FORCED PLUMES IN A STABLY STRATIFIED FLUID

By Dr. Eng., Mikio Hino, C.E. Member*

ABSTRACT

Predictions are given of the final height to which a fluid emitted from a finite actual source with buoyant force will rise in a stably stratified fluid. The effects of velocity and temperature of effluent are discussed. The results show that the increase in the discharging velocity does not necessarily contribute to the increase in the height of the plume top, reducing it in some cases of small mass flux and relatively slow velocity, although increasing the temperature is always effective. Also, virtual sources are determined which correspond to the actual sources of mass, momentum and buoyancy.

One of the practical problems to apply the proposed solution is an estimate of the necessary amount of temperature or velocity increase as required for stack gases of a thermal plant which is sufficient to penetrate the top of inversion layer in smog seasons and reach an upper atmosphere where they may be rapidly diffused by strong turbulence; another such problem will be encountered in sewage disposal in the marine environments.

INTRODUCTION

Recently, several problems concerning the air and water pollution are provoked owning to a rapid developments of industral activities in the urban districts, i.e., stack gases from thermal power plants and other industries[1),2),3),4),5)], coolant gases with radioactivity from atomic power stations[6)], detrimental wastes from chemical plants, discharge of sewage[7),8),9)], raw or treated, in ocean waters and so on. These problems are to be discussed from the standpoint of the diffusion phenomenon. This paper is the first report of a series of researches on the diffusion and the effects of velocity and density of an effluent from a source, treating the convection of plumes in a quiescent stratified environment.

* Technical Laboratory, Central Research Institute of Electric Power Industry, Komae-cho, Kitatama-Gun Tokyo, Japan.

In a stably stratified layer, the fluid released from a source, jet or plume, does not ascend infinitely as in a uniform environment. The increase in discharging velocity or temperature seems to be an effective practice[3),10),11)] to send an effluent gas penetrating the stable layer, for instance an inversion layer which prevails during smog season, and reaching more turbulent layer above. The solutions to this problm including plumes with negative buoyancy are presented in the following discussions.

Here, a historical review on the behaviours of jet and plume in a calm ambient body fluids will be described briefly. **Table** 1 shows a classified definition of jet and plume along with the authors of main achievements. The theoretical and experimental results of the studies have been summarized by Abraham[12)] (1960) for the case of a uniform environment, except the theory of Morton[13)] (1959) which was published shortly before his report.

The term *jet* means the motion of an effluent from a source without delivering buoyancy flux, in a quiescent environment whose properties may be equal to or different from those of the effluent; whereas the term *plume* the motion of an effluent with its initial buoyancy flux under the action of gravity or buoyancy force[13)]. In other words, the jet delivers mass and momentum, while the plume a flux of buoyancy beside them. If the plume generated from a source delivers a flux of mass and buoyancy only, it is nominated as a *pure plume*; on the other hand, the plume releasing mass, momentum and buoyancy from a source is called a *forced plume.* From a virtual point source where the momentum flow is finite, there can be no mass flow, because the momentum flux is of order (plume radius)^2 × (characteristic velocity)^2 and the mass flux (radius)^2 × (velocity) = (momentum flux)$^{1/2}$ × (radius).

In this Table, the underlines show that the works of these authors include the negative

Table 1

Ambient Fluid		Jet (no buoyancy)	Pure Plume (no momentum flux)	Forced Plume (mass, momentum, buoyancy)
Uniform	without gravity effect	Tollmien (1926) Pai (1955) (different gases)	———	———
	with gravity effect	Morton (1959)	W.Schmidt (1941) Yih (1951) Rouse et al. (1952) Morton et al. (1956) F.H. Schmidt (1957) Morton (1959)	Morton* (1959) Abraham* (1960) Lee-Emmons* (1961)
Straitified		Morton (1959) Hino* (1962)	Morton et al. (1956) Morton (1959) Hino (1962)	Morton (1959) Hino* (1962)

buoyancy case, while the asterisks show that they treated the actual cases with a flux of mass from a source.

Tollmien[14] (1926) was first to discuss the behaviour of a jet in a uniform environment. Since then, so many theoretical and experimental works[15),16] have been reported. The first consideration of a plume in a uniform ambient fluid has deen given by W. Schmidt[17] (1941), followed by Yih[18] (1951), Rouse et al.[19] (1952), and others. These authors based their theories on the assumptions that a similarity is preserved as to the velocity and density profiles in the cross-section of the jet or plume axis, and that the mixture length theory is applicable in order to determine the functional forms of velocity and density profiles. The treatment which represents the width, axial velocity and density variations, with simple powers of the distance from source is not always justified, for instance in the case of an upward plume with negative buoyancy, and also in particular in the case of the plumes in a stratified environment.

The solutins for a plume in a stratified fluid from a virtual point source, including a new approach for the plumes in a uniform fluid have been given by Priestley and Ball[21] (1955); Morton, Taylor and Turner[22] (1956), and Morton[13),23),24] (1957, 1959). They started from the conservation equations of mass, momentum and buoyancy, having the characteristic velocity, density and width of plume as fundamental variables. In general the plumes in a stratified layer have been shown to have final heights of ascent. This report discusses the motion of an actual forced plume which delivers a flux of mass, momentum and buoyancy from a source, extending the method by Morton[13] (1959), who solved forced flumes from a point source only.

THEORY

A. Assumptions

The analysis to be developed is based on the following assumptions about the nature of the plume. First (a), since the laminar plumes are so unstable to become disturbed quite close to the source[18], the flow in most plumes will be assumed to be effectively turbulent. (b) The profiles of velocity and those of density deficiency in any horizontal plane are assumed similar. The results of experiments[17),19),20] show that in a uniform environment the profiles of jets and plumes are Gaussian. However, in a stably stratified ambient fluid, the assumption of the top hat profiles, with constant velocity and density across a plume width and zero outside it, will be preferable, because the usefullness of this assumption has already been determined for the case of plumes in a uniform environment [22], and a Gaussian profile seems incompatible with this case. Of course, the similarity assumption cannot be maintained in regions near the source and the top of plumes. Nevertheless, no serious error may result from our assumption because the flow from a source rapidly developes turdulent and a relatively little entrainment will take place near the top of a plume due to the low axial velocity. (c) The local variation of density is small compared to some chosen reference density. (d) The fluid suffers no volume change on mixing with the amibent environment. Also, the fluid will be treated as incompressible, although the results thus obtained can be extended to include the convection in compressible atmosphere where use may be made of the conventional method of potential temperatures and densities. (e) The entrainment of ambient fluid into the plume is proportional to the characteristic velocity of the plume; the validity of this assumption has been investigated thoroughly by Ricou and Spalding[26] (1961).

B. Fundamental equations and analytical solution

Let X be the distance from a source; r the radial distance in a section of the plume; $U_1(X,r)$ and $U_2(X,r)$ the vertical and radial velocity, $\rho(X,r)$, ρ_0 and $\rho_e=\rho_e(X)$ the densities of the plume each at an arbitrary point, at the source and of the surrounding fluid, respectively; and τ the shearing stress.

Under the foregoing assumptions the fundamental equations of motion for an axisymmetrical plume are written as,

$$\frac{\partial U_1}{\partial X}+\frac{\partial U_2}{\partial r}+\frac{U_2}{r}=0 \quad\cdots\cdots(1)$$

$$\frac{\partial \rho U_1^2}{\partial X}+\frac{1}{r}\frac{\partial(r\rho U_1U_2)}{\partial r}=\frac{1}{r}\frac{\partial(r\tau)}{\partial r}-g(\rho-\rho_e) \quad\cdots\cdots(2)$$

$$U_1\frac{\partial\rho}{\partial X}+U_2\frac{\partial\rho}{\partial r}=-\frac{1}{r}\frac{\partial r(\overline{U_2\rho})}{\partial r} \quad\cdots\cdots(3)$$

which are the equations of conservation of volume, vertical momentum and mass (density deficiency), respectively.

By carrying out integration with respect to a plume section, equation (1) reduces to

$$\frac{\partial}{\partial X}\int_0^\infty 2\pi rU_1dr+\int_0^\infty 2\pi\frac{\partial(rU_2)}{\partial r}dr=0$$

Denoting the plume widths for momentum and mass transfer by B and λB respectively, the above equation becomes, with use of the assumptions (b) and (e),

$$\frac{d}{dX}[\pi B^2U_1]=-2\pi BU_2 \quad\cdots\cdots(4a)$$

$$\text{or}\quad \frac{d}{dX}[\pi B^2U_1]=2\pi\alpha BU_1 \quad\cdots\cdots(4)$$

wnere α is the entrainment constant.

Likewise, integration of equation (2) with respect to r, with use of equation (1), gives

$$\frac{d}{dX}\int_0^\infty 2\pi r\rho U_1^2dr=\int_0^\infty 2\pi rg(\rho_e-\rho)dr$$

which, by the assumptions of a small change in local density and similarity profiles, becomes

$$\frac{d}{dX}[\pi B^2U_1^2]=g\frac{\rho_e-\rho}{\rho_e}\cdot\pi\lambda^2B^2 \quad\cdots\cdots(5)$$

Similarly, equation (3) may be transformed as follows;

$$\int_0^\infty 2\pi rU_1\frac{\partial\rho}{\partial X}dr+\int_0^\infty 2\pi rU_2\frac{\partial\rho}{\partial r}dr=\int_0^\infty\frac{1}{r}\frac{\partial(\overline{rU_2\rho})}{\partial r}\cdot 2\pi r\cdot dr,$$

$$\int_0^\infty 2\pi rU_1\frac{\partial(\rho_0-\rho)}{\partial X}+\left\{\Big[2\pi rU_2(\rho_0-\rho)\Big]_0^\infty-\int_0^\infty 2\pi(\rho_0-\rho)\frac{\partial(rU_2)}{\partial r}dr\right\}=-2\pi\Big[\overline{rU_2\rho}\Big]_0^\infty,$$

$$\frac{\partial}{\partial X}\int_0^\infty 2\pi r\cdot U_1(\rho_0-\rho)dr+2\pi B[U_2(\rho_0-\rho(B)]=0$$

$$\frac{d}{dX}[\pi B^2U_1(\rho_0-\rho)]=-2\pi B\cdot U_2(\rho_0-\rho_e).$$

This equation can be further transformed with the use of (4a) as follows,

$$\frac{d}{dX}[\pi B^2U_1(\rho_0-\rho)]=(\rho_0-\rho_e)\frac{d}{dX}(\pi B^2U_1)=\frac{d}{dX}[\pi B^2U_1(\rho_0-\rho_e)]-\pi B^2U_1\frac{d}{dX}(\rho_0-\rho_e)$$

and then, with the assumption of (c), into the form

$$\frac{d}{dX}[\pi B^2U_1(\rho_e-\rho)]=\pi B^2U_1\frac{d\rho_e}{dX} \quad\cdots\cdots(6)$$

Introducing the transformations $V=BU_1$, $W=B^2U_1$, $F=B^2U_1g(\rho_e-\rho)/\rho_0$, and $G=-g\frac{d\rho_e/\rho_0}{dX}$ equations (4), (5) and (6) reduce to

$$\frac{dW}{dX}=2\alpha V \quad\cdots\cdots(7)$$

$$\frac{dV^4}{dX}=2\lambda^2FW, \quad\cdots\cdots(8)$$

$$\frac{dF}{dX}=-GW. \quad\cdots\cdots(9)$$

Here, it should be noticed that $\rho\pi V^2$ means the momentum flux, $\pi\rho W$ the mass flux and $\pi\rho_0F$ the buoyancy flux.

Usually, the excess heat flux or density deficiency at a source, F_0 is held constant independently of the discharge rate of both momentum and mass (V_0, W_0) at the source. Thus, as the bases most preferable for a non-dimensional transformation, we find F_0 and also the gradient of density variation of environment G. With the following transformation designed to remove all unnecessary coefficients,

$$\left.\begin{aligned}X&=2^{-5/8}\alpha^{-1/2}\lambda^{-1/4}|F_0|^{1/4}G^{-3/8}x\\V&=2^{1/4}\lambda^{1/2}|F_0|^{1/2}G^{-1/4}v\\W&=2^{5/8}\alpha^{1/2}\lambda^{1/4}|F_0|^{3/4}G^{-5/8}w\\F&=|F_0|f,\end{aligned}\right\}\cdots\cdots(10)$$

we have a set of non-dimentional forms of the conservation equation,

$$\frac{dw}{dx}=v, \quad \cdots\cdots(11)$$

$$\frac{dv^4}{dx}=fw, \quad \cdots\cdots(12)$$

$$\frac{df}{dx}=-w; \quad \cdots\cdots(13)$$

and the corresponding boundary conditions at $x=0$

$$\left.\begin{aligned} v&=v_0=V_0/\{2^{1/4}\lambda^{1/2}|F_0|^{1/2}G^{-1/4}\}\\ w&=w_0=W_0/\{2^{5/8}\alpha^{1/2}\lambda^{1/4}|F_0|^{3/4}G^{-5/8}\}\\ f&=\mathrm{sgn}\,F_0.\end{aligned}\right\} \quad (14)$$

From equations (12) and (13), we have

$$2v^4+f^2=2v_0{}^4+1=\frac{1}{1-\nu} \quad \cdots\cdots(15)$$

where the relationships

$$\nu=\frac{2v_0{}^4}{2v_0{}^4+1}=\frac{1}{1+\dfrac{\lambda^2F_0{}^2}{GV_0{}^4}} \quad \cdots\cdots(16)$$

has been introduced as a characteristic parameter in order to represent the effect of momentum at the source. Then, the non-dimensional buoyancy flux is given in terms of the variable v, as

$$f=\pm\left[\frac{1}{1-\nu}-2v^4\right]^{1/2}$$
$$=\pm(1-\nu)^{-1/2}(1-t)^{1/2} \quad \cdots\cdots(17)$$

where

$$t=2(1-\nu)v^4. \quad \cdots\cdots(18)$$

The sign in equation (17) should be determined in accordance with the direction of buoyancy force. Therefore, the equation (17) may be rewritten

$$f=[\mathrm{sgn}\,f](1-\nu)^{-1/2}(1-t)^{1/2} \quad \cdots\cdots(19)$$

From the first and the last of the conservation equations, (11) and (13), we obtain the relationship

$$dw^2=2[\mathrm{sgn}\,f]v\cdot df, \quad \cdots\cdots(20)$$

which, by integration along with (16), (17) and (18), becomes

$$w^2=w_0{}^2+2^{-1/4}(1-\nu)^{-3/4}\int_\nu^t[\mathrm{sgn}\,f]t^{1/4}(1-t)^{-1/2}dt. \quad \cdots\cdots(21)$$

Lastly, equation (13) gives

$$x=2^{-7/8}(1-\nu)^{1/8}\left\{\int_\nu^t[\mathrm{sgn}\,f]\times\frac{dt}{\sqrt{1-t}\sqrt{\dfrac{\tau}{1-\tau}(1-\nu)^{3/4}+\displaystyle\int_\nu^t[\mathrm{sgn}\,f]t^{1/4}(1-t)^{-1/2}dt}}\right\}, \quad \cdots\cdots(22)$$

where a new parameter, τ, is defined by

$$2^{1/4}w_0{}^2=\frac{\tau}{1-\tau}, \quad \cdots\cdots(23)$$

and represents the effects of the initial mass flow rate.

Equations (17), (18), (21) and (22) provide a parametric solution in terms of the variable t. These are rewritten in the following forms according to the sign of f.

(i) When the fluid discharged from the source is lighter than its surroundings and has an upward momentum ($F_0>0$ and $V_0>0$), f decreases steadily from its initial value $+1$ through zero-buoyancy section into the negative buoyancy region, finally reaching the top of the plume where the momentum flux becomes zero and $f=-(1-\nu)^{-1/2}$; the corresponding behaviour of the value for t is that t increases steadily from $t=\nu$ (≤ 1) at the source to the maximum $t=1$ where f vanishes and then decreases to zero at the top of the plume.

Thus, for $\nu\leq t\leq 1$

$$f=(1-\nu)^{-1/2}(1-t)^{1/2} \quad \cdots\cdots(24\text{ a})$$

$$w^2=2^{-1/4}(1-\nu)^{-3/4}\left\{\frac{\tau}{1-\tau}(1-\nu)^{3/4}+B(t)-B(\nu)\right\} \quad \cdots\cdots(25\text{ a})$$

$$x=2^{-7/8}(1-\nu)^{-1/8}\cdot\int_\nu^t\frac{dt}{\sqrt{1-t}\sqrt{\dfrac{\tau}{1-\tau}(1-\nu)^{3/4}+B(t)-B(\nu)}}, \quad \cdots(26\text{ a})$$

where $B(t)$ expresses the incomplete *beta* function with $p=5/4$ and $q=1/2$,

$$B(t)=\int_0^t t^{p-1}(1-t)^{q-1}dt. \quad \cdots\cdots(27)$$

For $1\geq\nu\geq 0$.

$$f=-(1-\nu)^{-1/2}(1-t)^{1/2} \quad \cdots\cdots(24\text{ b})$$

$$w^2=2^{-1/4}(1-\nu)^{-3/4}\left\{\frac{1}{1-t}(1-\nu)^{3/4}+2B(1)-B(\nu)-B(t)\right\} \quad \cdots\cdots(25\text{ b})$$

$$x=2^{-7/8}(1-\nu)^{-1/8}\cdot\left[\int_\nu^1\frac{dt}{\sqrt{1-t}\sqrt{\dfrac{\tau}{1-\tau}(1-\nu)^{3/4}+B(t)-B(\nu)}}+\int_t^1\frac{dt}{\sqrt{1-t}\sqrt{\dfrac{\tau}{1-\tau}(1-\nu)^{3/4}+2B(1)-B(\nu)-B(t)}}\right] \quad \cdots\cdots(26\text{ b})$$

(ii) When the fluid from the source is heavier than its environment and has upward momentum ($F_0<0$, $V_0>0$), t decreases from its initial value ν to zero, and correspondingly f varies from -1 to $-(1-\nu)^{-1/2}$, and

$$f=-(1-\nu)^{-1/2}(1-t)^{1/2}, \quad \cdots\cdots(28)$$

$$w^2=2^{-1/4}(1-\nu)^{-3/4}\left\{\frac{\tau}{1-\tau}(1-\nu)^{3/4}+B(\nu)-B(t)\right\}, \quad \cdots\cdots(29)$$

$$x=2^{-7/8}(1-\nu)^{-1/8}\int_t^\nu \frac{dt}{\sqrt{1-t}\sqrt{\frac{\tau}{1-\tau}(1-\nu)^{3/4}+B(\nu)-B(t)}} \quad (30)$$

(iii) When the fluid is discharged downwards, the behaviors of the plume are exactly the same as those with inverted sign of buoyancy at the source.

The final height of a plume is given by substituting $t=0$ in (26 b) and (30). The effects of discharge momentum and mass (or discharge velocity and density deficiency) at the source will be discussed later.

A comprehensive discussion including wholly the positive, negative and zero buoyancy plumes can be developed by applying the following transformation (31) to equations (7), (8) and (9) *

* $$\left.\begin{aligned} F&=2^{-5/6}\alpha^{-2/3}\lambda^{-1/3}W_0^{4/3}G^{5/6}\cdot f_*.\\ V&=2^{-1/6}\alpha^{-1/3}\lambda^{1/3}W_0^{2/3}G^{1/6}\cdot v_*.\\ W&=W_0\cdot w_*.\\ X&=2^{-5/6}\alpha^{-2/3}\lambda^{-1/3}W_0^{1/3}G^{-1/6}\cdot x_* \end{aligned}\right\} \cdots (31)$$

The reduced non-dimensional fundamental equations are exactly the same as those already derived in the preceding section,

$$\frac{dw_*}{dx_*}=v_* \quad \cdots\cdots(32)$$

$$\frac{d(v_*^4)}{dx_*}=f_*w_* \quad \cdots\cdots(33)$$

$$\frac{df_*}{dx_*}=-w_*, \quad \cdots\cdots(34)$$

with the corresponding boundary condition, at $x_*=0$

$$\left.\begin{aligned} f_*&=f_{*0}=2^{5/6}\alpha^{2/3}\lambda^{1/3}W_0^{-4/3}G^{-5/6}F_0\\ v_*&=v_{*0}=2^{1/6}\alpha^{1/3}\lambda^{-1/3}W_0^{-2/3}G^{-1/6}V_0\\ w_*&=1 \end{aligned}\right\} \quad (35)$$

The parametric solution is

$$v_*^4=\frac{1}{2}\left[\frac{\theta}{1-\theta}+\left(\frac{\varphi}{1-|\varphi|}\right)^2\right]t_* \quad \cdots\cdots(36)$$

$$f_*=[\mathrm{sgn}\, f_*]\left[\frac{\theta}{1-\theta}+\left(\frac{\varphi}{1-|\varphi|}\right)^2\right]^{1/2}(1-t_*)^{1/2} \cdots(37)$$

$$w_*^2=1+2^{-1/4}\left[\frac{\theta}{1-\theta}+\left(\frac{\varphi}{1-|\theta|}\right)^2\right]^{3/4}\int_\nu^{t_*}[\mathrm{sgn}\, f_*]t^{1/4}1-t)^{-1/2}dt \quad \cdots\cdots(38)$$

$$x_*=2^{-7/8}\left[\frac{\theta}{1-\theta}+\left(\frac{\varphi}{1-|\varphi|}\right)^2\right]^{1/8}\int_\nu^{t_*}[\mathrm{sgn}\, f_*]\frac{dt_*}{\sqrt{1-t_*}\sqrt{2^{1/4}\left[\frac{\theta}{1-\theta}+\left(\frac{\varphi}{1-|\varphi|}\right)^2\right]^{-3/4}+\int_\nu^{t_*}[\mathrm{sgn}\, f_*]t^{1/4}(1-t)^{-1/2}dt}} \quad \cdots\cdots(39)$$

where

$$\theta=\frac{2\,v_{*0}^4}{1+2\,v_{*0}^4}$$ (a characteristic parameter for discharge velocity)$\cdots$(40)

$$\varphi=\frac{f_{*0}}{1+|f_{*0}|}$$ (a characteristic parameter for buoyancy)$\cdots\cdots$(41)

and

$$\nu=\left[1+\frac{1-\theta}{\theta}\cdot\left(\frac{\varphi}{1-|\varphi|}\right)^2\right]^{-1}. \quad \cdots\cdots(42)$$

The relationships between the parameters introduced in these equations (θ and φ) and those in the previous ones (τ and ν) are represeneed

$$\frac{\theta}{1-\theta}=2^{1/3}\left(\frac{1-\tau}{\tau}\right)^{4/3}\left(\frac{\nu}{1-\nu}\right), \quad \cdots\cdots(43)$$

$$\frac{\varphi}{1-|\varphi|}=\pm 2^{1/6}\left(\frac{1-\tau}{\tau}\right)^{2/3}, \quad \cdots\cdots(44)$$

and x_* is related to x through the relationship

$$x_*=2^{1/24}\left(\frac{1-\tau}{\tau}\right)^{1/6}x. \quad \cdots\cdots(45)$$

C. Virtual point source

A plume generated from a source of finite size delivers a flux of buoyancy, momentum and mass. It will be shown in this section that there is always an equivalent point (or virtual) source of buoyancy and momentum only which produces the same flow as the extended source above the level $x=0$. It has already been noted that there can be no mass flow from a point source. Therefore a virtual source can be characterized by only one paramenter which is made of momentum and buoyancy flux(V_0' and F_0') ; i.e.

$$\sigma=1\Big/\left\{1+\frac{\lambda^2|F'_0|^2}{GV_0'^4}\right\} \quad \cdots\cdots(46)$$

which is equal to the value of t at a virtual source.

The equivalent virtual plume which is generated from a source $(F'_0, V'_0, 0)$ situated at a certain height $x=-x_\nu$, must satisfy the same non-dimensional equations as the forced plume from the source (F_0, V_0, W_0) and the modified boundary conditions $v=\gamma\, v_0$, $w=0$, $f=\mu$ at $x=-x_\nu$, and $v=v_0$, $w=w_0$, $f=\mathrm{sgn}\, f_0$ at $x=0$. The solution is the same as that for the forced plume except that each of the lower limits of integration, ν, should be replaced by σ.

Before proceeding to determine σ, the motion

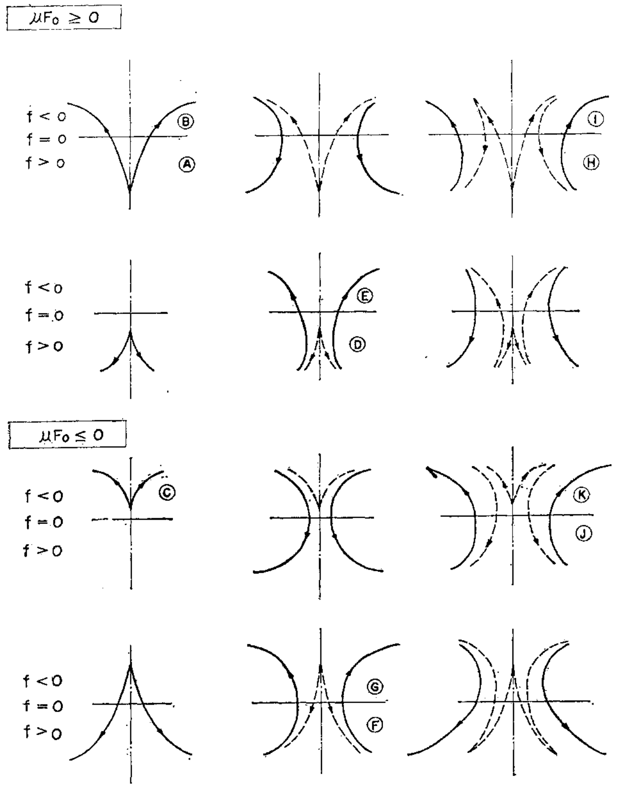

Fig. 1 The real and imaginary behaviours of plums.

of a plume from a point source will de discussed. The first stages of virtual plumes are represented in the left hand side of **Fig. 1**. The behaviour of an actual plume can be projected on one of these plumes from a point source with varying σ. However, for a plume with larger flow of mass and small momentum at the source, the equivalent source cannot be obtained by only considering the real behaviour of a plume, no matter how the source intensity may be increased. Thus, an imaginary point source is to be considered. The first order of a fictitious flow has an opposite direction to the actual flow until it spreads sideways at its final limit (when $t=0$ i.e. $v=0$), returns upon itself, and passes through the source with the appropriate rates of W_0. As the mass flow from an actual source is increased, we must further consider the higher orders of imaginary behaviour.

If we treat here only an upward delivered plume, the virtual source should be sought successively, as in the order shown below,

case A→case D→case F→case H→……($F_0>0$)
case C→case B→case E→case G→……($F_0<0$)

Correspondingly, the route of integration to determine σ must be traced in a complicated way.

For instance, for $F_0>0$

$$\left.\begin{aligned}
&\text{(case A)}\\
&\frac{\tau}{1-\tau}(1-\nu)^{3/4}=\int_\sigma^\nu t^{1/4}(1-t)^{-1/2}dt,\\
&\text{(case D)}\\
&\frac{\tau}{1-\tau}(1-\nu)^{3/4}=-\int_\sigma^0 t^{1/4}(1-t)^{-1/2}dt\\
&\qquad+\int_0^\nu t^{1/4}(1-t)^{-1/2}dt\\
&\text{(case F)}\\
&\frac{\tau}{1-\tau}(1-\nu)^{3/4}=\int_\sigma^1 t^{1/4}(1-t)^{-1/2}dt\\
&\qquad-\int_1^0 t^{1/4}(1-t)^{-1/2}dt\\
&\qquad+\int_0^\nu t^{1/4}(1-t)^{-1/2}dt,\\
&\quad\cdots\cdots
\end{aligned}\right\}\quad(47)$$

When σ is determined by either of these equations, the position of the virtual source, x_ν, can be found easily, and F_0' and V_0' are given respectively by

$$F_0'=F_0\left(\frac{1-\sigma}{1-\nu}\right)^{1/2}\ \cdots\cdots\cdots\cdots(48)$$

$$V_0'=V_0(\sigma/\nu)^{1/4}.\ \cdots\cdots\cdots\cdots(49)$$

The discussion on a virtual source will become significant when we treat the motion of plumes in multiple layered stratified fluid, which will be presented in a succeeding paper dealing with two-dimensional plumes accompanied by experimental investigations.

NUMERICAL CALCULATIONS AND DISCUSSIONS

In order to provide numerical values for the motions of plumes, it is necessary to modify the equations derived in the preceeding sections so as to give high precision with relatively facile procedures; because the integrands involved in these equations are not only so complicated to reject any mathematical method except for a numerical integration, but they diverge infinitely at the upper and/or lower limits of integration.

A. Incomplete BETA function

The incomplete *beta* function defined by equation (22) can be expanded into a power series; for relatively smaller values of t

$$B(t)=\int_0^t t^{1/4}(1-t)^{-1/2}dt$$
$$=\frac{4}{5}t^{5/4}F\left(\frac{5}{4},\frac{1}{2},\frac{9}{4},t\right)$$
$$=t^{5/4}\sum_{n=0}^{\infty}\frac{(2n)!}{\left(\frac{5}{4}+n\right)2^{2n}(n!)^2}t^n$$

and for values near unity

$$B(t)=\int_0^t t^{1/4}(1-t)^{1/2}dt$$
$$=\int_{t'}^1 t'^{1/2}(1-t')^{1/4}dt'$$
$$=B\left(\frac{1}{2},\frac{5}{4}\right)-\int_0^{t'} t'^{1/2}(1-t')^{1/4}dt'$$
$$=\beta\left(\frac{1}{2},\frac{5}{4}\right)-\sum_{n=0}^{\infty}\frac{\Gamma\left(n-\frac{1}{4}\right)}{\Gamma\left(-\frac{1}{4}\right)\left(\frac{1}{2}+n\right)n!}t'^{(n+1/2)} \qquad \cdots\cdots(50\text{ b})$$

where $t'=1-t$ $\cdots\cdots(51)$

B. Calculation of x

The formulae for x are modified by "integration by parts" in order that each of the integrands may not diverge within the range of integration;

(i a)

$$x=x_1(t)=2^{-7/8}(1-\nu)^{1/8}\Big[2\sqrt{1-\nu}\cdot\left\{\frac{\tau}{1-\tau}(1-\nu)^{3/4}\right\}^{-1/2}$$
$$-2\sqrt{1-\nu'}\cdot\left\{\frac{\tau}{1-\tau}(1-\nu)^{3/4}-B(\nu)+B(\nu)'\right\}^{-1/2}$$
$$+2t^{-1/4}\cdot\left\{\frac{\tau}{1-\tau}(1-\nu)^{3/4}-B(\nu)+B(t)\right\}^{1/2}$$
$$+2\nu'^{-1/4}\cdot\left\{\frac{\tau}{1-\tau}(1-\nu)^{3/4}-B(\nu)+B(\nu')\right\}^{1/2}$$
$$+\int_\nu^{\nu'} t^{1/4}\left\{\frac{\tau}{1-\tau}(1-\nu)^{3/4}-B(\nu)+B(t)\right\}^{-3/2}dt$$
$$+\int_{\nu'}^t t^{-5/4}\left\{\frac{\tau}{1-\tau}(1-\nu)^{3/4}-B(\nu)+B(t)\right\}^{1/2}dt\Big], \qquad \cdots\cdots(52\text{ a})$$

where an appropriate value should be chosen for ν' within the range $\nu\leq\nu'\leq t$ so that both of the integrands may not diverge too large, within the integration regions.

(i b) Similarly,

$$x=x_1(1)+2^{-7/8}(1-\nu)^{-1/8}\Big[2t'^{-1/4}$$
$$\cdot\left\{\frac{\tau}{1-\tau}(1-\nu)^{3/4}+B(\nu)-B(t')\right\}^{1/2}$$
$$-2\left\{\frac{\tau}{1-\tau}(1-\nu)^{3/4}+B(\nu)-B(1)\right\}^{1/2}$$
$$+2\sqrt{1-t'}\cdot\left\{\frac{\tau}{1-\tau}(1-\nu)^{3/4}+B(\nu)-B(t')\right\}^{-1/2}$$
$$-\frac{1}{2}\int_{t'}^1 t^{-5/4}\cdot\left\{\frac{\tau}{1-\tau}(1-\nu)^{3/4}+B(\nu)-B(t)\right\}^{1/2}dt$$
$$+\int_t^{t'} t^{1/4}\cdot\left\{\frac{\tau}{1-\tau}(1-\nu)^{3/4}+B(\nu)-B(t)\right\}^{-3/2}dt \qquad \cdots\cdots(52\text{ b})$$

where t' should be $t\leq t'\leq 1$.

(ii)

$$x=2^{-7/8}(1-\nu)^{-1/8}\Big[2t''^{-1/4}$$
$$\cdot\left\{\frac{\tau}{1-\tau}(1-\nu)^{3/4}+B(\nu)-B(t'')\right\}^{1/2}$$
$$-2\nu^{-1/4}\cdot\left\{\frac{\tau}{1-\tau}(1-\nu)^{3/4}\right\}^{1/2}+2\sqrt{1-t}$$
$$\cdot\left\{\frac{\tau}{1-\tau}(1-\nu)^{3/4}+B(\nu)-B(t)\right\}^{-1/2}$$
$$-2\sqrt{1-t''}\cdot\left\{\frac{\tau}{1-\tau}(1-\nu)^{3/4}+B(\nu)-B(t'')\right\}^{-1/2}$$
$$-\frac{1}{2}\int_{t''}^{\nu} t^{-5/4}\cdot\left\{\frac{\tau}{1-\tau}(1-\nu)^{3/4}+B(\nu)-B(t)\right\}^{1/2}dt$$
$$+2\int_t^{t''} t^{1/4}\cdot\left\{\frac{\tau}{1-\tau}(1-\nu)^{3/4}+B(\nu)-B(t)\right\}^{-3/2}dt\Big]$$

where t'' should be also chosen so that the integrands of the equation may not diverge too large.

C. Results and discussions

The non-dimensinal final heights of plumes ($\bar{x}$ and $\bar{x}_*$) have been calculated from the equation given above with $t=0$, by applying the Gaussian method of numerical intrgration.

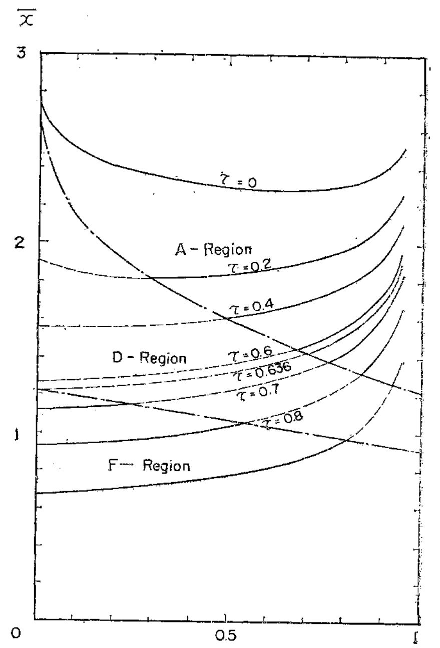

Fig. 2 (a) The final height of plume with positive initial buoyancy; ν represents a parameter characteristic of discharge velocity, whereas τ a parameter of discharge volume. [Eq. (26 b)]

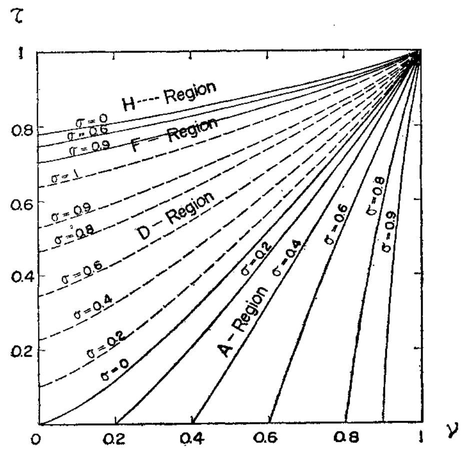

Fig. 2 (b)

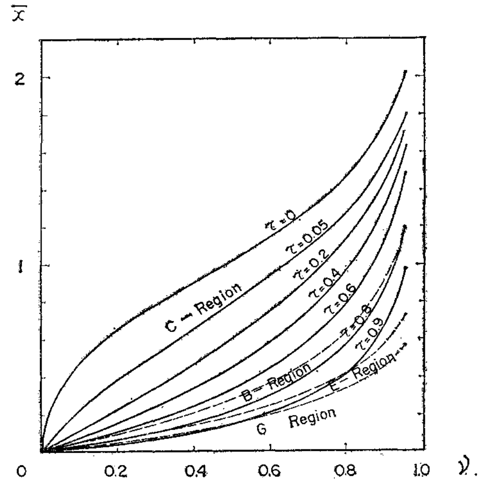

Fig. 3 (a) The final height of plume with negative initial buoyancy. [Eq. (30)]

The results are illustrated in **Figs. 2** to **5**, which demonstrate clearly that the increase in discharge velocity of a plume cannot always be effective to affect the ascent of a plume in a stratified environment; for relatively smaller mass flux (τ) of positive buoyancy, the top of the plume decreases at first steeply and then gradually as the discharge momentum (ν) is inereased from zero, and only for extremely high value of ν it increases rapidly towards infinity. On the other hand, for zero and negativy buoyant plumes, the increase of ν is always effective. In general, the heat excess, F_0, is kept constant. Hence, the figures also show that the increase of mass flux from the source and therefore the increase in density or the decrease

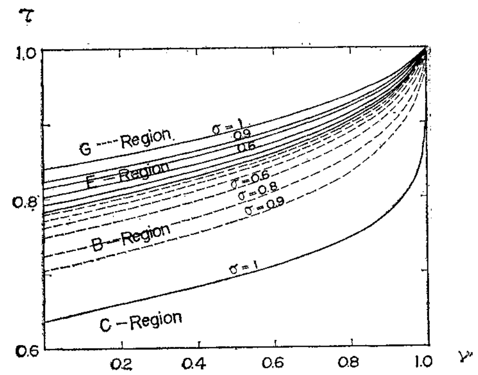

Fig. 3 (b)

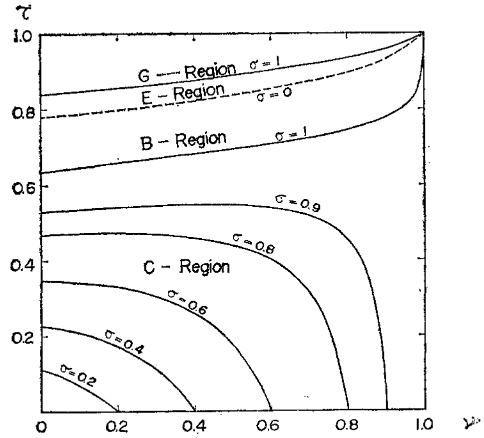

Fig. 3 (c)

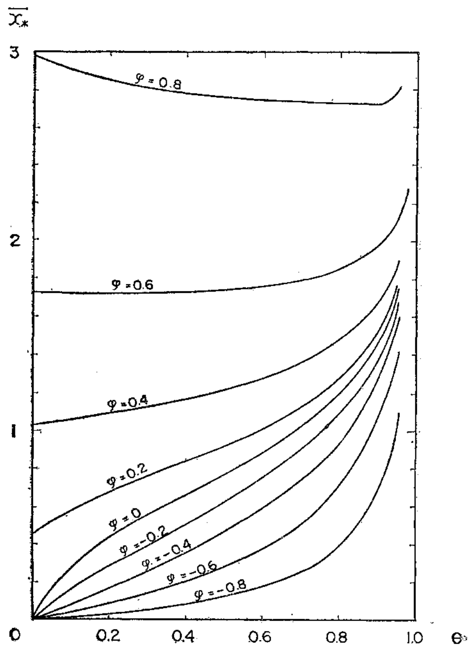

Fig. 4 The final height of plume. [Eq. (39)]

in temperature of plumes gives rise, in all cases, to unfavorable effects on the ascent of plumes.

This behaviour of the plumes may be explained physically by the fact that, e.g. for the buoyant plumes, the increase in the discharge velocity results in the increase in the entrainment near the source of the less buoyant ambient fluid, thus reducing the energy of buoyancy per unit volume which has been possessed originally by the plume.

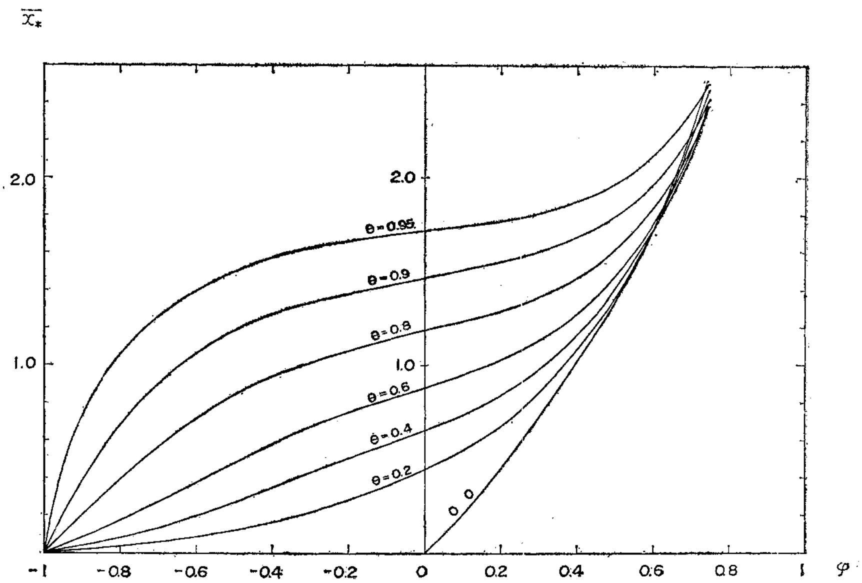

Fig. 5

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