CHARACTERISTICS OF LAMINAR FLOW THROUGH RECTANGULAR CHANNEL

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Introduction

When the laminar flow is a uniform flow, if an appropriate coordinates are applied, the Navier-Stokes' equation may be reduced to the Poisson's differential equation. The same can be said with the torsional problem of elastic bar. In the case of a channel of rectangular section the result is the same as what has been found by the Saint Venant's principle[1]. As a different solution, it has also been found, although under a certain condition, that the energy dissipation of the flow owing to the viscosity becomes minimum[2].

In this paper, the author, putting the solution in a system of orthogonal functions, simply derived its exact solution, in the same manner as Navier did in the problem of the deflection on the simply supported rectangular thin plate[3]. Its results becomes the same to the latter above mentioned. The discharge, the Darcy-Weisbach's friction factor, the momentum correction factor and boundary shear stress are calculated here. And then, the effect of the side wall on the flow in the wide channel is reported. All these calculations have been carried out for the open channels. In this paper, since the flow are treated as the laminar flow, all these results may be applied for the Reynolds number smaller than 500 or so.

I. Velocity Distribution

In the uniform flow, the Navier-Stokes' formula is:

$$\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=-\frac{wI}{\mu} \quad \cdots\cdots\cdots\cdots\cdots(1)$$

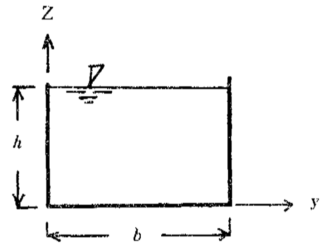

Fig. 1

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where the boundary conditions are $u=0$ at $y=0, y=b$ and $z=0$, and $\partial u/\partial z=0$ at $z=h$ (in which u : local velocity, w : specific weight of water, I : surface slope, μ : coefficient of viscosity). As a solution of this equation, let us assume the following equation :

$$u=\sum_m\sum_n \frac{A_{mn}}{\left(\frac{m\pi}{b}\right)^2+\left\{\left(n+\frac{1}{2}\right)\frac{\pi}{h}\right\}^2}\sin\left(\frac{m\pi y}{b}\right)\cdot\cos\left\{\left(n+\frac{1}{2}\right)\frac{\pi}{h}(z-h)\right\} \quad \cdots\cdots(2)$$

in which $m=1, 2, 3, \cdots\infty$, and $n=0, 1, 2, \cdots\infty$. Obviously, Eq. (2) satisfies all of the boundary conditions. Calculating the terms of $\partial^2 u/\partial y^2$ and $\partial^2 u/\partial z^2$ by using Eq. (2), and substituting them in Eq. (1), we obtain :

$$A_{mn}=\frac{(-1)^n 8\,wI}{\mu m\left(n+\frac{1}{2}\right)\pi^2}$$

where $m=1, 3, 5\cdots\infty$, and $n=0, 1, 2\cdots\infty$. Also putting $2n+1=n'$, Eq. (2) becomes

$$u=\sum_m\sum_{n'} \frac{(-1)^{(n'-1)/2}\, 64\, wIh^2}{\mu m n'\pi^4\left\{\left(\frac{2h}{b}m\right)^2+n'^2\right\}}\sin\left(\frac{m\pi}{b}y\right)\cdot\cos\left\{\frac{n'\pi}{2h}(z-h)\right\}$$

Anew, we represent n instead of n', and taking into account

$$\cos\left\{\frac{n\pi}{2h}(z-h)\right\}=(-1)^{(n-1)/2}\sin\left(\frac{n\pi z}{2h}\right),$$ we get:

$$u=\sum_m\sum_n \frac{64\, wIh^2}{\mu m n\pi^4\left\{\left(\frac{2h}{b}m\right)^2+n^2\right\}}\sin\left(\frac{m\pi}{b}y\right)\cdot\sin\left(\frac{n\pi}{2h}z\right) \cdots\cdots\cdots\cdots\cdots(3)$$

where $m, n=1, 3, 5, \cdots\infty$.

(1) When $h/b\leq 0.5$

In this double series, since $\sum_m$ converges slower than $\sum_n$, reducing $\sum_m$ according to the Fourier series, we transformed the series into the single infinite series of $\sum_n$. Since

$$\frac{4}{\pi}\sum_m\frac{1}{m}\sin\left(\frac{m\pi}{b}y\right)$$

is the Fourier expansion of the following functions

$$\underset{-b \qquad\qquad\qquad 0 \qquad\qquad\qquad b}{\overset{-1 \qquad\qquad\qquad +1}{\overline{\qquad\qquad\qquad\qquad\qquad\qquad}}}$$

$$\frac{4}{\pi}\cosh\left(\frac{\alpha\pi}{2}\right)\sum_m \frac{m}{m^2+\alpha^2}\sin\left(\frac{m\pi y}{b}\right)$$

is the Fourier expansion of the following functions (where α is constant)

$$\underset{-b \qquad\qquad\qquad\qquad 0 \qquad\qquad\qquad\qquad b}{\overset{-\cosh\left\{\alpha\left(\frac{\pi}{2}+\frac{\pi y}{b}\right)\right\} \qquad +\cosh\left\{\alpha\left(\frac{\pi}{2}-\frac{\pi y}{b}\right)\right\}}{\overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}}},$$

and also

$$\frac{8}{\pi}\sum_n \frac{1}{n^3}\sin\left(\frac{n\pi z}{2h}\right)$$

is the Fourier expansion of the following functions

$$\underset{-2h \qquad\qquad\qquad\qquad 0 \qquad\qquad\qquad\qquad 2h}{\overset{\frac{\pi^2 z}{2h}\left(1+\frac{z}{2h}\right) \qquad \frac{\pi^2 z}{2h}\left(1-\frac{z}{2h}\right)}{\overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}}},$$

therefore, Eq. (3) becomes as follows :

$$u=\frac{wI}{\mu}\left[z\left(h-\frac{z}{2}\right)-\frac{16h^2}{\pi^3}\sum_n\frac{1}{n^3}\cdot\frac{\cosh\left\{\frac{n\pi b}{4h}\left(1-\frac{2y}{b}\right)\right\}}{\cosh\left(\frac{n\pi b}{4h}\right)}\sin\left(\frac{n\pi z}{2h}\right)\right]\cdots(4)$$

(2) When $h/b \geq 0.5$

In Eq.(3), since $\sum_n$ converges slower than $\sum_m$, reducing $\sum_n$ according to the Fourier series similarly to the above, and transforming it into the infinite series of $\sum_m$ only, we get :

$$u=\frac{wI}{\mu}\left[\frac{y}{2}(b-y)-\frac{4b^2}{\pi^3}\sum_m\frac{1}{m^3}\frac{\cosh\left\{\frac{m\pi h}{b}\left(1-\frac{z}{h}\right)\right\}}{\cosh\left(\frac{m\pi h}{b}\right)}\cdot\sin\left(\frac{m\pi y}{b}\right)\right]\cdots\cdots(5)$$

In the case of $h/b=0.5$, evaluating the velocity from Eqs. (4) and (5), the some equi-velocity lines are plotted as shown in Fig. 2.

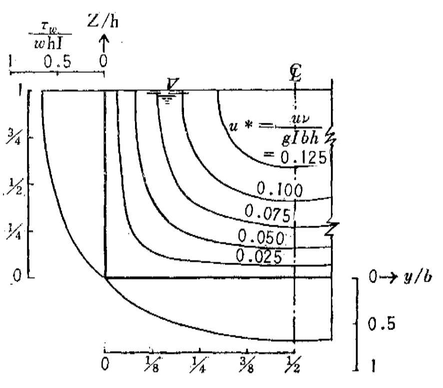

Fig. 2 Velocity and boundary shear stress distribution $h/b=0.5$.

II. Discharge

The discharge which flows through the rectangular channel is $Q=\int_0^b\int_0^h u\cdot dy\cdot dz$; and as a dimensionless discharge let us assume that $Q^*=\frac{Q\mu}{wIA^2}$, then we have

$$Q^*=\frac{128}{\pi^6}\sum_m\sum_n\frac{2h/b}{m^2n^2\left\{\left(\frac{2h}{b}m\right)^2+n^2\right\}} \quad\cdots\cdots(6)$$

where we used Eq. (3) for the velocity distribution. Either using Eq. (4) or transforming Eq. (6) by the Fourier series, we obtain

$$Q^*=\frac{1}{6}\left(\frac{2h}{b}\right)-\frac{32}{\pi^5}\left(\frac{2h}{b}\right)^2\sum_n\frac{1}{n^5}\tanh\left(\frac{n\pi b}{4h}\right) \quad\cdots\cdots\cdots\cdots\cdots(7)$$

And using Eq.(5), or transforming Eq. (6), we get

$$Q^*=\frac{1}{6}\left(\frac{b}{2h}\right)-\frac{32}{\pi^5}\left(\frac{b}{2h}\right)^2\sum_m\frac{1}{m^5}\tanh\left(\frac{m\pi h}{b}\right) \quad\cdots\cdots\cdots\cdots\cdots(8)$$

In Eq. (6), let us put $2h/b=X$, we have $Q^*=\frac{128}{\pi^6}f(X)$. When the value of X takes to zero or infinity, the discharge is obviously zero. And

$$f'(X)=\sum_m\sum_n\frac{n^2-(Xm)^2}{m^2n^2\{(Xm)^2+n^2\}^2}$$

and this function converges uniformly. When $X<1$, $f'(X)>0$ and the dimensionless discharge is an increasing function; when $X>1$, $f'(X)<0$ and the dimensionless discharge is a decreasing function; and when $X=1$, $f'(X)=0$ and the dimensionless discharge has its maximum value. Because $f(X)=f(1/X)$, the values of the function at the variable X and its inverse values are the same; for this reason, the value of the function at $X=1$ is found also to be maximum or minimum. In the same way,

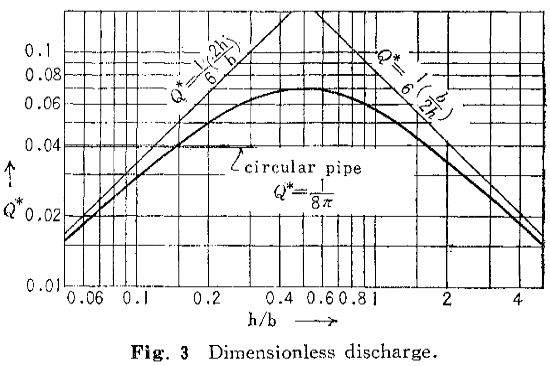

Fig. 3 Dimensionless discharge.

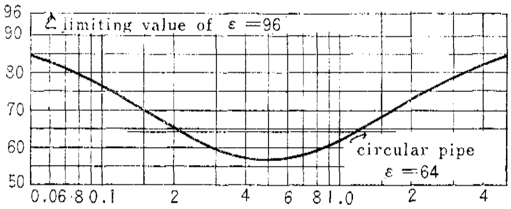

Fig. 4 ε vs. h/b curve.

using a geometrical progression as the X-scale of the coordinates, we can understand easily that the functions are symmetry with respect to $X=1$.

III. Darcy-Weisbach's Friction Factor

The Darcy-Weisbach equation is $I=\lambda\cdot\frac{1}{4R}\cdot\frac{u_m^2}{2g}$, where λ : friction factor, R : hydraulic radius, u_m : mean velocity, g : gravitational acceleration. Substituting $u_m=\frac{Q^*wIA}{\mu}$ in the above equation, we obtain

$$\lambda=\frac{8R\nu}{Q^*Au_m}=\frac{32\,h/b}{(2\,h/b+1)^2Q^*}\cdot\frac{1}{R_e}=\frac{\varepsilon}{R_e}\cdots(9)$$

in which R_e is the Reynolds number $\frac{4Ru_m}{\mu/\rho}$. If we use a geometrical progression as h/b-scale of the coordinates, λ will be symmetry with respect to $h/b=0.5$. ε evaluated by using the above equation (9) is as shown in Fig. 4 which is the same as given by L.G. Straub and others[4].

IV. Momentum Correction Factor

Let us now denote η for momentum correction factor, then it becomes

$$\eta=\frac{1}{A}\int_A\left(\frac{u}{u_m}\right)^2 dA=\frac{\mu^2}{w^2I^2A^3Q^{*2}}\int_A u^2dA\cdots(10)$$

Using Eq. (3) for u, and taking the following into account

$$\int_0^L\sin\left(\frac{M\pi}{L}X\right)\cdot\sin\left(\frac{N\pi}{L}X\right)dX=0 \text{ if } M\neq N$$
$$\frac{L}{2} \text{ if } M=N$$

we get

$$\eta=\frac{256}{\pi^8Q^{*2}}\sum_m\sum_n\frac{\left(\frac{2h}{b}\right)^2}{m^2n^2\left\{\left(\frac{2h}{b}m\right)^2+n^2\right\}^2}\cdots\cdots(11)$$

And then, using the following relations that

$$\sum_m\frac{1}{m^2}=\frac{\pi^2}{8},\quad \sum_n\frac{1}{n^6}=\frac{\pi^6}{960},\quad \sum_m\frac{1}{a^2+m^2}=\frac{\pi}{4a}\tanh\left(\frac{\pi a}{2}\right)$$

and $\sum_m\frac{1}{(a^2+m^2)^2}=\frac{\pi}{8a^3}\tanh\left(\frac{a\pi}{2}\right)-\frac{\pi^2}{16a^2}\cdot\text{sech}^2\left(\frac{a\pi}{2}\right)$, and putting

$a=\frac{bn}{2h}$ into Eq. (11)

we have

$$\eta=\frac{1}{Q^{*2}}\left[\frac{1}{30}\left(\frac{2h}{b}\right)^2-\frac{96}{\pi^7}\left(\frac{2h}{b}\right)^3\sum_n\frac{1}{n^7}\tanh\left(\frac{bn\pi}{4h}\right)+\frac{16}{\pi^6}\left(\frac{2h}{b}\right)^2\sum_n\frac{1}{n^6}\text{sech}^2\left(\frac{bn\pi}{4h}\right)\right]\cdots\cdots(12)$$

And, in the same relations, changing m for n, and putting $a=\frac{2hm}{b}$, we have

$$\eta=\frac{1}{Q^{*2}}\left[\frac{1}{30}\left(\frac{b}{2h}\right)^2-\frac{96}{\pi^7}\left(\frac{b}{2h}\right)^3\sum_m\frac{1}{m^7}\tanh\left(\frac{m\pi h}{b}\right)+\frac{16}{\pi^6}\left(\frac{b}{2h}\right)^2\sum_m\frac{1}{m^6}\text{sech}^2\left(\frac{m\pi h}{b}\right)\right]\cdots\cdots(13)$$

For the different ways, using Eqs. (4) and (5) as the velocity distribution, we can induce Eqs. (12) and (13) respectively. The curve in Fig. 5 is calculated by using the above equations.

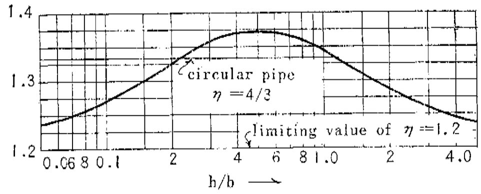

Fig. 5 Momentum correction factor.

V. Boundary Shear Stress

The shear stress upon the channel bed is computed through the relation of $\tau_b=\mu\left(\frac{\partial u}{\partial z}\right)_{z=0}$. Accordingly by using Eq (4), we obtain

$$\tau_b=wIh\left[1-\frac{8}{\pi^2}\sum_n\frac{1}{n^2}\frac{\cosh\left\{\frac{bn\pi}{4h}\left(1-\frac{2y}{b}\right)\right\}}{\cosh\left(\frac{bn\pi}{4h}\right)}\right]\cdots\cdots(14)$$

Similarly, by using Eq. (5), we get

$$\tau_b=wIb\frac{4}{\pi^2}\sum_m\frac{1}{m^2}\tanh\left(\frac{mh\pi}{b}\right)\cdot\sin\left(\frac{m\pi y}{b}\right)\cdots\cdots(15)$$

The shear stress upon the channel wall is computed through the relation of $\tau_w=\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$. Then, by using Eq. (4), we obtain

$$\tau_w=wI\frac{8h}{\pi^2}\sum_n\frac{1}{n^2}\tanh\left(\frac{bn\pi}{4h}\right)\cdot\sin\left(\frac{n\pi z}{2h}\right)\cdots(16)$$

and from Eq. (5), we obtain

$$\tau_w=wI\frac{b}{2}\left[1-\frac{8}{\pi^2}\sum_m\frac{1}{m^2}\frac{\cosh\left\{\frac{m\pi h}{b}\left(1-\frac{z}{h}\right)\right\}}{\cosh\left(\frac{m\pi h}{b}\right)}\right]\cdots\cdots(17)$$

In Eqs. (15) and (16), if the value of the hyperbolic tangent is nearly unity, we had better use the following equation that converges more quickly;

$$\sum_M \frac{1}{M^2}\sin(M\pi Y)=\frac{\pi}{2}Y\left\{1-\ln\left(\frac{\pi Y}{2}\right)\right\}-\frac{1}{9}\left(\frac{\pi Y}{2}\right)^3-\frac{7}{450}\left(\frac{\pi Y}{2}\right)^5-\frac{62}{19\,845}\left(\frac{\pi Y}{2}\right)^7-\cdots \qquad (18)$$

where $Y=y/b$ and $M=m$ in Eq. (15), $Y=z/2h$ and $M=n$ in Eq. (16), and $m,\ n=1,3,5,\cdots\infty$. In the case of $h/b=0.5$, the boundary shear stress is plotted (on the normal direction with each boundary surface) in Fig. 2. The mean shear stress computed through these equations becomes of course the well-known form of wIR.

VI. Effect of the Side Wall on Wide Channel

In the Eq. (3), putting $\frac{m\pi}{b}=B$, and assuming $b\to\infty$ as in Fig. 6, we get $\frac{2\pi}{b}=dB$ and $\frac{1}{m}=\frac{\pi}{dB}=\frac{dB}{2B}$; so that we have

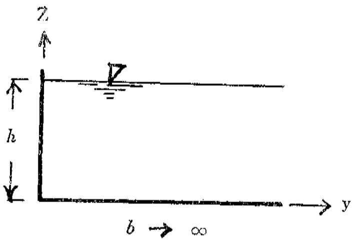

Fig. 6

$$\lim_{b\to\infty} u=\frac{8wI}{\mu\pi^2}\sum_n \frac{1}{n}\sin\left(\frac{n\pi z}{2h}\right)\cdot\int_0^\infty \frac{\sin(By)}{B\left\{B^2+\left(\frac{n\pi}{2h}\right)^2\right\}}dB$$

Because the integration in this equation becomes

$$\int_0^\infty \frac{\sin(By)}{B\left\{B^2+\left(\frac{n\pi}{2h}\right)^2\right\}}dB=\left(\frac{2h}{n\pi}\right)^2\left\{\frac{\pi}{2}-\frac{\pi}{2}\exp\left(-\frac{n\pi y}{2h}\right)\right\}$$

so that

$$u=\frac{16\,wIh^2}{\mu\pi^3}\sum_n\frac{1}{n^3}\left\{1-\exp\left(-\frac{n\pi y}{2h}\right)\right\}\cdot\sin\left(\frac{n\pi z}{2h}\right)$$
$$=\frac{wI}{\mu}\left\{z\left(h-\frac{z}{2}\right)-\frac{16h^2}{\pi^3}\sum_n\frac{1}{n^3}\exp\left(-\frac{n\pi y}{2h}\right)\cdot\sin\left(\frac{n\pi z}{2h}\right)\right\} \qquad (19)$$

Let us assume that $y\to\infty$, then we get the well-known equation as follows

$$u=\frac{wI}{\mu}z\left(h-\frac{z}{2}\right)$$

Using Eq. (19), the shear stress on the channel bed becomes

$$\tau_b=wIh\left\{1-\frac{8}{\pi^2}\sum_n\frac{1}{n^2}\exp\left(-\frac{n\pi y}{2h}\right)\right\} \qquad (20)$$

and similarly, the shear stress on the channel wall becomes

$$\tau_w=wI\frac{8h}{\pi^2}\sum_n\frac{1}{n^2}\sin\left(\frac{n\pi z}{2h}\right) \qquad (21)$$

In the above equation, when evaluating the infinite series, we had better use Eq. (18) that converges more quickly. Fig. 7 is plotted the velocity and the boundary shear stress distribution by using these equations.

For the purpose of investigating the effect of the channel wall on the velocity distribution, let us assume that $[u]_{y\to\infty}=u_i$, and we obtain

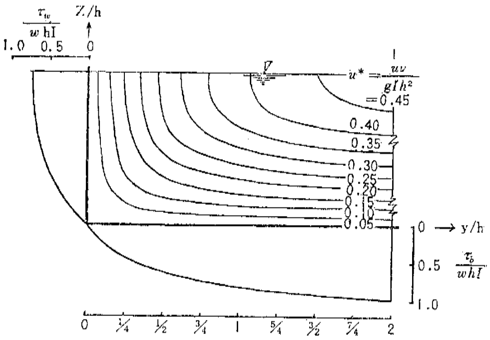

Fig. 7 Velocity and boundary shear stress distribution on wide L-shape channel.

$$\frac{u_i-u}{u_i}=\frac{1}{z/h(1-z/2h)}\frac{16}{\pi^3}\sum_n\frac{1}{n^3}\exp\left(-\frac{n\pi y}{2h}\right)\cdot\sin\left(\frac{n\pi z}{2h}\right)=N \qquad (22)$$

Similarly, in order to investigate the effect of the channel wall on the shear stress distribution on the channel bed, let us assume that $[\tau_b]_{y\to\infty}=\tau_{bi}$, then we get

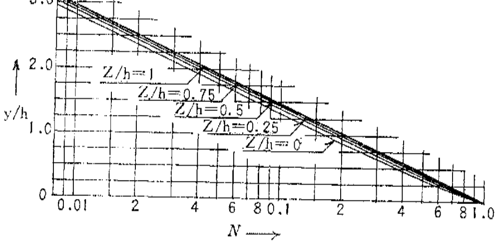

Fig. 8 Effect of the side wall on wide channel.

$$\frac{\tau_{bi}-\tau_b}{\tau_{bi}}=\frac{8}{\pi^2}\sum_n\frac{1}{n^2}\exp\left(-\frac{n\pi y}{2h}\right)=N \cdots\cdots\cdots(23)$$

This equation is equal to $z\to0$ in Eq. (22). Evaluating N on the basis of these equations, it becomes as shown in Fig. 8. The value of N takes the greater part at the first term of the infinite series, and especially the larger y/h becomes, the more correct. Accordingly, as shown in Fig. 8, taking logarithmic scale for N, the N vs. y/h curve becomes roughly a straight line for each value of z/h.

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