$\min \quad I = F[x(t_1)] = \sum_{i=1}^{m} x_i(t_1), \quad (3\text{-}21)$$

subject to

differential constraints (3-12); equality constraints (3-13a)-(3-13d); inequality constraints

$$S_{i+n+m} = -u_i \leqq 0, \quad i=1,\dots,m; \quad (3\text{-}22)$$

boundary conditions (3-14); and side conditions (3-15).

In this kind of singular control problem, the control inequality constraints are generally kept as a set of important conditions for optimality solutions. The Hamiltonian is

$$H(x,y,u,v,t,\lambda,\rho) = \sum_{i=1}^{2m} \lambda_i f_i + \sum_{i=1}^{2m+n} \rho_i S_i. \quad (3\text{-}23)$$

The necessary conditions on H is

$$\partial H/\partial u = \lambda^T f_u + \rho^T S_u = 0, \quad (3\text{-}24a)$$

$$\partial H/\partial v = \lambda^T f_v + \rho^T S_v = 0, \quad (3\text{-}24b)$$

which have the similar form as (3-18) with the additional requirement that

$$\rho_i \geqq 0, \quad S_i = 0, \quad i=1+n+m,\dots,n+2m, \quad (3\text{-}24c)$$

$$\rho_i = 0, \quad S_i < 0, \quad i=1+n+m,\dots,n+2m. \quad (3\text{-}24d)$$

Since the Hamiltonian is linear in the control variables u(t) (but nonlinear in the state variables), extremal arcs ($H_u=0$) occur on which the matrix H_{uu} is singular in program (C). For such system, the coefficient of the linear control term in H vanishes identically on a singular arc; thus, the control is not determined in terms of the state and adjoint variables, x and $\{\lambda, \rho\}$, by the necessary condition $H_u=0$ (or minimizing H) along the singular arc. Instead, the control is determined by the requirement that the coefficient of these linear terms remain zero on the singular arc; i.e., the time derivatives of H_u must be zero. By incorporating equation (3-12), the following 2m algebraic optimal conditions are obtained.

$$\frac{d^2(H_u)}{dt^2} = \ddot{\lambda} + \ddot{\rho}^T S_u + 2\dot{\rho}^T \dot{S}_u + \rho^T \ddot{S}_u = 0, \quad (3\text{-}25a)$$

$$\frac{d^2(H_v)}{dt^2} = \ddot{\lambda} + \ddot{\rho}^T S_v + 2\dot{\rho}^T \dot{S}_v + \rho^T \ddot{S}_v = 0. \quad (3\text{-}25b)$$

The other first-order optimality conditions are

$$\dot{\lambda}^T = -H_x = -\lambda^T f_x - \rho^T S_x, \quad (3\text{-}26a)$$

$$\dot{\lambda}^T = -H_y = -\lambda^T f_y - \rho^T S_y, \quad (3\text{-}26b)$$

(2m differential equations)

and 2m optimality boundary conditions (3-20).

We replace $d\lambda/dt$ in time derivatives $d^2(H_u)/dt^2$ with optimality conditions (3-26), differential equations (3-25) will have explicit dependence on control variables {u(t),v(t)}. The equations (3-25) determine the 2m-vector {u(t),v(t)} with the control inequality constraints (3-24c,d).

In summary, necessary conditions for program (C) to have an optimal value are: 2m differential state equations (3-12); 2m differential optimal condition equations (3-26); 2m algebraic optimal condition equations (3-25); (n+m) equality constraint algebraic equations (3-13a)-(3-13d); m multipliers constraints (3-24c,d); 4m boundary conditions (3-14),(3-20) and s side conditions (3-15).

4. Solution Approaches of DSO Traffic Assignment

Unless the state equations, the performance index, and the constraints are quite simple, we must employ numerical methods to solve DSO traffic assignment problems. In section 3, the DSO traffic assignment problem (A) are transformed into two kinds of forms (B) and (C) similar to the canonical optimal control problem (3-1). The corresponding necessary conditions are also presented. Therefore the algorithms for solving this kind of standard optimal control problems can be adopted. One of these algorithms is simply explained in the following in order to give an insight into the solution of the DSO traffic assignment problems.

4.1 Solution for Normal DSO Traffic Assignment

Over the past several years, a family of sequential gradient-restoration algorithms (SGRA) for solving the conical program (3-1) have been developed by A. Miele and his associates [6]. Therefore it's possible to apply SGRA to solve our normal DSO traffic assignment problems. The idea of these algorithms is explained in the following.

Approximate Methods. In general, the differential system (3-11)-(3-15) is nonlinear, and approximate methods must be used to seek a solution iteratively. In order to define convergence in seeking the solution iteratively, let the norm of a vector z be defined as

$$N(z) = z^T z.$$

Then, let P denote the norm squared of the errors associated with the constraints (3-12)-(3-15), and let Q denote the norm squared of the errors associated with the optimality conditions (3-18)-(3-20).

$$P = \int_{t_0}^{t_1} N(\dot{x} - f)dt + \int_{t_0}^{t_1} N(S)dt + N(\phi)_{t1}, \tag{4-1a}$$

$$Q = \int_{t_0}^{t_1} N(\dot{\lambda} + f_x \lambda + f_{ox} + S_x \rho)dt + \int_{t_0}^{t_1} N(f_u \lambda + f_{ou} + S_u \rho)dt + N(\lambda + \phi_x \mu)_{t1}. \tag{4-1b}$$

For the exact optimal solution, one must have

$$P \equiv 0, \quad Q \equiv 0. \tag{4-2}$$

For an approximation to the optimal solution, the following relations are to be satisfied:

$$P \leqq \varepsilon_1, \quad Q \leqq \varepsilon_2, \tag{4-3}$$

where ε_1 and ε_2 are preselected, small, positive numbers.

Sequential Gradient-Restoration Algorithm.

Sequential gradient-restoration algorithms involve a sequence of two-phase cycles, each cycle including the gradient phase and the restoration phase. In the gradient phase, the value of the augmented functional J is decreased, while avoiding excessive constraint violation; in the restoration phase, the constraint error is decreased, while avoiding excessive change in the value of the functional. In a complete gradient-restoration cycle, the value of the functional is decreased, while the constraints are satisfied to a predetermined accuracy. Hence, this sequential gradient-restoration algorithm, with complete restoration (SGRA-CR), produces a succession of suboptimal solutions, which is important for the engineering purposes. The algorithmic details of SGRA-CR can be found in Ref. 6.

4.2 Solution for Singular DSO Traffic Assignment

For the singular DSO traffic assignment problems, we have two kinds of methods to solve them. One is the first-order, or gradient, approach suggested in this paper. In this kind of approach, the necessary conditions $H_u=0$ which have no explicit dependence on control u(t), are transformed into necessary conditions which provide information about u(t); i.e., the time derivatives of H_u must be zero. Then, the gradient algorithm used in normal DSO traffic assignment can also be used to solve the singular DSO traffic assignment problems.

As pointed out by Bell and Jacobson [1], the convergence rate of gradient algorithm sometimes is low for solving singular problems. Therefore they present a method which converts the singular problem into a sequence of nonsingular ones by adding a term

$$\varepsilon \int_{t_0}^{t_1} u^T u \, dt$$

into the cost functional (3-21). As ε is progressively reduced toward zero, we find that the solution of the ε-problem tends to that of the original one. This method is called ε-algorithm. Its application in our singular DSO traffic assignment problems will be studied in future work.

5. Analysis of Dynamic Traffic Assignment Models

5.1 Existence of Solution and Uniqueness of Solution

The existence of solution of the above mathematical programs can generally be conducted from many existence theorems of optimal control. The strict mathematical reasoning of the existence of solutions will be discussed in another paper. In singular DSO traffic assignment programs another important problem of solution remains to be answered. The stationary solutions or singular arcs sometimes may not be minimizing. This point also needs to be studied in future work.

For the uniqueness conditions of solution, we have the following general statement [2]: If the function $f_0[x(t),u(t),t]$ in cost functional is smooth, coercive(i.e., grows quickly at infinity), and convex with respect to dx/dt, this problem has exactly one solution for almost every end condition. But for the actual problems, the constraints to the programs, such as the OD time-varying conditions and initial-end boundary conditions, together with the assumptions of cost functions and exit functions, will be the main factors determining the solution to be unique or not.

5.2 Comparisons With Other Dynamic Traffic Assignment Models

Merchant and Nemhauser[4][5] present a dynamic system optimal traffic assignment model which is applied only to a many-to-one case. The cost function assumed in their model is only dependent with state variable x(t). In fact, the assumption in our presentation that cost function depends on both state and control variables is more general than that in their model. As pointed out by Merchant and Nemhauser, it's difficult to extend their own model to the general many-to-many cases. The necessary conditions in their model are conducted by using Kuhn-Tucker conditions. As it is well known that Kuhn-Tucker conditions are not effective to deal with dynamic problems, their approaches are not preferred for general DSO traffic assignment problems.

Matsui[3] uses the discrete maximum principle to treat the dynamic problem. But his results only apply to one OD network with parallel links. In his analysis of dynamic user equilibrium, he introduces the Lagrangian function. But the constraints of state equations $dx/dt=f(x,u,t)$ don't appear in his Lagrangian function. If the constraints which shouldn't be omitted are put into the Lagrangian function, it's impossible to get the similar results with his one. In his application of discrete maximum principle, an approximate form of cost function is used through the transformation of triangle approximation. But his method is not applicable to a general situation.

The dynamic models have several characteristics distinct with the static models. The concept of path flow is difficult to define due to the differences of link flows of one path even at the same time. The in-flows into links become important in determining the solvability of dynamic problems.

As an important characteristics of dynamic problems, two groups of variables, control variables and state variables, exist in order to decide the dynamic system. This has brought out great difficulties in conducting the similar user equilibrium patterns to static case due to the coupling function of control variables and state variables.

6. Conclusions

This paper represents a new direction in the development of dynamic traffic assignment theory by introducing the optimal control theory. The general DSO traffic assignment problems are formulated as the Bolza type optimal control problems. The normal and singular DSO traffic assignment problems are transformed into a conical form and the first-order necessary conditions are presented. Therefore it's possible to apply the sequential gradient-restoration algorithm to solve our dynamic problems.

There are many theoretical, computational, and empirical questions that remain to be answered. Foremost among them is the quality of the model. In particular, the assumptions of cost function and exit function need to be checked empirically. The uniqueness of solution should be considered carefully and suitable conditions for uniqueness of solution should de conducted under strict mathematical programming. As the further research directions, the efficient algorithms of solving DSO traffic assignment programs should be established and the practical computational tests should be done. Furthermore, the feasibility of applying DSO models to more realistic networks should be studied.

The dynamic problems are quite different from the static problems. The aims and approaches should vary with the characteristics of dynamic problems. It's important to study the dynamic traffic problems by applying the corresponding dynamic approaches.

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Appendix

Notes of Symbols. The transpose of a vector or matrix is indicated by a superscript following the symbol, e.g., x^T or λ^T. The partial derivative of a scalar or vector about one scalar or vector is indicated by a underscript following the symbol, e.g., H_u or ϕ_x.

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