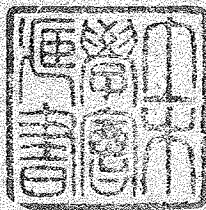TRANSACTIONS
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ON WATER-HAMMER PRESSURE DUE TO PERIODIC OPENING AND CLOSURE OF VALVE

(弁の周期的開閉による水撃圧について)

By Sukeyuki Shima, C.E. Member, and Yoshio Ogihara, C.E. Member

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Synopsis : A water-hammer action develops in a pipe when the valve is periodically opened and closed at the end of an elastic pipe. Since this phenomenon is similar in appearance to an elastic vibration, there arises a question if a resonant vibration could be introduced at a certain period of alternate opening and closure of the valve to render the resulting water-hammer unstable. Questions have also been aroused as to the spatial and temporal distribution of the water-hammer occurring under such conditions. The present study has treated of these problems both in theory and experiment.

1. INTRODUCTION

As a recent trend a regulating valve of a hydroelectric plant has to be opened and closed periodically in response to the demand of, such as, a strip mill of an iron work where power must be fed periodically. When a flow section is thus periodically varied at an end of an elastic pipe, the flow velocity changes accordingly and as a result a water-hammer action develops inside the pipe depending on the elastic characteristics and shape of the pipe and the elastic characteristics of water. Not much study has been done in respect to this type of water-hammer action either theoretically and experimentally, and key problems remain unsolved. This is due theoretically to nonlinearity of the boundary conditions and complexity of the computation procedures involved, and experimentally to limitations imposed on the pipe length and a special device needed for the structure of a valve model which must be opened and closed in rapid succession.

Fortunately, we have succeeded in overcoming these difficulties and marking a step forward both in theory and experiment. Our original purpose consisted in examining the question if this type of water-hammer could be rendered unstable by periodic opening and closure of the valve as a resonance commonly observed in an elastic vibration. We also planned to study the effects of viscosity on divergence of the water-hammer action. However, the results of our theoretical and experimental study have shown that this type of water-hammer action is stable, and further that the effects of viscosity is all but negligible.

2. THEORETICAL CONSIDERATION

The denotations used in this paper are defined below in reference to **Fig. 1**.

(1) **Denotation**

x = distance from the reservoir along the pipe axis t = time

$P_1(x,t)$ = pressure inside the pipe at $x=x$ and $t=t$

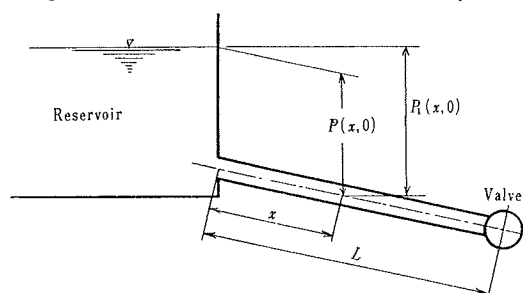$V(x,t)$ = average flow velocity at $x=x$ and $t=t$

ϕ = potential of external force

$P(x,t) = P_1(x,t) - \rho\phi$ D = I.D. of the pipe

L = entire length of the pipe

d = thickness of the pipe

Fig. 1 Generale view of water-hammer system.

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$K=$ coefficient of compressibility of water

$E=$ coefficient of elasticity of pipe material

$\rho=$ density of water

$C=$ Chézy's constant

$a=$ wave velocity of water-hammer pressure

$g=$ acceleration due to gravity

$\Psi(t)=$ gate ratio function representing opening ratio of the valve

$T=$ period of opening and closure movement of the valve

(2) Fundamental equations

Considering infinitesimal terms of relatively higher orders, the equations of motion and continuity for a water-hammer pressure are given, respectively,[1]

$$\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}=-\frac{4gV^2}{C^2D}-\frac{1}{\rho}\frac{\partial p}{\partial x}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(1)$$

$$\frac{\partial V}{\partial x}=-\frac{1}{\rho a^2}\left(\frac{\partial p}{\partial t}+V\frac{\partial p}{\partial x}\right)\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(2)$$

where

$$a=\sqrt{\frac{1}{\rho}\Big/\left(\frac{1}{\kappa}+\frac{D}{dE}\right)}$$

These nonlinear equations are linearized on the assumptions below.

(1) The pressure due to water-hammer is relatively small as compared with the hydrostatic pressure.

(2) Changes in flow velocity due to water-hammer is small as compared with the initial velocity.

(3) The density of water ρ and wave velocity of the water-hammer a are constant.

(4) Variation in gate ratio which represents the ratio of opening of the valve is small.

(5) The following relationship hold,

$$\left.\begin{array}{l}P(x,t)=P(0,0)+p_1(x,0)+p_2(x,t)\\V(x,t)=V(0,0)+v_1(x,0)+v_2(x,t)\end{array}\right\}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(3)$$

Now that the initial conditions $P(0,0)$ and $V(0,0)$ are already known, our remaining problem is to determine $p_1(x,0)$, $v_1(x,0)$, $p_2(x,t)$, and $v_2(x,t)$. The terms $p_1(x,0)$ and $v_1(x,0)$ are the stationary solutions. Hence they can be determined on the assumptions already made and by further assuming that these stationary terms are small as compared with $P(0,0)$ and $V(0,0)$.

$$\left.\begin{array}{l}v_1(x,0)=\frac{1}{2}V(0,0)\left\{\exp.\left(-\frac{8gV(0,0)^2}{(a^2-v(0,0)^2C^2D}x\right)-1\right\}\\p_1(x,0)=-\frac{\rho a}{2}\left\{1-\exp.\left(\frac{8gV(0,0)^2}{(a^2-V(0,0)^2)C^2D}x\right)\right\}\end{array}\right\}\cdots\cdots\cdots(4)$$

Substituting these results and rewriting the fundamental equations (1) and (2) with p_2 and v_2, we obtain linear fundamental equations in respect to p_2 and v_2, as follows.

$$\frac{\partial v_2}{\partial t}+V_0\frac{\partial v_2}{\partial x}=-\frac{1}{\rho}\frac{\partial p_2}{\partial x}-\frac{8gV_0}{C^2D}v_2\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(5)$$

$$\frac{\partial v_2}{\partial x}=-\frac{1}{\rho a^2}\left\{\frac{\partial p_2}{\partial t}+V_0\frac{\partial p_2}{\partial x}\right\}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(6)$$

where

$$V_0=V(0,0)\quad\text{and}\quad P_0=P(0,0).$$

We may rewrite these fundamental equations by using 10 non-dimensional terms shown below.

$$\left.\begin{array}{l}u=\frac{v_2}{V_0},\quad h=\frac{p_2}{P_0}\\\xi=\frac{x}{L},\quad z=\frac{t}{T}\end{array}\right\}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(7)$$

$$2\,\rho = \frac{\rho\,a\,V_0}{P_0} = \frac{a\,V_0}{g\,H_0}, \quad 2\,\theta = \frac{a\,T}{L}$$

$$f = \frac{\rho\,V_0{}^2}{P_0} = \frac{V_0{}^2}{g\,H_0}, \quad \mu = \frac{8\,g\,V_0{}^2\rho\,L}{C^2 D P_0} = \frac{8\,V_0{}^2 L}{C^2 D H_0}$$

$$m = \frac{V_0}{a}, \quad f = 2\,\rho\,m$$

$$\varphi = \frac{\psi(t)}{\Psi(0)}$$

\quad\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (8)

where

$$\Psi(t) = \Psi(0) + \psi(t), \quad H(x,t) = \frac{1}{\rho\,g} p(x,t)$$

As a result the relationships (5) and (6) are reduced to

$$\frac{\rho}{\theta}\frac{\partial u}{\partial z} + f\frac{\partial u}{\partial \xi} + \frac{\partial h}{\partial \xi} + \mu\,u = 0 \dots\dots\dots\dots\dots\dots\dots\dots (9)$$

$$4\,\rho\theta\frac{\partial u}{\partial \xi} + \frac{\partial h}{\partial z} + 2\,m\,\theta\frac{\partial h}{\partial \xi} = 0 \dots\dots\dots\dots\dots\dots (10)$$

Further, the initial and boundary conditions may be rewritten by using the same non-dimensional terms as follows.

Initial conditions :

$$u(\xi,0) = 0, \quad \left(\frac{\partial u}{\partial z}\right)_{z=0} = 0$$
$$h(\xi,0) = 0, \quad \left(\frac{\partial h}{\partial z}\right)_{z=0} = 0$$

\quad\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (11)

Boundary conditions :

At the reservoir

$$h(0,z) = 0 \quad\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (12)$$

If it is assumed that the Torricelli's theorem holds, $V(L,t) = \Psi(t)\sqrt{P(L,t)}$, and based on the previous assumptions, this is linearized in a non-dimensional form as

where

$$u(1,z) = \nu\left\{\varphi(z) + \frac{h(1,z)}{2\,\kappa}\right\}$$

$$\nu = \frac{V(L,0)}{V(0,0)} = 1 + \frac{1}{2}\left\{e^{\frac{m^2}{1-m^2}\frac{\mu}{f}} - 1\right\}$$

$$\kappa = \frac{P(L,0)}{P(0,0)} = 1 + \frac{\rho}{m}\left\{1 - e^{\frac{m^2}{1-m^2}\frac{\mu}{f}}\right\}$$

\quad\dots\dots\dots\dots\dots\dots (13)

In ordinary cases, the value of m is very small, so that we may put $\nu = 1$, $\kappa = 1 - \frac{\mu}{2}$. Therefore, Eq. (13) is reduced to

$$u(1,z) = \varphi(z) + \frac{h(1,z)}{2\left(1 - \frac{\mu}{2}\right)} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (13')$$

Further, if we neglect the effect of friction, Eq. (13') becomes

$$u(1,z) = \varphi(z) + \frac{h(1,z)}{2} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (13'')$$

Taking into consideration the relationships (11), (12) and (13), and applying the Laplace transform[2] to the equations (9) and (10) in term of z, and then rewriting in term of pressure, we generally obtain

$$\bar{h}(\xi,s) = \frac{-\nu\,\bar{\varphi}\,e^{a\xi}\,\mathrm{sh}\,b\,\xi}{e^a\left[\left\{\frac{\theta}{\rho\,s+\mu\theta}\left(\frac{2\,\rho-fm}{2\,\rho}a - \frac{fs}{4\,\rho\theta}\right) + \frac{\nu}{2\,\kappa}\right\}\mathrm{sh}\,b + \frac{\theta}{\rho\,s+\mu\theta}\frac{2\,\rho-fm}{2\,\rho}b\,\mathrm{ch}\,b\right]} \quad\dots\dots (14)$$

where

$$a = \frac{sf + 2\rho ms + 2m\theta\mu}{4\theta(2\rho - fm)}$$

$$b = \frac{\sqrt{(sf + 2\rho ms + 2m\theta\mu)^2 + 8(2\rho - fm)(\rho s^2 + \mu s\theta)}}{4\theta(2\rho - fm)}$$

The result function $\bar{h}(\xi, s)$, corresponding to the boundary condition Eq. (13'), is reduced to

$$\bar{h}(\xi, s) = \frac{-\bar{\varphi}\, e^{a\xi}\,\mathrm{sh}\, b\xi}{e^a\left[\left(\dfrac{1}{2-\mu} - \dfrac{a\theta}{\rho s + \mu\theta}\right)\mathrm{sh}\, b + \dfrac{b\theta}{\rho s + \mu\theta}\mathrm{ch}\, b\right]} \quad\cdots\cdots\cdots\cdots (14')$$

where

$$a = \frac{fs}{8\rho\theta}$$

$$b = \sqrt{\left(\frac{fs}{8\rho\theta}\right)^2 + \frac{\rho s^2 + \mu\theta s}{4\rho\theta^2}}$$

Further, when we assume that the value of f is negligible as in the case of ordinary water-hammer analysis, Eq. (14') is simplified as

$$\bar{h}(\xi, s) = \frac{-\bar{\varphi}\,\mathrm{sh}\, b\xi}{\dfrac{1}{2-\mu}\mathrm{sh}\, b + \dfrac{\theta}{\rho s + \mu\theta}b\,\mathrm{ch}\, b} \quad\cdots\cdots\cdots\cdots\cdots\cdots (14'')$$

where

$$b = \sqrt{\frac{\rho s^2 + \mu\theta s}{4\rho\theta^2}}$$

In Eqs. (14), (14') and (14''), s denotes the parameter of transformation and $-$ a transformed result function. The inverse transform of these relationships is too complex to obtain through an ordinary procedure of computation.

(3) In case where the flow velocity varies periodically at the position of the valve.

Here, we may use as the boundary conditions, instead of Eq. (13),

$$u(1, z) = \varphi(z) \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (15)$$

If we assume that m and f are small as in the case of Eq. (14'') the Laplace transform in respect to pressure is

$$\bar{h}(\xi, s) = \frac{-\bar{\varphi}\,\mathrm{sh}\, b\xi}{\dfrac{\theta}{\rho s + \mu\theta}b\,\mathrm{ch}\, b} \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (16)$$

The inverse transform can be obtained and expressed as follows.

$$\bar{h}(\xi, z) = -\mu\xi\varphi(z) - 8\rho\theta\int_0^z e^{-\frac{\mu\theta}{2\rho}(z-\mathfrak{z})}\sum_{n-1}^{\infty}\frac{(-1)^{n-1}\sin\dfrac{(2n-1)}{2}\pi\xi}{(2n-1)\pi}$$

$$\times\left[\sin(2n-1)\pi\theta(z-\mathfrak{z}) - \frac{\mu}{(2n-1)\pi\rho}\cos(2n-1)\pi\theta(z-\mathfrak{z})\right]\frac{\partial\varphi}{\partial\mathfrak{z}}d\mathfrak{z} \quad\cdots\cdots (17)$$

Here, $\varphi(z)$ can be an arbitrary function, but, if so, the integration is rendered considerably difficult. We have carried out the computation of Eq. (17) for a particular condition expressed by

$$\varphi(z) = B\sin 2\pi z \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (18)$$

As a result the following relationship has been obtained

$$h(\xi, z) = -\frac{16 B\rho}{\pi\theta}\mathfrak{F}_1\cos 2\pi z - \mu B\left(\xi + \frac{64}{\pi^2\theta^2}\mathfrak{F}_2\right)\sin 2\pi z$$

$$+ \frac{8 B}{\pi^2\theta}e^{-\frac{\mu\theta}{2\rho}z}\{2\rho\pi\mathfrak{F}_3 + \mu\mathfrak{F}_4\} \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots (19)$$

where

$$\mathfrak{F}_1 = \sum_{n=1}^{\infty} (-1)^{n-1} L_n \sin \frac{C_n}{2\theta} \xi$$

$$\mathfrak{F}_2 = \sum_{n=1}^{\infty} (-1)^{n-1} M_n \sin \frac{C_n}{2\theta} \xi$$

$$\mathfrak{F}_3 = \sum_{n=1}^{\infty} (-1)^{n-1} L_n \sin \frac{C_n}{2\theta} \cos C_n z$$

$$\mathfrak{F}_4 = \sum_{n=1}^{\infty} (-1)^{n-1} N_n \sin \frac{C_n}{2\theta} \xi \sin C_n z$$

$$\left. \right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (20)$$

$$L_n = \frac{\left(\frac{\mu}{2\rho\pi}\right)^2 \left\{(2n-1)^2 + \frac{8}{\theta^2}\right\} + (2n-1)^2 \left\{\frac{4}{\theta^2} - (2n-1)^2\right\}}{-(2n-1)^2 \left[2\left(\frac{\mu}{2\rho\pi}\right)^2 \left\{(2n-1)^2 + \frac{4}{\theta^2}\right\} + \left\{(2n-1)^2 - \frac{4}{\theta^2}\right\}^2\right]}$$

$$M_n = \frac{(2n-1)^2 - \frac{2}{\theta^2} - \frac{1}{2}\left(\frac{\mu}{2\rho\pi}\right)^2}{(2n-1)^2 \left[2\left(\frac{\mu}{2\rho\pi}\right)^2 \left\{(2n-1)^2 + \frac{4}{\theta^2}\right\} + \left\{(2n-1)^2 - \frac{4}{\theta^2}\right\}^2\right]}$$

$$N_n = \frac{3\left(\frac{\mu}{2\rho\pi}\right)^2 + \left\{3(2n-1)^2 - \frac{4}{\theta^2}\right\}}{(2n-1)\left[2\left(\frac{\mu}{2\rho\pi}\right)^2 \left\{(2n-1)^2 + \frac{4}{\theta^2}\right\} + \left\{(2n-1)^2 - \frac{4}{\theta^2}\right\}^2\right]}$$

$$\left. \right\} \quad \cdots\cdots\cdots\cdots (21)$$

$$C_n = (2n-1)\pi\theta \cdots\cdots\cdots\cdots\cdots (22)$$

This relationship enables us to evaluate the pressure due to water-hammer at an arbitrary position and arbitrary time for the case where the effect of viscosity is considered. The results are shown in **Fig. 2** in terms of Manning's roughness coefficient n vs. maximum water-hammer pressure at $z \to \infty$. The initial state of water-hammer pressure derived also from this relationship is shown in **Fig. 6** in comparison with other examples of computation. By computing Eq. (19) it has been found that the maximum amplitude of water-hammer pressure occurs at $\frac{1}{\theta} = \frac{1}{2} + i$ ($i = 0, 1, 2, \cdots$).

Neglecting the effect of viscosity in Eq. (16) we obtain

Fig. 2 Relationship between maximum water-hammer pressure and roughness for the pipe system of our experiment.

$$D = 0.10\,\text{m}$$
$$L = 100\,\text{m}$$
$$a = 1000\,\text{m/sec}$$
$$H_0 = 15\,\text{m}$$

$$\bar{h}(\xi, s) = \frac{-2\rho\bar{\varphi}\,\text{sh}\,\frac{s}{2\theta}\xi}{\text{ch}\frac{s}{2\theta}} = -2\rho\bar{\varphi}\sum_{n=0}^{\infty} (-1)^n \left\{e^{-\frac{s}{2\theta}(2n+1-\xi)} - e^{-\frac{s}{2\theta}(2n+1+\xi)}\right\} \quad \cdots\cdots\cdots\cdots (23)$$
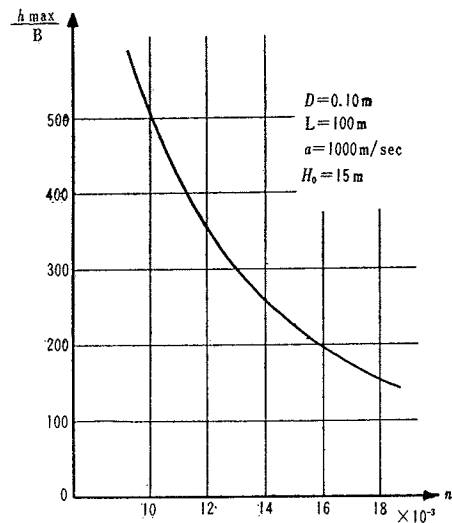
The inverse transform is also obtained

$$h(\xi, z) = -2\rho \sum_{n=0}^{\infty} (-1)^n \left[\varphi\left\{z - \frac{1}{2\theta}(2n+1-\xi)\right\} - \varphi\left\{z - \frac{1}{2\theta}(2n+1+\xi)\right\}\right] \cdots\cdots\cdots\cdots (24)$$

where

$$\left. \begin{array}{l} \varphi(z-\alpha) = 0, \quad z < \alpha \\ = \varphi(z-\alpha), \quad z \geqq \alpha \end{array} \right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (25)$$

such that the computation is carried out relatively with ease when the effect of viscosity is neglected. However, since these serial expressions converge poorly, the computation is considerably complex for a

large value of z. The physical significance and computed examples of Eq. (24) will be discussed later.

(4) In case the valve is opened and closed periodically.

The water-hammer due to periodic opening and closure of the valve can be determined by computing the inverse transform of Eq. (14). However, since this is generally impossible, we have confined our objective to such cases where the effect of viscosity is negligible. Eq. (14″) then becomes

$$\bar{h}(\xi,s) = \frac{-2\,\rho\bar{\varphi}\,\mathrm{sh}\dfrac{s}{2\,\theta}\xi}{\rho\,\mathrm{sh}\dfrac{s}{2\,\theta}+\mathrm{ch}\dfrac{s}{2\,\theta}} \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (26)$$

which corresponds to the boundary condition Eq. (13″). It is all but impossible to obtain the imverse transform of this equation for the arbitrary gate ratio function $\varphi(z)$ by using the residue theorem just as done by George R. Rich for the special case shown below. However, our method has turned out a result through a relatively simple procedure and also allowed a clear representation of its physical significance. Let us briefly review the Rich's method[3]. For a particular condition $\varphi(z)=B\sin 2\pi z$, where a stationary state prevails after disappearance of a relatively earlier water-hammer, i.e., for the limit condition where $z\to\infty$, the imverse transform of Eq. (26) becomes

$$h(\xi,\infty) = -2\,\rho\,B\sin\frac{\pi}{\theta}\xi\,\sin(2\,\pi z+\beta) \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (27)$$

$$h(1,\infty)_{\max}=2\,B\,|\cos\beta| \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (28)$$

where

$$\beta=\tan^{-1}\left\{\frac{\cot\dfrac{\pi}{\theta}}{\rho}\right\}$$

The above method is not available for giving the water-hammer pressure at the earlier stage, but suffices to give stable and unstable conditions of the water-hammer pressure which is generated by periodic opening and closure of the valve.

We have succeeded in obtaining a relationship which provides the water-hammer pressure at the earlier stage and for an arbitrary function of $\varphi(z)$, as follows.

Eq. (26) is rewritten in exponential terms

$$\bar{h}(\xi,s) = -\frac{2\,\rho\bar{\varphi}\,(e^{\frac{s}{2\theta}\xi}-e^{-\frac{s}{2\theta}\xi})}{(1+\rho)\,e^{\frac{s}{2\theta}}+(1-\rho)\,e^{-\frac{s}{2\theta}}}$$

Considering further that ρ is a positive value and $\dfrac{\rho-1}{\rho+1}<1$, the relationship is developed in terms of $\dfrac{\rho-1}{\rho+1}$.

$$\bar{h}(\xi,s) = -\frac{2\rho}{1+\rho}\bar{\varphi}\sum_{n=0}^{\infty}\left(\frac{\rho-1}{\rho+1}\right)^{n}\left\{e^{-\frac{s(2n+1-\xi)}{2\theta}}-e^{-\frac{s(2n+1+\xi)}{2\theta}}\right\} \quad\cdots\cdots\cdots\cdots (29)$$

Thus the inverse transform of this equation is obtained as follows.

$$h(\xi,z) = -\frac{2\,\rho}{1+\rho}\sum_{n=0}^{\infty}\left(\frac{\rho-1}{\rho+1}\right)^{n}\left[\varphi\left\{z-\frac{1}{2\,\theta}(2\,n+1-\xi)\right\}-\varphi\left\{z-\frac{1}{2\theta}(2\,n+1+\xi)\right\}\right] \quad\cdots\cdots (30)$$

where

$$\begin{aligned}\varphi(z-\alpha)&=0,\quad z<\alpha\\&=\varphi(z-\alpha),\quad z\geqq\alpha\end{aligned}\Bigg\} \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (25)$$

By using Eq. (30) the water-hammer pressure at an arbitrary position and time can be computed for and arbitrary function of $\varphi(z)$. Moreover, since ρ is relatively approximate to unity, convergence of this series is extremely good. When compared with Eq. (24) which has been derived for the condition where the flow varies periodically, the physical significance of Eq. (30) is also evident.

The physical significance of Eqs. (24) and (30) is illustrated in **Fig. 3**. The first term represents the pressure at the instant when the water-hammer wave initiated at the position of the valve passes the position $\xi = \xi$; the second term the pressure at the instant when the same wave passes $\xi = \xi$ after being reflected from the reservoir; and then the third term the pressure at the instant when this reflected wave once again passes $\xi = \xi$ after being reflected at the valve and propagates as a progressive wave. The subsequent terms show that these phenomena are repeated in succession. Since $\varphi(z)$ is generally not such a simple function as $B \sin 2\pi z$, it is necessary to determine $\varphi(z)$ by the actual measurement in advance and from this to obtain the water-hammer pressure diagrammatically through Eq. (30). An example is given in **Fig. 4**.

For $\xi = 1$ at Eq. (30):

(a) if $\rho > 1$, $h(1, z)$

$$= -\frac{2\rho}{1+\rho} \Sigma \left(\frac{\rho-1}{\rho+1}\right)^n \left[\varphi\left(z - \frac{n}{\theta}\right)\right.$$

$$\left. - \varphi\left(z - \frac{n+1}{\theta}\right)\right] \cdots\cdots\cdots (31)$$

(b) if $\rho = 1$, $h(1, z)$

$$= -\varphi(z) + \varphi\left(z - \frac{1}{\theta}\right) \cdots\cdots (32)$$

(c) if $1 > \rho > 0$, $h(1, z) =$

$$-\frac{2\rho}{1+\rho} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1-\rho}{1+\rho}\right)^n$$

$$\left[\varphi\left(z - \frac{n}{\theta}\right) - \varphi\left(z - \frac{n+1}{\theta}\right)\right] \cdots (33)$$

By using these formulae the temporal distribution of the water-hammer pressure at the position of the valve can be determined.

Let us now consider the spatial distribution of the water-hammer pressure. At the early stage of time it is as shown in **Figs. 5** (a), (b) and (c). For $z \to \infty$, i.e., after a considerable length of time the number of the terms of Eq. (3) becomes infinite, which implies that

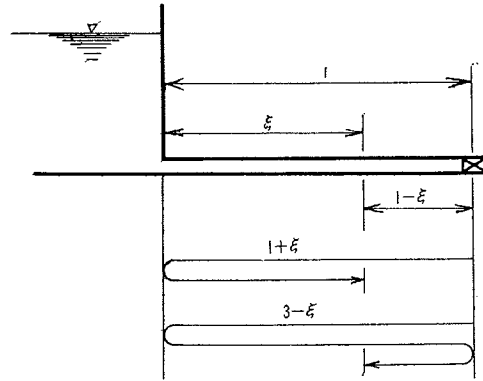**Fig. 3** Progressive and reflected waves of point ξ.

Fig. 4 Water-hammer pressure at $\xi = 1$, obtained diagrammatically through Eq. (30).
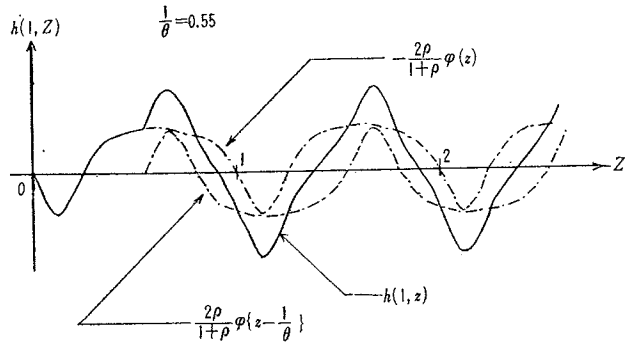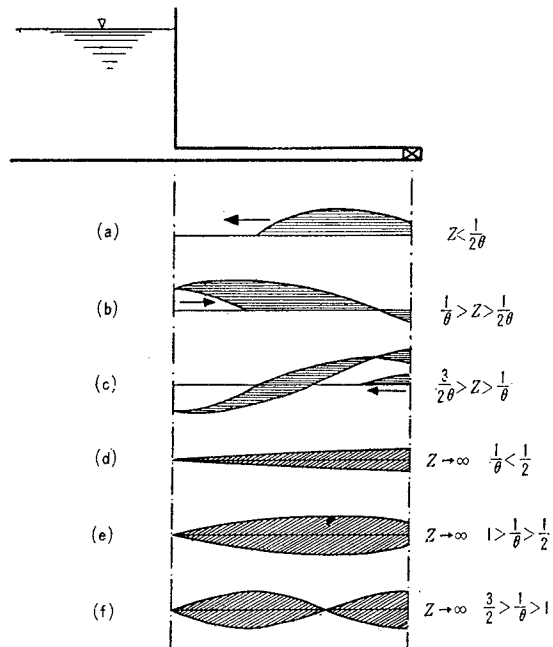

Fig. 5 Transient and stationary pressure distribution along pipe axis.

physically the pressure distribution co-
mes to be stationary. Let us examine
Eq. (30) for such condition. Since $\varphi(z)$
is a function of the period equal to
unity, we obtain the condition on which
the relationship $h(\xi, \infty) = 0$ holds inde-
pendently of the value of z, as follows.

$$\xi = i\theta \quad (i = 0, 1, 2, \cdots\cdots) \quad\cdots\cdots\cdots(34)$$

Such value of ξ which gives the max-
imum $h(\xi, \infty)$ can not be obtained
unless the expression of $\varphi(z)$ is determ-
ined. For a particular case where $\varphi(z)$
$= B \sin 2\pi z$, Eq. (30) is rewritten

Fig. 6　Numerical results obtained by three different
equations.

$$h(\xi, \infty) = -\frac{4\rho B}{1+\rho} \sin \frac{\pi\xi}{\theta} \sum_{n=0}^{\infty} \left(\frac{\rho-1}{\rho+1}\right)^n \cos 2\pi \left\{z + \frac{2n+1}{2\theta}\right\} \quad\cdots\cdots\cdots\cdots\cdots(35)$$

giving the same expression as Eq. (27) which has already been discussed. Hence the value of ξ giving
the maximum $h(\xi, \infty)$ is, in this case, evidently,

$$\xi = \left(\frac{1}{2} + i\right)\theta \quad (i = 0, 1, 2, \cdots\cdots)\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(36)$$

Taking into account these results, the spatial distribution of the water-hammer pressure is illustrated as
shown in Figs. 5 (d), (e), and (f), which are quite similar to the vibration pattern of a string or a
tuning fork.

Let us now consider the water-hammer pressure at the position of the valve for the condition $\varphi(z)$
$= B \sin 2\pi z$. The results are illustrated in Fig. 6 for the following three cases.　　(a) Water-hammer
pressure due to periodic variation of the flow velocity, considering the effect of viscosity ($n = 0.014$);
(b) Water-hammer pressure due to periodic variation of the flow velocity, neglecting the effect of
viscosity; (c) Water-hammer pressure due to periodic opening and closure of the valve, neglecting the
effect of viscosity. As a result, it is found that the water-hammer pressure is extremely unstable when
the flow velocity varies periodically at the position of the valve, while it is stable when the valve is
opened and closed periodically. In practice, however, such phenomenon where the flow velocity varies
periodically seldom occurs.

3.　EXPERIMENTS

In the preceding chapter a theoretical analysis of the water-hammer pressure has been discussed for
the cases where the valve is opened and closed periodically. We will now proceed to verify the theor-
etical results through experiments.

One of the greatest difficulties encountered was the structure of a valve model which is durable to
high-frequency vibration. After a series of futile efforts we have fortunately succeeded in designing a
valve of special type. Outline of the experimental equipments and a part of the experimental results
will be discussed below.

Fig. 7 gives a schematical representation of the experimental equipments, which includes (1) pipeline,
(2) high-head tank, (3) low-head tank, (4) notch weir, (5) lift pump, (6) main valve, (7) cock,
(8) motor and speed regulator, (9) pressure gauge, (10) strainmeter and oscillograph, (11) manometer
for calibration and pressure unit, (12) valve for calibration, (13) terminal contact, (14) signal for cali
bration, (15) air valve.

First the lift pump　(5) is operated to send water from the low-head tank　(3) to the high-head tank
(2), and then by opening the valve (12) water is passed down the pipeline (1). The discharge is
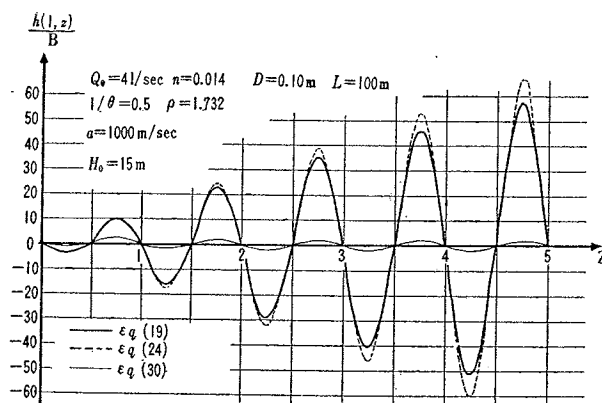
measured by the 60°-notch weir (4). Now, the main valve (6) is started by the motor and speed regulator (8) to produce a water-hammer wave inside the pipeline (1). The wave is recorded by the pressure gauge (9) and the oscillograph-amplifier unit (10). In the mean time the opening and closure movement of the main valve (6) is transmitted electrically by the terminal contact (13) to the oscillograph recorder. The calibration procedure consists of first cutting the flow by closing the cook (7) completely, filling the pipe with water by closing the valve (12), raising the pressure inside the pipe to a known valuw with the pressure unit (11), and then recording with the pressure gauge (9) and the strainmeter-oscillograph unit (10) the pressure which is being gradually diminished. In the mean time the water pressure is measured by the mano-meter (11) and transformed into an electric signal by the key (14), which is then transmitted to the oscillograph (10) and recorded there simultaneo-usly with the water pressu-re. The air inside the pipe is removed through the air valve (15). The phase vel-ocity of the water-hammer pressure is determined from the wave profiles resulting from the instant closure of the valve. The pipeline is made of steel and of a circular cross-section 10.5 cm I.D., 0.45 cm thick, 97.60 cm in total length, and 16.7 m inl head.

The arrangement of the main valve and the motor and speed regulator is illustrated in **Fig. 8**, showing in position   (1) motor, (2) speed regulator,   (3) pulley for arm con-nection,   (4) arm,   (5) holes for connecting arm,   (6) main body of valve, (7) contact for revolution indicator,   (8) cook and (9) pipe.

Power is supplied from the motor   (1) to the speed regulator   (2) which controls the period of the opening and closure movement of the main valve through the pulley (3) plus the arm (5). The pulley (3) is equipped with two holes, denoted No. 1 and 2, where the

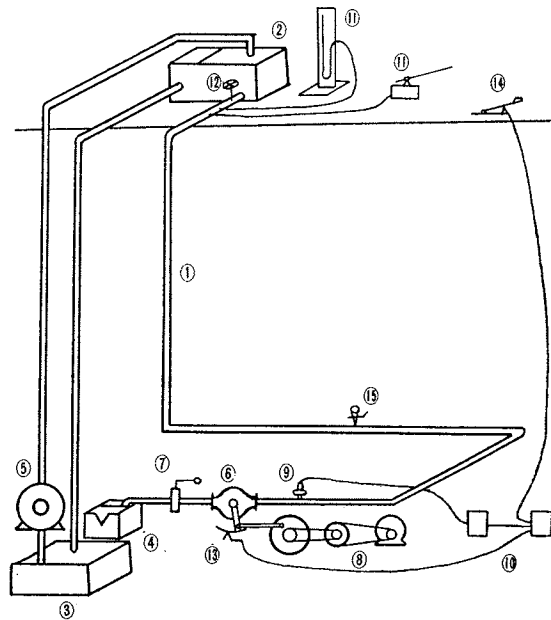**Fig. 7** Schematic representation of the experimental equipment.

Fig. 8 Main valve, motor and speed regulator.
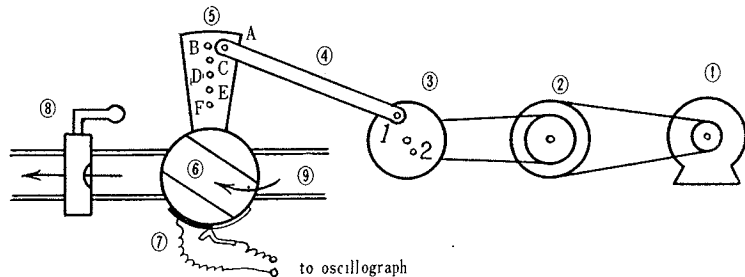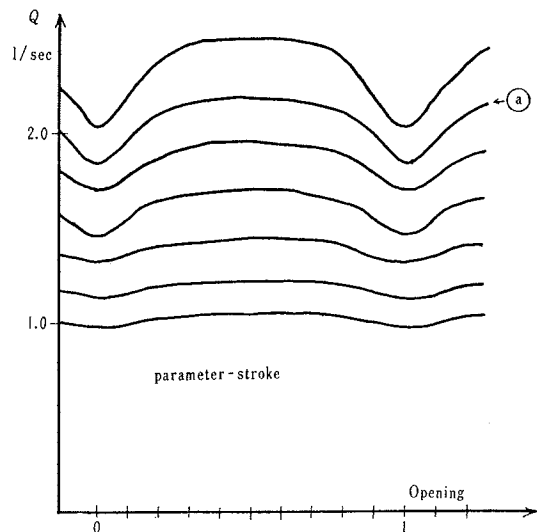

to oscillograph

Fig. 9 Discharge characteristic for different stroke and opening.

arm is connected and each hole gives a different length of stroke. The holes (5), denoted A, B, C, D, E, and F, can produce six different angles of revolution of the main valve to control the variation ratio of the discharge through the section of the flow, i.e., the gate ratio.

In order to determine the expression of the gate ratio $\Psi(t)$, a discharge under the hydrostatic head was measured for the variable angle of revolution of the pulley (3) and the expression of $\Psi(t)$ was estimated from the following relationship.

$$Q = \alpha \Psi(t) \sqrt{2gH_0} = \text{const} \times \Psi(t)$$

This relationship shows that the expressions of $\Psi(t)$ and Q are similar. The results are shown in **Fig. 9**.

Measurements of the water-hammer wave were conducted by using a strainmeter and an oscillograph. **Fig. 10** shows the profiles of the water-hammer wave for the boundary conditions of the valve shown in **Fig. 9 (a)**. A theoretical solution, based on Eq. (33), is also plotted for the purpose of comparison. The amplitude of this water-hammer wave varies according to the period of the opening and closure movement of the valve, and the relationship between the period and the amplitude is shown in **Fig. 11**. This figure shows that the water-hammer wave, without diverging due to periodic opening and closure of the valve, merely attains a maximum at a certain valve of frequency. The figure also plots the theoretical values derived from Eq. (33) as well as the Rich's theory. Comparing these plottings it is understood that our theory which satisfies the boundary conditions of the valve gives a better agreement with the experimental results.

4. CONCLUSIONS

The results of our theoretical and experimental study on the water-hammer pressure phenomenon have been briefly discussed, particularly dealing with the case where the valve is periodically opened and closed. It has been demonstrated that our theory and experiment agrees favorably.

Hence it follows that, when the

Fig. 10 Comparisons between theoretical and experimental water-hammer waves.
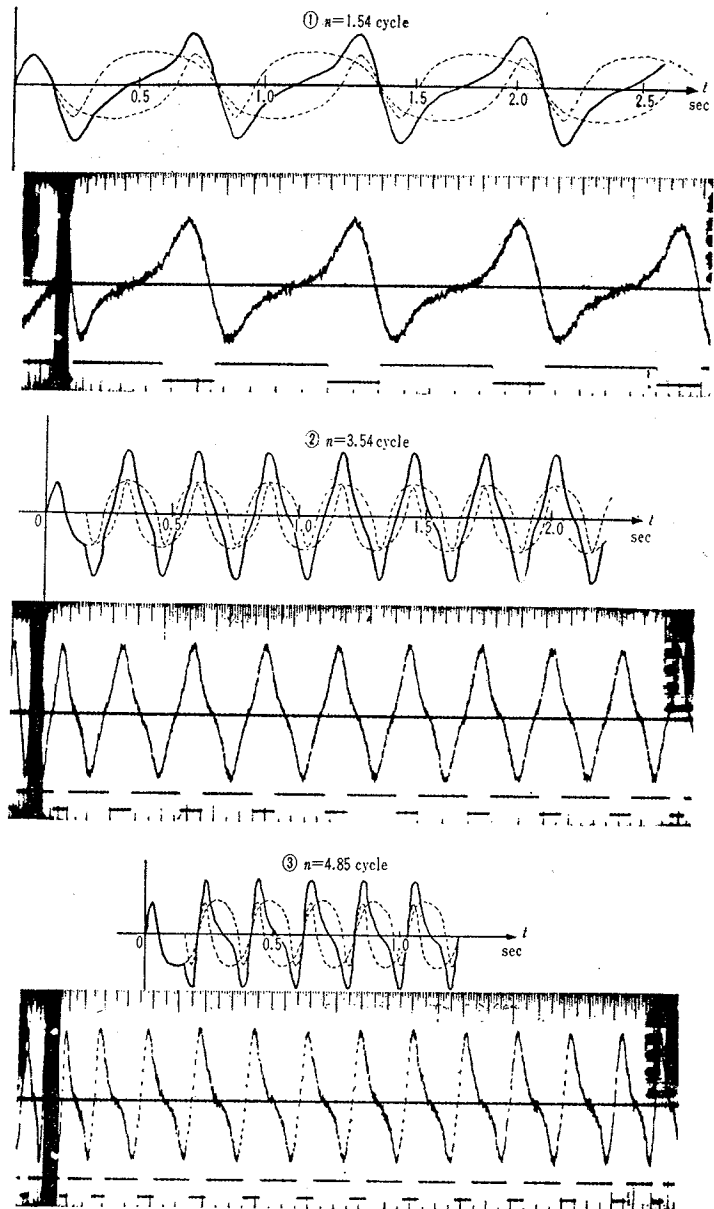
opening and closure movement of the valve located at the end of the pipe is relatively small and as a result the cross-sectional area of the flow varies by as much as $\varphi(z)$, the average flow velocity accordingly varies by $u = \dfrac{1}{1+\rho}$ $\varphi(z)$. It is evident that this variation of the flow velocity gives rise to a water-hammer pressure $h = -2\rho u = -\dfrac{2\rho}{1+\rho}\varphi(z)$ as would be shown also by the Joukowsky formula. The water-hammer pressure thus produced propagates toward upstream and is reflected at the position of the reservoir. If a total reflection occurs, the symbol is reversed and as a result the water-hammer pressure $\dfrac{2\rho}{1+\rho}\varphi(z)$ travels backward from the reservoir toward the valve. The wave then reaches the valve and is reflected there again; but this time there occurs a partial, instead of total, reflection depending on the value of ρ. It is evident from our treatment that this
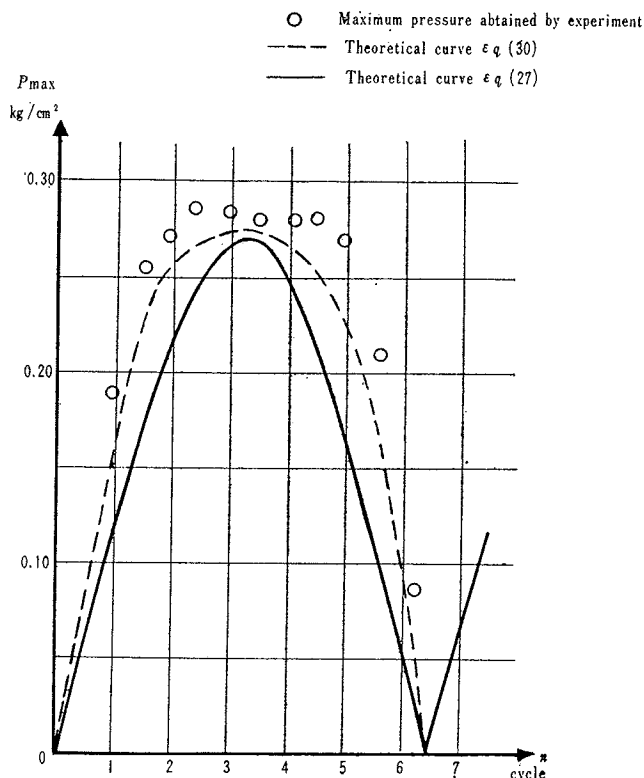

Fig. 11 Comparison of the maximum pressure between theory and experiment.

O   Maximum pressure abtained by experiment
— — —   Theoretical curve εq (30)
————   Theoretical curve εq (27)

coefficient of reflection is expressed by $\dfrac{\rho-1}{\rho+1}$. If the valve of ρ is approximate to unity, the term $\dfrac{\rho-1}{\rho+1}$ is nearly equal to 0, i.e., the pressure wave is barely reflected at the valve but virtually absorbed there. Our experimental results seem to verify this assumption. It also follows from the preceding discussion that this type of pressure can not be reduced unstable at a certain period of the opening and closure movement of the valve.

However, a further study is necessary for such cases where the opening and closure movement of the valve is so large that the resulting water-hammer pressure is also large as compared with the hydrostatic head, and where the mass and the elastic and electric supporting conditions of the valve must be taken into account.

ACKNOWLEDGEMENTS

References

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