THE ENERGY LEVEL OF A LINEAR DYNAMIC SYSTEM UNDER RANDOM EXCITATION

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Summary : The expected energy level of a linear dynamic system is expressed in terms of the power input and energy dissipation rate. Simple expressions are obtained relating the energy level to the time-dependent autocorrelation function of the disturbance.

The non-stationary response of a linear dynamic system to a suddenly applied random disturbance has been discussed by Caughey and Stumpf[1], and a paper by Sawaragi, Sunahara, and Soeda[2] describes the response of non-linear systems. The present paper will employ a somewhat simpler approach based on the energy level of the system that provides a physical interpretation of the transient process and may be applied to certain cases of non-stationary excitatation.

Consider a liner spring-mass-damper system described by the differential equation

$$M\ddot{x}+C\dot{x}+Kx=f(t)$$

where $f(t)$ is an external disturbing force that begins abruptly at $t=0$. If the system is initially at rest, the velocity at any instant is given by the superposition integral

$$\dot{x}=\int_0^t V(\tau)f(t-\tau)d\tau \qquad (1)$$

where $V(\tau)$ is the velocity response to a unit force impulse,

$$V(\tau)=e^{-\frac{C}{2M}\tau}\left(\cos q\tau-\frac{C}{2Mq}\sin q\tau\right)$$

and

$$q=\sqrt{\frac{K}{M}-\frac{C^2}{4M^2}}$$

The instantaneous power input to the system is the product of the velocity and the external disturbing force

$$p=f(t)\dot{x}$$

or

$$p=\int_0^t V(\tau)f(t)f(t-\tau)d\tau$$

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For an ensemble of N systems, or for N experiments on a single system, the ensemble average power input is

$$P=\int_0^t V(\tau)\left[\frac{1}{N}\sum_1^N f_i(t)f_i(t-\tau)\right]d\tau$$

or

$$P=\int_0^t V(\tau)R(t,\ t-\tau)d\tau \qquad (2)$$

where

$$R(t,t-\tau)=\frac{1}{N}\sum_1^N f_i(t)f_i(t-\tau)$$

denotes an ensemble average autocorrelation function.

The rate at which energy is dissipated in the damper is

$$d=c\dot{x}^2 \qquad (3)$$

Using the expression for $\dot{x}$ from equation (1),

$$d=c\left[\int_0^t V(\tau)f(t-\tau)d\tau\right]^2$$

or

$$d=c\int_0^t\int_0^t V(\tau)V(\lambda)f(t-\tau)f(t-\lambda)d\tau\, d\lambda$$

and the ensemble average rate of energy dissipation is

$$D=c\int_0^t\int_0^t V(\tau)V(\lambda)R(t-\tau,\ t-\lambda)d\tau\, d\lambda \qquad (4)$$

where

$$R(t-\tau,\ t-\lambda)=\frac{1}{N}\sum_1^N f_i(t-\tau)f_i(t-\lambda)$$

The rate of change of the energy level of a single system is

$$\frac{de}{dt}=p-d$$

and the ensemble average rate of increase in energy level is

$$\frac{dE}{dt}=\frac{1}{N}\sum_1^N(p_i-d_i)$$

Thus

$$\frac{dE}{dt}=\frac{1}{N}\sum_1^N p_i-\frac{1}{N}\sum_1^N d_i$$

or

$$\frac{dE}{dt}=P-D \qquad (5)$$

Using expressions (2) and (4),

$$\frac{dE}{dt}=\int_0^t V(\tau)R(t,\ t-\tau)d\tau-c\int_0^t\int_0^t$$

$$V(\tau)V(\lambda)R(t-\tau,\ t-\lambda)d\tau\, d\lambda$$

and

$$E=\int_0^t\int_0^t V(\tau)R(t,\ t-\tau)d\tau\, dt-c\int_0^t\int_0^t\int_0^t V(\tau)V(\lambda)R(t-\tau,\ t-\lambda)d\tau\, d\lambda dt \cdots\cdots\cdots(6)$$

If the disturbance $f(t)$ is random, the awkward integrations in equation (6) may be avoided by employing an approximation valid for systems with small amounts of damping. For lightly damped systems, the motion caused by a random force is approximately sinusoidal at the natural frequency q, with random variations in the amplitude, X, and the phase angle, θ. Thus, the motion of each member of the ensemble may be described by

$$x_i=X_i\sin(qt+\theta_i)$$

and

$$\dot{x}_i=q\, X_i\cos(qt+\theta_i)$$

The individual rates of energy dissipation are

$$d_i=c\dot{x}_i^2=cq^2X_i^2\cos^2(qt+\theta_i)$$

and the ensemble average rate of energy dissipation is

$$D=cq^2\frac{1}{N}\sum_1^N X_i^2\cos^2(qt+\theta_i)$$

If the amplitude X_i and phase angle θ_i are independent variables,

$$D=cq^2\left[\frac{1}{N}\sum_1^N X_i^2\right]\left[\frac{1}{N}\sum_1^N\cos^2(qt+\theta_i)\right]$$

and if all values of θ are equally probable,

$$D=\frac{cq^2}{2}\cdot\frac{1}{N}\sum_1^N X_i^2 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(7)$$

Moreover, the energy level of each system is

$$e_i=\frac{K}{2}X_i^2$$

and the ensemble average energy level is

$$E=\frac{K}{2}\frac{1}{N}\sum_1^N X_i^2$$

Thus,

$$\frac{1}{N}\sum_1^N X_i^2=\frac{2E}{K}$$

and from equation (7)

$$D=\frac{cq^2}{K}E \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(8)$$

Using expression (8) for the average rate of energy dissipation in equation (5),

$$\frac{dE}{dt}=P-\frac{cq^2}{K}E\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(9)$$

Remembering that P is a function of t given by equation (2), a first-order linear differential equation in E may be written

$$\frac{dE}{dt}+\frac{qc^2}{K}E=P(t)\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(10)$$

For systems initially at rest the solution of (10) is

$$E=e^{-\frac{q^2c}{K}t}\int_0^t e^{\frac{q^2c}{X}t}\ P(t)dt \cdots\cdots\cdots\cdots\cdots\cdots(11)$$

A further simplification may be made if the disturbance is stationary and ergodic. In this case the autocorrelation function is even and independent of t, and time averages may be used instead of ensemble averages. Moreover, for a random disturbance the autocorrelation function

$$R(\tau)=R(-\tau)=R(t,\ t-\tau)$$

vanishes for large τ. Thus, from equation (2)

$$P=\int_0^T V(\tau)R(\tau)d\tau=\int_0^\infty V(\tau)R(\tau)d\tau=\text{constant}. \cdots\cdots\cdots\cdots\cdots(12)$$

for all values of t greater than the time T beyond which the autocorrelation function is negligible. The expected energy level thus approaches

$$E=\frac{PK}{q^2C}\left(1-e^{-\frac{q^2c}{K}t}\right) \cdots\cdots\cdots\cdots\cdots\cdots\cdots(13)$$

for large values of t.

In the case of white-noise excitation with spectral density G in radian per second dimensions,

$$R(\tau)=\frac{\pi G}{2}\delta(\tau)$$

where $\delta(\tau)$ is the delta function, and the power input is

$$P=\int_0^t V(\tau)\frac{\pi G}{2}\delta(\tau)d\tau=\frac{\pi G}{2}V(O)$$

Since $v(O)=1/M$,

$$P=\frac{\pi G}{2M}$$

and

$$E=\frac{\pi GK}{2q^2cM}\left(1-e^{-\frac{q^2c}{K}t}\right)$$

For lightly damped systems,

$$q^2\cong\omega_0^2=\frac{K}{M}$$

Hence

$$E=\frac{\pi G}{2C}\left(1-e^{-\frac{\omega_0^2c}{K}t}\right) \cdots\cdots\cdots\cdots\cdots\cdots\cdots(14)$$

is the expected energy level of a lightly damped system subjected to stationary white-noise excitation of spectral density G.

For completely undamped systems equation (14) is indeterminate, but in the absence of dissipation the expected energy level is

$$E=\int_0^t P(t)dt \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(15)$$

where $P(t)$ is given by equation (2). For a stationary whit-noise excitation the expected energy level of an undamped system is thus

$$E=\frac{\pi G}{2M}t \quad \cdots\cdots(16)$$

The same equations may be applied to the relative motion of a system disturbed by random motion of the foundation. In this case the equation of relative motion is

$$M\ddot{x}+C\dot{x}+Kx=M\ddot{x}_0$$

where $\ddot{X}_0$ is the foundation acceleration. The appropriate autocorrelation function is

$$R(t,\ t-\tau)=\frac{M^2}{N}\sum_1^N \ddot{x}_0(t)\ddot{x}_0(t-\tau)$$

and the energy level is referred to a system of coordinates moving with the foundation.

It is tempting to speculate on the application of equations (6), (12), and (15) to the survival of structures under earthquake shocks. Examination of strong-motion earthquake records[3] shows that many are characterized by several shocks of large intensity and short duration, followed by a longer period of disturbance at a lower level. During the initial stages of the motion, transient phenomena in the response may be important in determining the damage done to the structure and its ability to survive the remainder of the disturbance. For these cases, analyses based on the timedependent autocorrelation function might be informative. Unfortunately, the motion of the ground during the initial period of an earthquake is strongly dependent on the nature of the soil beneath the structure, and this effect must be accounted for before meaningful time-dependent autocorrelation functions can be obtained and applied to real situations. Perhaps the methods used by Kanai[4] in correlating velocity spectra with soil characteristics may be applied to the problem of non-stationary disturbances.

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REFERENCES

1) T.K. Caughey and H.J. Stumpf: Transient Response of a Dynamic System under Random Excitation, Journal of Applied Mechanics, Vol. 28, Series E, No. 4, pp. 563-566, December, 1961.
2) Y. Sawaragi, Y. Sunahara, and T. Soeda: Statistical Studies on the Response of Non-Linear Control Systems Subjected to a Suddenly Applied Stationary Random Input, Technical Reports of the Engineering Research Institute, Kyoto University, Vol. XII, No. 1, Report No. 91, February, 1962.
3) Strong-Motion Earthquake Records in Japan, Vol. I, Earthquake Research Institute, University of Tokyo, March, 1960.
4) K. Kanai: An Empirical Formula for the Spectrum of Strong Earthquake Motions, Bulletin of the Earthquake Research Institute, Vol. 39, pp. 85-95, 1961.

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