

SIMILARITY AND DESIGN METHODS OF RIVER MODELS WITH MOVABLE BED*

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SYNOPSIS

In this paper, the similarity, various similarity ratios and design methods of distorted river models with movable bed in consideration of new developments in many researches for flowing water and sediment transport in open channels are investigated. The results obtained would contribute much to the designs and experiments of distorted or un-distorted river models with movable bed.

1. Introduction

Up to this time, the quantitative results on sediment transportation could not be obtained, on account of the complex phenomena of sediment transport, from the results of hydraulic experiments by the use of river models with movable bed.

In 1944, H.A.Einstein published a short account¹⁾ of how compatible systems of scales can be found. He proposed in that paper the derivation of model scales from empirical working equations rather than from the underlying differential equations of motion. It was pointed out there that equations must be used which are applicable in the same form both to model and prototype, and which should preferably have the form of power functions²⁾.

In 1954, H.A.Einstein and N.Chien³⁾ developed the similarity of distorted river models with movable bed by the use of this approach. Their procedure in obtaining the various similarity ratios is tedious procedure for the sake of including a trial and error method. And also, their procedure includes the zero sediment-load criterion which is the condition of similar flow conditions at the beginning of sediment motion (the condition of critical tractive force). Therefore, it is known that the similarity in the

vicinity of the critical tractive force will be satisfied, but the similarity in the case of flow having large sediment transport will be not satisfied. For this reason, the condition of the critical tractive force in obtaining the various similarity ratios of river models with movable bed should be excluded from among the similarity conditions for sediment transport. In obtaining the various similarity ratios of river models with movable bed in the vicinity of a critical tractive force, the condition of the critical tractive force should be taken into consideration.

Excluding the condition of the critical tractive force, and using the different method against their approach, the similarity and various similarity ratios for distorted river models with movable bed were obtained³⁾ by solving simultaneously the equations of similarity conditions which were derived from the equations of motion and continuity for flowing water and sediment transport, respectively, and the equation of the resistance law for flowing water with sediment transport.

Last of all, the example of a distorted river model with movable bed was added to demonstrate the design methods and experimental results of the river model with movable bed. Comparing the results which were transferred to the prototype by the use of the similarity ratio of bed-load rate with the results which were computed by some different methods; the Modified Einstein procedure, Brown's formula and Kalinske's formula, the author indicated that the results obtained from the hydraulic experiments of river models can be transferred to the prototype by the use of these similarity ratios.

2. Equations of Similarity Conditions

A) Equations of Similarity Conditions for Flowing Water

Taking the x-axis in the down-stream direction along the river-bed, and let U be the mean

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velocity, h the water depth, R the hydraulic radius, g the gravity acceleration, t the hydraulic time, i the bed slope, τ the shearing stress on bed, and ρ the density of water, the equation of motion for flowing water is

$$\frac{\partial h}{\partial x} + \frac{1}{g} \frac{\partial U}{\partial t} + \frac{\partial}{\partial x} \left(\frac{U^2}{2g} \right) = i - \frac{\tau}{\rho g R} \dots\dots(1)$$

On the other hand, introducing an equivalent roughness k_s , corresponded to the increasing of bed roughness due to movement of sand on movable bed, the equation of resistance law of flowing water with sediment transport can be expressed, in general, as the following equation :

$$\frac{U}{\sqrt{(\tau/\rho)}} = A_p' - 2.5 + 5.75 \log \left(\frac{R}{k_s} \right) \dots\dots\dots(2)$$

where A_p' is 8.5 when the effects of cross-sectional shapes are not taken into consideration, and A_p' is 8.75 when they are taken into consideration. We would like to use this equation as an equation among similarity conditions, but in the case of solving with another various similarity conditions, the form of Eq. (2) can be thought having many difficulties, therefore the following equation which is expressed in the exponential form can be used in a region of (R/k_s) as an equation approximately equal to Eq. (2) :

$$\frac{U}{\sqrt{(\tau/\rho)}} = E \left(\frac{R}{k_s} \right)^q \dots\dots\dots(2)'$$

where E is a constant, and q a dimensionless exponent. The value of E can also be given as the following equation :

$$E = k_s^q / n g^{1/2} \dots\dots\dots(3)$$

where n is Manning's roughness factor. The equation that E considered approximately constant was introduced by H.A.Einstein and N.L. Barbarossa⁴⁾ from Manning's formula and Strickler's formula, namely,

$$\frac{U}{\sqrt{(\tau/\rho)}} = 7.76 \left(\frac{R}{k_s} \right)^{1/6}$$

From various discussions, thus q is approximately 1/6, the value of E can be expressed by the value of 7~8 in the region of $(R/k_s) = 1 \sim 1000$.

The equivalent roughness k_s is expressed generally with a function of (k_s/d) and a parameter $(\tau/\rho)/\{(\sigma/\rho)-1\}gd$, which denotes the dimensionless tractive force, as the following equation :

$$\left(\frac{k_s}{d} \right) = K \left[\frac{(\tau/\rho)}{\{(\sigma/\rho)-1\}gd} \right]^m \dots\dots\dots(4)$$

where d is the mean diameter of sediment particles, σ and ρ are the densities of sediment particles and water, respectively. K is a constant, and m the dimensionless exponent pertaining to the tractive force. In sediment hydraulics, since the mean diameter of sediment particles is approximately equal to the median diameter, the median diameter is used frequently instead of the mean diameter.

Dr. Y.Iwagaki⁵⁾ proposed the following equation from his experiments performed under the conditions of smooth bed and fairly steep bed slopes :

$$\left(\frac{k_s}{d} \right) = 10 \left[\frac{(\tau/\rho)}{\{(\sigma/\rho)-1\}gd} \right]^{0.769}$$

It is verified that this equation can be applied to natural streams which have fairly steep bed slopes and are assumed to have smooth bed.

And also, from the field data in the several natural streams, Dr. T.Tsubaki and A.Furuya⁶⁾ proposed the following equation :

$$\log \left(\frac{k_s}{d} \right) = 3.48 \left[1 - 0.225 \left\{ \frac{(\tau/\rho)}{\{(\sigma/\rho)-1\}gd} \right\}^{-1/2} \right]$$

Since these data were collected from natural streams which had fairly gentle bed slopes and bed material of fine size, it is verified that this equation can be applied to natural streams, the beds of which are covered with ripples and dunes. By calculating the value of m from this equation, a value of $m=3$ was obtained.

As to the smooth bed in a transitional region from dunes to antidunes and the smooth bed under the tractive force near a critical point,

J.Matsunashi⁷⁾ proposed the following equation from G.K.Gilbert's data and his data :

$$\left(\frac{k_s}{d} \right) = K \left[\frac{(\tau/\rho)}{\{(\sigma/\rho)-1\}gd} \right]^3$$

On the other hand, the investigations have been scarcely presented on the value of (k_s/d) in natural streams, in which the bed slope and the sediment size are comparatively large. At any rate, the investigations of (k_s/d) in natural streams will remain hereafter as one of the most important themes among the sediment problems.

Considering from various discussions, the value of m can be expressed as follows :

(1) It is considered that the value of $m=3$ can be applied to natural streams and river models with movable bed, the whole beds of

which are covered with ripples and dunes.

(2) It is considered that the values of $m = 0.769$ and $m = 0$ can be applied to natural streams and river models with movable bed, the whole of which are covered with smooth bed and plane bed.

Generally speaking, it seems good to use $m = 0$ as the value of m for the similarity of river models with movable bed. In this paper, therefore, the author deals mainly with the case of $m = 0$.

The friction term in Eq. (1) can be rewritten as the following equation by the use of Eq. (2)':

$$\frac{\tau}{\rho g R} = \frac{U^2 k_s^{2q}}{g E^2 R^{(2q+1)}} \dots\dots\dots (5)$$

and (τ/ρ) is

$$(\tau/\rho) = g R i_f \dots\dots\dots (6)$$

where i_f is the friction slope.

Substituting Eq. (5) in Eq. (1), the right side of Eq. (1) is

$$\frac{\partial h}{\partial x} + \frac{1}{g} \frac{\partial U}{\partial t} + \frac{\partial}{\partial x} \left(\frac{U^2}{2g} \right) = i - \frac{U^2 k_s^{2q}}{g E^2 R^{(2q+1)}} \dots\dots\dots (7)$$

The equation of continuity for flowing water is

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x} (A \cdot U) = 0 \dots\dots\dots (8)$$

where A is the cross-sectional area of flowing water. Assuming that the river has a rectangular cross-section, the cross-sectional area of flowing water, A , can be expressed by $A = Bh$, in which B is the breadth of the river.

The symbols with the subscripts, P and M , indicate quantities in prototype and model, respectively. By using this symbols, Eqs. (3), (4), (6), (7) and (8) are expressed as follows, re-

spectively :

$$\frac{k_{sp}}{d_p} = K_p \left[\frac{(\tau/\rho)_p}{\{(\sigma/\rho) - 1\}_p g d_p} \right]^m \dots\dots\dots (9)$$

$$\frac{k_{sM}}{d_M} = K_M \left[\frac{(\tau/\rho)_M}{\{(\sigma/\rho) - 1\}_M g d_M} \right]^m \dots\dots\dots (10)$$

$$E_P = k_{sP}^q / n_P g^{1/2} \dots\dots\dots (11)$$

$$E_M = k_{sM}^q / n_M g^{1/2} \dots\dots\dots (12)$$

$$(\tau/\rho)_P = g R_P i_{fP} \dots\dots\dots (13)$$

$$(\tau/\rho)_M = g R_M i_{fM} \dots\dots\dots (14)$$

$$\frac{\partial h_P}{\partial x_P} + \frac{1}{g} \frac{\partial U_P}{\partial t_P} + \frac{\partial}{\partial x_P} \left(\frac{U_P^2}{2g} \right) = i_P - \frac{U_P^2 k_{sP}^{2q}}{g E_P^2 R_P^{(2q+1)}} \dots\dots\dots (15)$$

$$\frac{\partial h_M}{\partial x_M} + \frac{1}{g} \frac{\partial U_M}{\partial t_M} + \frac{\partial}{\partial x_M} \left(\frac{U_M^2}{2g} \right) = i_M - \frac{U_M^2 k_{sM}^{2q}}{g E_M^2 R_M^{(2q+1)}} \dots\dots\dots (16)$$

$$\frac{\partial A_P}{\partial t_P} + \frac{\partial}{\partial x_P} (A_P \cdot U_P) = 0 \dots\dots\dots (17)$$

$$\frac{\partial A_M}{\partial t_M} + \frac{\partial}{\partial x_M} (A_M \cdot U_M) = 0 \dots\dots\dots (18)$$

In order that the flow in the model may be dynamically similar to that in the prototype, it is necessary that all corresponding terms in Eqs. (9) and (10), Eqs. (11) and (12), Eqs. (13) and (14), Eqs. (15) and (16), and Eqs. (17) and (18), respectively, should be proportional. The ratio of corresponding values in the two scales will be denoted by a subscript r . For instance, the ratio of depth scales is defined as $h_r = h_P/h_M$. Thus, the equations of similarity conditions are as follows :

$$k_{sr} d_r^{(m-1)} K_r^{-1} \tau_r^{-m} (\sigma - \rho)_r^m = 1 \dots\dots\dots (19)$$

$$U_r^2 k_{sr}^{2q} E_r^{-2} R_r^{-(2q+1)} i_r^{-1} = 1 \dots\dots\dots (20)$$

$$i_r x_r h_r^{-1} = 1 \dots\dots\dots (21)$$

$$U_r^2 h_r^{-1} = 1 \dots\dots\dots (22)$$

$$u_r t_r x_r^{-1} = 1 \dots\dots\dots (23)$$

$$\tau_r R_r^{-1} i_{fr}^{-1} = 1 \dots\dots\dots (24)$$

$$n_r E_r k_{sr}^{-q} = 1 \dots\dots\dots (25)$$

Eq. (22) is well known as Froude's law. In open channel flows, the Froude law of similarity is one of the main similarity conditions.

The ratio of hydraulic radius, R_r , can be expressed as

$$R_r = \beta h_r \dots\dots\dots (26)$$

and

$$\beta = 1 + 2\alpha (h_P/B_P) \dots\dots\dots (26)'$$

where $\alpha = x_r/h_r$, in which α is called the degree of distortion, h_P is the water depth of the prototype and B_P the breadth of the prototype. The variation of β with (h_P/B_P) can be shown in Fig. 1 by choosing α as a parameter.

Substituting Eq. (26) in Eqs. (20) and (24), and rearranging,

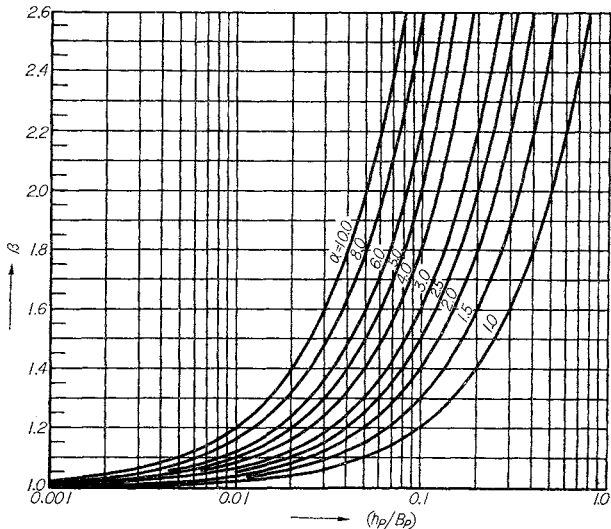


Fig. 1 Variation of β with (h_P/B_P) and α

$$U_r^2 k_{sr}^{2q} E_r^{-2\beta - (2q+1)} h_r^{-(2q+1)} i_r^{-1} = 1 \dots\dots\dots(20)'$$

$$\tau_r \beta^{-1} h_r^{-1} i_r^{-1} = 1 \dots\dots\dots(24)'$$

B) Equations of Similarity Conditions for Sediment Transport

(a) Equation of Motion for Sediment Transport

In the following discussions, the symbol with subscript *s* indicates quantities pertaining to the sediment transport. In this paper, the same form as the following formula proposed by A.A.Kalinske and C.B.Brown⁸⁾ is adopted :

$$\frac{q_s}{\sqrt{(\tau_s/\rho)d}} = a_s \left[\frac{(\tau_s/\rho)}{\{(\sigma/\rho) - 1\}gd} \right]^p \dots\dots\dots(27)$$

where *q_s* is the rate of sediment transport in volume of material per unit time and width, including the rate of suspended sediment transport. *τ_s* is the shearing stress on bed pertaining to the sediment transport. The values of *p* and *a_s* are constants. According to the expression by A.A.Kalinske and C.B.Brown⁸⁾, the values of *p* and *a_s* are *p*=2 and *a_s*=10.

(b) Equation of Continuity for Sediment Transport

Let *h_s* be the river-bed elevation measured from any datum surface, then for the rectangular cross-section of width *B*, the equation of continuity pertaining to the sediment transport is expressed as

$$\frac{\partial h_s}{\partial t_s} + \frac{1}{B(1-\lambda)_s} \frac{\partial(q_s \cdot B)}{\partial x_s} = 0 \dots\dots\dots(28)$$

where *t_s* is the time pertaining to the sediment transport and *λ* the porosity of bed material. In soil mechanics, porosity can be given by the following equation⁹⁾ :

$$\lambda = \left\{ 1 - \left(\frac{\tau_d}{G_s \tau_w} \right) \right\} \dots\dots\dots(29)$$

where *τ_d* is the dry density of the bed material (g/cm³), *G_s* the specific gravity of particles in the bed material and *τ_w* the specific weight of water. The value of *τ_w* is equal to 1.00 g/cm³ under natural conditions and in most soil mechanics work the value 1.00 g/cm³ is sufficiently accurate. The values of dry density for different median diameters increase hyperbolically with increasing median diameter on the semi-logarithmic paper as shown in Fig. 2. The data on Fig. 2 were obtained from the REFERENCES (10), (11), (12), (13) and the author's measurements in the Nagara River located in central Japan. Using these data, the author obtained, approximately, the following equation⁹⁾ for uncon-

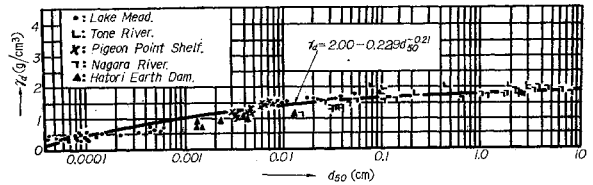


Fig. 2 Relationship between the dry density *γ_d* and the median diameter *d₅₀* unconsolidated sediments in saturated state.

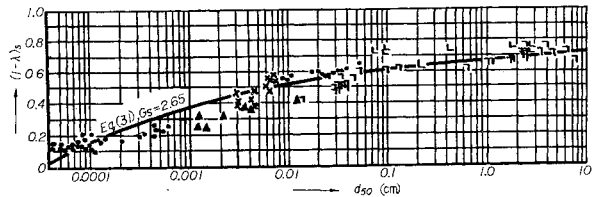


Fig. 3 Relationship between $(1-\lambda)_s$ and the median diameter *d₅₀* of unconsolidated sediments in saturated state.

solidated sediments in saturated state :

$$\tau_d = 2.00 - 0.229 d_{50}^{-0.21} \dots\dots\dots(30)$$

(*d₅₀* in cm)

Substituting Eq. (30) in Eq. (29), the following equation is obtained :

$$(1-\lambda)_s = \frac{1}{G_s \tau_w} (2.00 - 0.229 d_{50}^{-0.21}) \dots\dots\dots(31)$$

(*d₅₀* in cm)

Therefore, the value of $(1-\lambda)_s$ can be calculated by using Eq. (31). The value of $(1-\lambda)_s$ can also be obtained from the porosity tests of the sediment under consideration. The relationship between $(1-\lambda)_s$ and *d₅₀* in Eq. (31) can be shown in Fig. 3 by choosing *G_s* as a parameter. The data obtained from the REFERENCES (9), (10), (11), (12) and the author's measurements in the Nagara River were plotted on Fig. 3 with respect to $(1-\lambda)_s$ and *d₅₀*.

The curve on Fig. 3 indicates the relationship of Eq. (31) in the case of *G_s*=2.65.

(c) Equations of Similarity Conditions for Sediment Transport

By using the same method as used for flowing water, the equations of similarity conditions for sediment transport can be obtained from Eqs. (27) and (28) as the following equations :

$$q_{sr} a_{sr}^{-1} \tau_{sr}^{-(p+1/2)} d_r^{(p-1)} (\sigma - \rho) r^p = 1 \dots\dots\dots(32)$$

$$x_{sr} h_{sr} t_{sr}^{-1} q_{sr}^{-1} (1-\lambda)_{sr} = 1 \dots\dots\dots(33)$$

3. Solutions of the Equations of similarity Conditions

A) When the Values of *x_r* and *h_r* are Cho-

sen Arbitrarily. In order that the flow in the model may be dynamically similar to that in the prototype, it is necessary that τ_{sr} is equal to τ_r . On the other hand, if we choose as the same element x_{sr} and x_r , h_{sr} and h_r , and i_{sr} and i_r , respectively, the equations of similarity conditions for sediment transport and Eq. (24)' can be rewritten as follows :

$$\tau_r \beta^{-1} h_r^{-1} i_r^{-1} = 1 \dots\dots\dots(24)''$$

$$q_{sr} a_{sr}^{-1} \tau_r^{-(p+1/2)} d_r^{(p-1)} (\sigma - \rho)_r^p = 1 \dots\dots\dots(32)'$$

$$x_r h_r t_{sr}^{-1} q_{sr}^{-1} (1 - \lambda)_{sr} = 1 \dots\dots\dots(33)'$$

Let us assume that x_r and h_r can be given arbitrarily in the nine equations (19), (20)', (21), (22), (23), (24)'', (25), (32)' and (33)', and then, the six ratios β , E_r , K_r , a_{sr} , $(\sigma - \rho)_r$ and $(1 - \lambda)_{sr}$ can be chosen as known ratios. The following nine ratios remain as unknown ratios: i_r , U_r , t_r , τ_r , n_r , k_{sr} , q_{sr} , d_r and t_{sr} .

The solutions (similarity ratios) of the nine equations can be obtained by solving these nine equations for the nine unknown ratios simultaneously. The solutions are as follows :

$$i_r = x_r^{-1} h_r \dots\dots\dots(34)$$

$$U_r = h_r^{1/2} \dots\dots\dots(35)$$

$$t_r = x_r h_r^{-1/2} \dots\dots\dots(36)$$

$$\tau_r = x_r^{-1} h_r^{2\beta} \dots\dots\dots(37)$$

$$n_r = x_r^{-1/2} h_r \left(\frac{1+2q}{2} \right) \beta \left(\frac{1+2q}{2} \right) \dots\dots\dots(38)$$

$$k_{sr} = x_r^{-\frac{1}{2q}} h_r \left(\frac{1+2q}{2q} \right) \beta \left(\frac{1+2q}{2q} \right) E_r^{1/q} \dots\dots\dots(39)$$

$$q_{sr} = x_r^{-\frac{(3mq-2pq-q+p-1)}{2q(m-1)}} h_r^{\frac{(6mq-2pq-4q+p-1)}{2q(m-1)}} \cdot \beta^{\frac{(3mq-3q+p-1)}{2q(m-1)}} a_{sr} E_r^{\frac{(p-1)}{q(m-1)}} K_r^{\frac{(p-1)}{(m-1)}} (\sigma - \rho)_r^{-\frac{(m-p)}{(m-1)}} \dots\dots\dots(40)$$

$$d_r = x_r^{-\frac{(2mq-1)}{2q(m-1)}} h_r^{\frac{(4mq-2q-1)}{2q(m-1)}} \beta^{\frac{(2mq-2q-1)}{2q(m-1)}} E_r^{-\frac{1}{q(m-1)}} \cdot K_r^{\frac{1}{(m-1)}} (\sigma - \rho)_r^{-\frac{m}{(m-1)}} \dots\dots\dots(41)$$

$$t_{sr} = x_r^{-\frac{(5mq-2pq-3q+p-1)}{2q(m-1)}} h_r^{\frac{(4mq-2pq-2q+p-1)}{2q(m-1)}} \cdot \beta^{-\frac{(3mq-3q+p-1)}{2q(m-1)}} a_{sr}^{-1} \cdot E_r^{-\frac{(p-1)}{q(m-1)}} K_r^{\frac{(p-1)}{(m-1)}} (\sigma - \rho)_r^{\frac{(m-p)}{(m-1)}} \cdot (1 - \lambda)_{sr} \dots\dots\dots(42)$$

The discharge of flowing water, Q , is approximately expressed by the following equation; $Q = AU \div BhU$, therefore the ratio Q_r can be expressed as $Q_r = x_r h_r U_r$. By using Eq. (35), the ratio Q_r is

$$Q_r = x_r h_r^{3/2} \dots\dots\dots(43)$$

Substituting $q=1/6$, $p=2$ and $x_r = \alpha h_r$ in Eqs (34)~(43), and rearranging,

$$i_r = \alpha^{-1} \dots\dots\dots(34)'$$

$$U_r = h_r^{1/2} \dots\dots\dots(35)'$$

$$t_r = \alpha h_r^{1/2} \dots\dots\dots(36)'$$

$$\tau_r = \alpha^{-1} h_r \beta \dots\dots\dots(37)'$$

$$n_r = \alpha^{-1/2} h_r^{1/6} \beta^{2/3} \dots\dots\dots(38)'$$

$$k_{sr} = \alpha^{-3} h_r \beta^4 E_r^6 \dots\dots\dots(39)'$$

$$q_{sr} = \alpha^{-\frac{(3m+1)}{2(m-1)}} h_r^{3/2} \beta^{\frac{2(m-1)}{3}} a_{sr} E_r^{\frac{6}{(m-1)}} K_r^{-\frac{1}{(m-1)}} \cdot (\sigma - \rho)_r^{-\frac{(m-2)}{(m-1)}} \dots\dots\dots(40)'$$

$$d_r = \alpha^{-\frac{(m-3)}{(m-1)}} h_r \beta^{\frac{(m-4)}{(m-1)}} E_r^{-\frac{6}{(m-1)}} K_r^{\frac{1}{(m-1)}} \cdot (\sigma - \rho)_r^{-\frac{m}{(m-1)}} \dots\dots\dots(41)'$$

$$t_{sr} = \alpha^{-\frac{(5m-1)}{2(m-1)}} h_r^{1/2} \beta^{-\frac{3(m+1)}{2(m-1)}} E_r^{\frac{6}{(m-1)}} K_r^{-\frac{1}{(m-1)}} \cdot (\sigma - \rho)_r^{\frac{(m-2)}{(m-1)}} (1 - \lambda)_{sr} \dots\dots\dots(42)'$$

$$Q_r = \alpha h_r^{5/2} \dots\dots\dots(43)'$$

If the flows in the model and prototype are hydraulically similar, the values of a_{sr} , E_r and K_r are equal to unity.

The similarity ratios in the case of $m=0$ are as follows :

$$i_r = \alpha^{-1} \dots\dots\dots(44)$$

$$U_r = h_r^{1/2} \dots\dots\dots(45)$$

$$t_r = \alpha h_r^{1/2} \dots\dots\dots(46)$$

$$\tau_r = \alpha^{-1} h_r \beta \dots\dots\dots(47)$$

$$n_r = \alpha^{-1/2} h_r^{1/6} \beta^{2/3} \dots\dots\dots(48)$$

$$k_{sr} = \alpha^{-3} h_r \beta^4 \dots\dots\dots(49)$$

$$q_{sr} = \alpha^{1/2} h_r^{3/2} \beta^{-3/2} \dots\dots\dots(50)$$

$$d_r = \alpha^{-3} h_r \beta^4 \dots\dots\dots(51)$$

$$t_{sr} = \alpha^{1/2} h_r^{1/2} \beta^{3/2} (\sigma - \rho)_r^2 (1 - \lambda)_{sr} \dots\dots\dots(52)$$

$$Q_r = \alpha h_r^{5/2} \dots\dots\dots(53)$$

Note that the similarity ratios of un-distorted river models can be given by substituting $\alpha=1$ in all equations of each cases. And also, it should be noted that Eqs. (44)~(48) and (53) can be used as the similarity ratios for distorted or un-distorted river models with a fixed bed.

B) When the Values of x_r and d_r are Chosen Arbitrarily.

Let us assume that the values of x_r and d_r can be given arbitrarily in the nine equations (19), (20)', (21), (22), (23), (24)'', (25), (32)' and (33)', and then, the six ratios β , E_r , K_r , a_{sr} , $(\sigma - \rho)_r$ and $(1 - \lambda)_{sr}$ can be chosen as known ratios. The following nine ratios remain as unknown ratios: i_r , U_r , t_r , τ_r , n_r , h_r , k_{sr} , q_{sr} and t_{sr} . As described previously, the solutions of the nine equations can be obtained by solving these nine equations for the nine unknown ratios simultaneously.

Substituting $q=1/6$ and $p=2$ in this solutions

of the nine equations, and rearranging, we obtain

$$i_r = x_r \frac{(m-1)}{2(m-2)} d_r \frac{(m-1)}{2(m-2)} \beta \frac{(m-4)}{2(m-2)} K_r \frac{1}{2(m-2)} K_r \frac{1}{2(m-2)} \cdot E_r \frac{3}{(m-2)} (\sigma - \rho) \frac{m}{2(m-2)} \dots (54)$$

$$U_r = x_r \frac{(m-3)}{4(m-2)} d_r \frac{(m-1)}{4(m-2)} \beta \frac{(m-4)}{4(m-2)} K_r \frac{1}{4(m-2)} E_r \frac{3}{2(m-2)} \cdot (\sigma - \rho) \frac{m}{4(m-2)} \dots (55)$$

$$t_r = x_r \frac{(3m-5)}{4(m-2)} d_r \frac{(m-1)}{4(m-2)} \beta \frac{(m-4)}{4(m-2)} K_r \frac{1}{4(m-2)} E_r \frac{3}{2(m-2)} \cdot (\sigma - \rho) \frac{m}{4(m-2)} \dots (56)$$

$$\tau_r = x_r \frac{1}{(m-2)} d_r \frac{(m-1)}{(m-2)} \beta \frac{2}{(m-2)} K_r \frac{1}{(m-2)} E_r \frac{6}{(m-2)} \cdot (\sigma - \rho) \frac{m}{(m-2)} \dots (57)$$

$$n_r = x_r \frac{m}{6(m-2)} d_r \frac{(m-1)}{3(m-2)} \beta \frac{m}{3(m-2)} K_r \frac{1}{3(m-2)} E_r \frac{2}{(m-2)} \cdot (\sigma - \rho) \frac{3m}{3(m-2)} \dots (58)$$

$$k_{sr} = x_r \frac{m}{(m-2)} d_r \frac{2(m-1)}{(m-2)} \beta \frac{2m}{(m-2)} K_r \frac{2}{(m-2)} E_r \frac{6m}{(m-2)} \cdot (\sigma - \rho) \frac{2m}{(m-2)} \dots (59)$$

$$q_{sr} = x_r \frac{5}{2(m-2)} d_r \frac{(2m-1)}{2(m-2)} \beta \frac{5}{(m-2)} K_r \frac{5}{2(m-2)} E_r \frac{15}{(m-2)} \cdot (\sigma - \rho) \frac{(m+8)}{2(m-2)} a_{sr} \dots (60)$$

$$h_r = x_r \frac{(m-3)}{2(m-2)} d_r \frac{(m-1)}{2(m-2)} \beta \frac{(m-4)}{2(m-2)} K_r \frac{1}{2(m-2)} E_r \frac{3}{(m-2)} \cdot (\sigma - \rho) \frac{m}{2(m-2)} \dots (61)$$

$$t_{sr} = x_r \frac{(3m-2)}{2(m-2)} d_r \frac{m}{(m-2)} \beta \frac{(m+6)}{2(m-2)} K_r \frac{2}{(m-2)} E_r \frac{12}{(m-2)} \cdot (\sigma - \rho) \frac{4}{(m-2)} (1 - \lambda)_{sr} a_{sr} \dots (62)$$

And also, the ratio Q_r can be obtained by substituting Eqs. (55) and (61) in $Q_r = x_r h_r U_r$ namely,

$$Q_r = x_r \frac{(7m-17)}{4(m-2)} d_r \frac{3(m-1)}{4(m-2)} \beta \frac{3(m-4)}{4(m-2)} K_r \frac{3}{4(m-2)} E_r \frac{9}{2(m-2)} \cdot (\sigma - \rho) \frac{3m}{4(m-2)} \dots (63)$$

As explained earlier, if the flows in the model and prototype are hydraulically similar, the values of a_{sr} , E_r and K_r are equal to unity.

The similarity ratios in the case of $m=0$ are as follows:

$$i_r = x_r^{-1/4} d_r^{1/4} \beta^{-1} \dots (64)$$

$$U_r = x_r^{3/8} d_r^{1/8} \beta^{-1/2} \dots (65)$$

$$t_r = x_r^{5/8} d_r^{-1/8} \beta^{1/2} \dots (66)$$

$$\tau_r = x_r^{1/2} d_r^{1/2} \beta^{-1} \dots (67)$$

$$n_r = d_r^{1/6} \dots (68)$$

$$k_{sr} = d_r \dots (69)$$

$$q_{sr} = x_r^{5/4} d_r^{1/4} \beta^{-5/2} (\sigma - \rho) r^{-2} \dots (70)$$

$$h_r = x_r^{3/4} d_r^{1/4} \beta^{-1} \dots (71)$$

$$t_{sr} = x_r^{1/2} \beta^{3/2} (\sigma - \rho) r^2 (1 - \lambda)_{sr} \dots (72)$$

$$Q_r = x_r^{17/8} d_r^{3/8} \beta^{-3/2} \dots (73)$$

4. Determination Methods of Model sand and Scales

When sediment ripples do not occur at all in a prototype, if the sediment-ripple formations occur in the model correspond to this prototype under experiments, this model will not become similar geometrically and dynamically to the prototype. Therefore it is necessary that the sediment-ripple formations are prevented as much as possible. It is said in general that sediment ripples cannot occur when the bed material size d is greater than about 2 mm.

In order to prevent the sediment-ripple formations, it is necessary that the following equation is satisfied in general:

$$R_{e*} = \frac{U_* d_M}{\nu} \geq 100 \dots (74)$$

where R_{e*} is the shear velocity Reynolds number (sometimes called the bed Reynolds number), U_* the shear velocity, and ν the kinematic viscosity of water.

The so-called critical tractive force in the region of Eq. (74) can be expressed as Eq. (75):

$$U_{*c}^2 = \frac{\tau_c}{\rho} = 0.05 \{ (\sigma/\rho) - 1 \} g d_M \dots (75)$$

where U_{*c} is the critical shear velocity and τ_c the critical shear stress on bed. Combining Eqs. (74) and (75), and indicating for d_M ;

$$d_M \geq \left[\frac{R_{e*}^2 \nu^2}{0.05 \{ (\sigma/\rho) - 1 \} g} \right]^{1/3} \dots (76)$$

Substituting the following values in Eq. (76);

$R_{e*} = 100$, $\nu = 0.01 \text{ cm}^2/\text{sec}$, $(\sigma/\rho) = 2.65$ and $g = 980 \text{ cm}/\text{sec}^2$, we obtain $d_M \geq 2.3 \text{ mm}$.

Therefore it is understood that sediment ripples cannot occur when the bed material size d_M is greater than 2.3 mm. But in most river models with movable bed, the bed material size d_M has been used the model sand of $d_M \geq 0.1 \text{ mm}$ for the sake of the relations of scales and the magnitude of bed material size d_P in prototypes.

In order to obtain the relationship between scales and model sand in the case of $m=0$, let us use Eq. (51):

$$d_r = \alpha^{-3} h_r \beta^4 \dots (51)$$

Note that the relation of Eq. (51) can also be introduced from Eq. (71). By putting the value

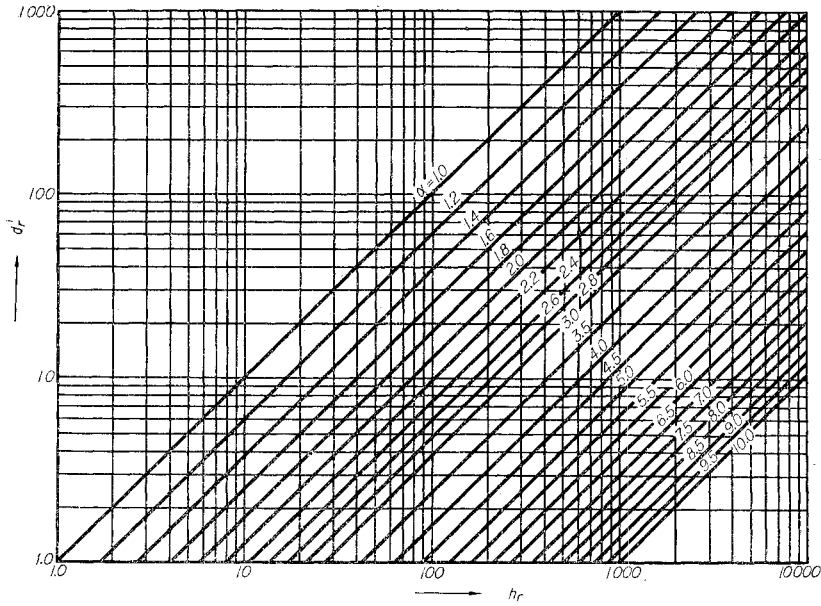


Fig. 4 Variation of d_r' with h_r and α .

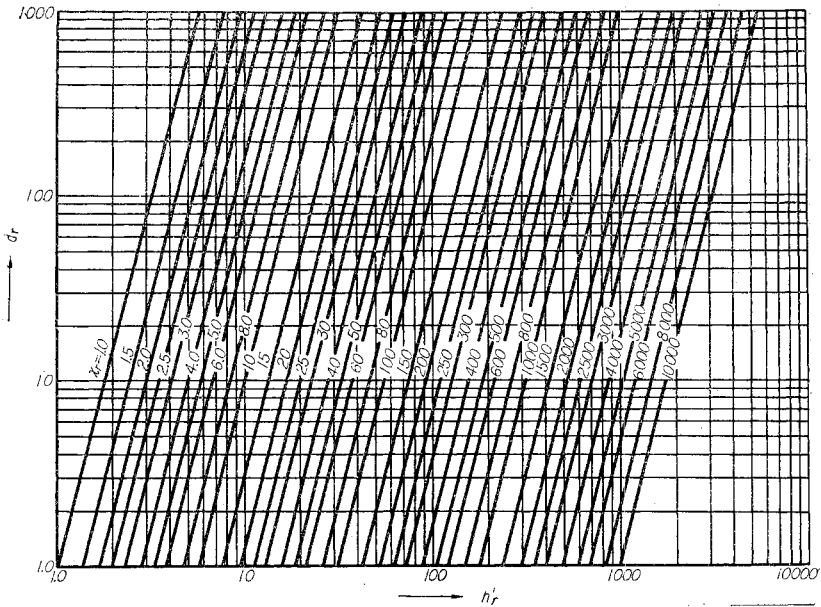


Fig. 5 Variation of d_r with h_r' and x_r

of d_r equal to $d_r'\beta^4$

$$d_r' = \alpha^{-2} h_r \dots\dots\dots (77)$$

where $d_r = d_r'\beta^4$. The relationship between d_r' and h_r in the case of $m=0$ can be shown in Fig. 4 by choosing α as a parameter.

On the other hand, in order to obtain the relationships x_r , h_r and d_r in the case of $m=0$, let us use Eq. (71);

$$h_r = x_r^{3/4} d_r^{1/4} \beta^{-1} \dots\dots\dots (71)$$

By putting the value of h_r equal to $h_r'\beta^{-1}$:

$$h_r' = x_r^{3/4} d_r^{1/4} \dots\dots\dots (78)$$

where $h_r = h_r'\beta^{-1}$. The relationship between h_r' and d_r can be shown in Fig. 5 by choosing x_r as a parameter.

For the convenience of the determination of model sand and scales, let us express the relationships between n_r and h_r , and also, n_r and d_r . By putting the value of n_r equal to $n_r'\beta^{2/3}$, Eq. (48) is

$$n_r' = \alpha^{-1/2} h_r^{1/6} \dots\dots\dots (79)$$

where $n_r = n_r'\beta^{2/3}$. The relationship between n_r' and h_r can be shown in Fig. 6 by choosing α as

a parameter.

In order to obtain the relationship between n_r and d_r , combining Eqs. (48) and (51), the following equation is obtained:

$$n_r = d_r^{1/6} \dots\dots\dots (80)$$

It is understood that the relationship of Eq. (80) is equal to Eq. (68). The relationship between n_r and d_r in the case of $m=0$ can be shown in Fig. 7.

The determination methods of model sand and scales in the case of $m=0$ can be divided into two cases

A) When the Values of x_r and h_r are Chosen Arbitrarily

The determination method of model sand and scales in this case can be divided into eight steps.

(1) In general, the value of x_r has the several restrictions in the point of view of an ground area for construction of a model, costs for constructions and experiments, several equipments for experiments and so on, but the value of x_r must be determined so as to take the smallest value that the value of x_r is approximately equal to h_r , as much as possible.

(2) The value of h_r is assumed so as to take the value of $d_M' \geq 2.3$ mm as much as possible by using Fig. 4.

(3) The value of α can be calculated from the assumed h_r and determined x_r .

(4) The value of β can be obtained from Fig. 1 by using the values of α and (h_p/B_P) derived from several cross-sections of the river under consideration. The value of β can also be calculated from Eq. (26)'.
 (5) By using these values x_r , h_r , α and β , the similarity ratios of i_r , U_r , t_r , τ_r , n_r , k_{sr} , q_{sr} , d_r and Q_r are calculated from Eqs. (44) ~ (51) and (53).

(6) The value of $(\sigma - \rho)_r$ can be obtained from the density tests of the determined model sand and bed material samples of the prototype.
 (7) The value of $(1 - \lambda)_{sr}$ can be calculated from Eq. (31) by using the values of $\gamma_w = 1.00$ g/cm³, G_{sr} , d_r calculated from Eq. (51) and d_p which is obtained from the sieve analyses of bed material samples collected in the proto-

type; that is, by using the values of $\gamma_w = 1.00$ g/cm³, G_{sr} , d_p and $d_M (= d_p/d_r)$. The value of $(1 - \lambda)_{sr}$ can also be obtained from the porosity tests of the model sand to be used and the river-bed sediments in the prototype under consideration.

(8) The similarity ratio t_{sr} can be calculated from Eq. (52) by using these determined values x_r , h_r , α , β , $(\sigma - \rho)_r$ and $(1 - \lambda)_{sr}$.

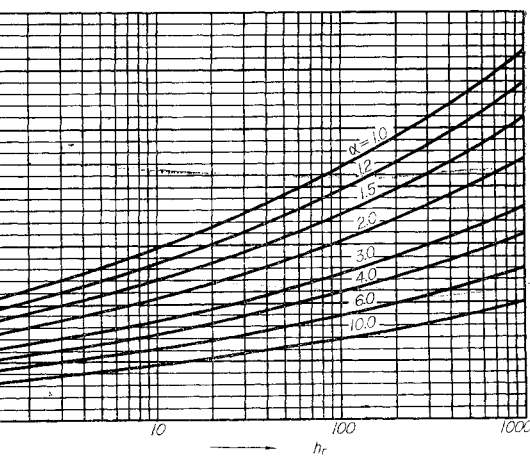


Fig. 6 Variation of n_r' with h_r and α

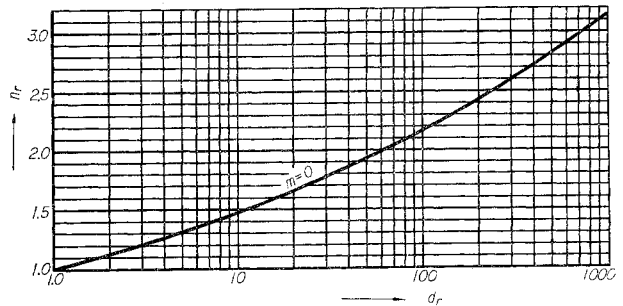


Fig. 7 Variation of n_r with d_r

B) When the Values of x_r and d_r are Chosen Arbitrarily.
 The determination method of model sand and scales in this case can be divided into eleven steps.
 (1) The value of x_r is determined so as to take the smallest value as possible, considering construction and operating costs and the benefits that may be derived.
 (2) The value of d_r is assumed so as to take the value of $d_M \geq 2.3$ mm as much as possible. The value of d_p is obtained from the sieve analyses of bed material samples collected in the prototype under consideration, therefore, when the known model sand d_M is used, the value

of d_r can be obtained from the values of d_p and d_M .

(3) The value of h_r' can be obtained from Fig. 5 by using this d_r .

(4) The value of α' can be approximately obtained from the determined x_r and obtained h_r' .

(5) The value of β' can be obtained from Eq. (26)' or Fig. 1 by using the values of α' and (h_p/B_p) derived from several cross sections of the prototype.

(6) The value of h_r can be determined from the equation of $h_r \doteq h_r'/\beta'$.

(7) The value of α is calculated by using this h_r .

(8) The value of β can be obtained from Eq. (26)' or Fig. 1 by using the values of α and (h_p/B_p) .

(9) The value of $(\sigma - \rho)_r$ can be obtained from the density tests of the determined model sand and bed material samples of the prototype.
 (10) The value of $(1 - \lambda)_{sr}$ can be calculated from Eq. (31) by using the values of $\tau_w = 1.00 \text{ g/cm}^3$, G_{sr} , d_p and $d_M (= d_p/d_r)$.

The value of $(1 - \lambda)_{sr}$ can also be obtained from the porosity tests of the model sand to be used and river-bed sediments in the prototype.
 (11) By using these values of x_r , d_r , β , $(\sigma - \rho)_r$ and $(1 - \lambda)_{sr}$, the various similarity ratios can be calculated from Eqs. (64)~(73).

5. An Example of Model Experiment

The author conducted the experiments on sediment discharge by the movable bed model of the Makita River, Gifu Prefecture, Japan, which was 30 m long and about 2 m wide, with scales of, horizontal 1 : 250, vertical 1 : 130, at the Hydraulics Laboratory, Gifu University.

The determinations of x_r , h_r and d_r were conducted by the determination method (B). The determination procedures of the model sand, scales and various similarity ratios were conducted as follows :

(1) As to the value of x_r , it was determined to take the value of $x_r = 250$ due to the restriction of an ground area for the construction of this model.

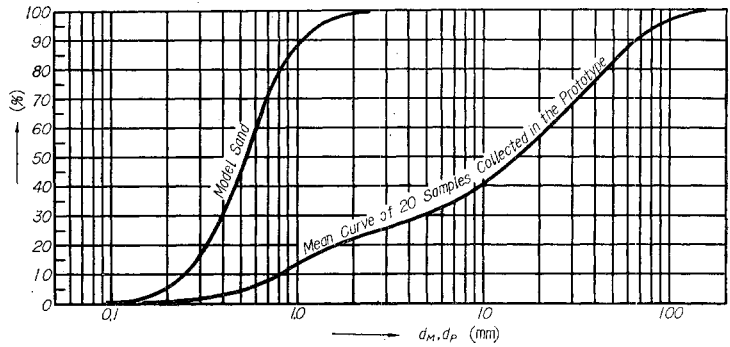


Fig. 8 Size distributions of bed material samples of the model and prototype

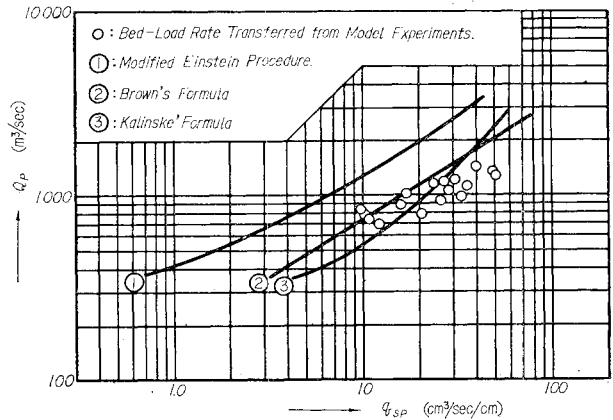


Fig. 9 A comparison of the computed bed-load rate with the bed-load rate transferred from model experiments

(2) As to the value of d_p , a value of $d_p = 15.3 \text{ mm}$ as the mean value of median diameters was obtained from the sieve analyses of 20 samples collected in the Makita River. As to the model sand, the author used the mixture sand which had a value of $d_M = 0.54 \text{ mm}$ (median diameter), which could be obtained easily in large quantities in this prototype. Therefore, as to the value of d_r , a value of $d_r = 28.3$ was obtained from the equation $d_r = d_p/d_M$. The results of the sieve analyses of the model sand used in this model experiments and of 20 samples collected in the Makita River shown in Fig. 8.

(3) From Fig. 5, a value of $h_r' = 144$ was obtained by using the values of $x_r = 250$ and $d_r = 28.3$.

(4) By calculating the ratio of (x_r/h_r') , a value of $\alpha' = 1.74$ was obtained.

(5) A value of $(h_p/B_p) = 0.03$ was obtained from several cross sections of this prototype.

Therefore, by using the values of $\alpha'=1.74$ and $(h_p/B_p)=0.03$, a value of $\beta'=1.11$ was obtained from Fig. 1.

(6) The value of $h_r \doteq 130$ was obtained from the equation of $h_r \doteq h_r'/\beta'$.

(7) By using this value $h_r=130$, a value of $\alpha=1.923$ was obtained.

(8) From Eq. (26)', a value of $\beta=1.115$ was obtained by using the values of $\alpha=1.923$ and $(h_p/B_p)=0.03$.

(9) Since the model sand of same density obtained from this prototype was used, a value of $(\sigma-\rho)_r=1.0$ was obtained.

(10) A value of $(1-\lambda)_{sr}=1.135$ was calculated from Eq. (31) by using the values of $\gamma_w=1.00 \text{ g/cm}^3$, $G_{sr}=1.0$ ($G_{sP}=G_{sM}=2.59$), $d_p=1.53 \text{ cm}$ and $d_M=0.054 \text{ cm}$.

(11) By substituting the values of $x_r=250$, $d_r=28.3$, $\beta=1.115$, $(\sigma-\rho)_r=1.0$, and $(1-\lambda)_{sr}=1.135$ in Eqs. (64)~(73), the similarity ratios were calculated as follows :

$$\begin{array}{ll} i_r = 0.52 & k_{sr} = 28.30 \\ U_r = 11.40 & q_{sr} = 1.75 \times 10^3 \\ t_r = 21.93 & h_r = 130.05 \\ \tau_r = 75.44 & t_{sr} = 21.13 \\ n_r = 1.75 & Q_r = 3.71 \times 10^5 \end{array}$$

For comparison purposes, let us express the results in the case of the determination method (A).

Since the values of x_r and h_r can be determined arbitrarily, the author determined as $x_r=250$ and $h_r=130$. By using the values of $x_r=250$ and $h_r=130$, the values of $\alpha=1.923$ and $\beta=1.115$ was obtained. The value of $(\sigma-\rho)_r=1.0$ was obtained from the density tests of bed material samples in the model and prototype. The similarity ratios in the case of $m=0$ by the determination method (A) were as follows :

$$\begin{array}{ll} i_r = 0.52 & k_{sr} = 28.26 \\ U_r = 11.40 & q_{sr} = 1.75 \times 10^3 \\ t_r = 21.93 & d_r = 28.26 \\ \tau_r = 75.38 & t_{sr} = 21.13 \\ n_r = 1.75 & Q_r = 3.71 \times 10^5 \end{array}$$

The value of $(1-\lambda)_{sr}$ was calculated as described previously from Eq. (31) by using the values of $\gamma_w=1.00 \text{ g/cm}^3$, $G_{sP}=G_{sM}=2.59$, $d_p=1.53 \text{ cm}$ and $d_M=0.054 \text{ cm}$. Therefore the ratio of $t_{sr}=21.13$ was obtained by substituting the values of α , h_r , β , $(\sigma-\rho)_r$ and $(1-\lambda)_{sr}=$

1.135 in Eq. (52).

The test results which performed in the model to obtain the bed-load rate at a point of this prototype were shown in Fig. 9 as an example. Comparing the results which were transferred to the prototype by the use of the similarity ratio of bed-load rate with the results which were computed by some different methods; the Modified Einstein procedure, Brown's formula and Kalinske's formula, the author obtained the results as shown in Fig. 9. Fig. 9 indicates that the results derived from the experiments of river models can be transferred to the prototype by the use of these various similarity ratios.

6. Conclusions

As described previously, various similarity ratios can be expressed in general as Eqs. (34)'~(43)' or Eqs. (54)~(63).

As regards the value of m for the similarity of river models with movable bed, in general case; it can be concluded that the value of m ought to take the value of $m=0$ from Eq. (69) or the comparison Eq. (39)' with Eq. (41)'. In special case, the value of m can be taken the value of $m=3$ according to the bed configurations.

The similarity ratios in the case of $m=0$ can be expressed as Eqs. (44)~(53) or Eqs. (64)~(73).

The similarity ratios in the case of distorted or un-distorted river models with a fixed bed can also be expressed as Eqs. (44)~(48) and (53).

The determinations of model sand, scales and various similarity ratios can be easily conducted by either the determination method (A) or (B).

The results derived from the experiments of river models can be transferred to the prototype by the use of these various similarity ratios as illustrated previously.

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APPENDIX

Symbols

- A = cross-sectional area of the flowing water;
 a_s = constant;
 B = breadth of the prototype;
 d = mean diameter of the bed material;
 d_{50} = median diameter of the bed material;
 E = constant;
 g = gravity acceleration;
 G_s = specific gravity of particles in the bed material;
 h = mean depth of the flowing water, and vertical lengths;
 h_s = river-bed elevation measured from any datum surface;
 i = bed slope;
 i_f = friction slope;
 K = constant;
 k_s = equivalent roughness;
 m = dimensionless exponent pertaining to the tractive force;
 n = Manning's roughness factor;
 p = constant;
 Q = discharge of the flowing water;
 q_s = rate of sediment transport in volume of material per unit time and width, including the rate of suspended sediment transport;
 q = dimensionless exponent pertaining to (R/k_s) ;
 R = hydraulic radius;
 Re_* = shear velocity Reynolds number (sometimes called the bed Reynolds number);
 t = hydraulic time;
 t_s = sedimentation time (time pertaining to the sediment transport);
 U = mean velocity of the flowing water;
 U_* = shear velocity;
 U_{*c} = critical shear velocity;
 x = horizontal lengths;
 α = degree of distortion ($\alpha = x_r/h_r$);
 β = dimensionless coefficient ($\beta = 1 + 2\alpha(h_p/B_p)$);
 γ_d = dry density of the bed material;
 γ_w = specific weight of water;
 λ = porosity of the bed material;
 ν = kinematic viscosity of water;
 ρ = density of water;
 σ = density of the bed material;
 τ = shearing stress on bed;
 τ_c = critical shear stress on bed.
- Subscripts
- P = refers to prototype;
 M = refers to model;
 r = denotes the ratio of corresponding values in model and prototype;
 s = denotes quantity pertaining to the sediment transport.