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ON THE GENERALIZATION OF CERRUTI'S PROBLEM IN AN ELASTIC HALF-SPACE

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A singular boundary-value problem of an elastic half-space subjected to a force vector at one point of the surface is solved. The force vector has three components which are two tangential and one normal forces to the surface. Solutions to the problem are expressed in orthogonal curvilinear coordinates and are applied to rectangular Cartesian, cylindrical and spherical coordinates, as examples of the orthogonal curvilinear coordinates. The expressions for displacement and stress components are demonstrated in these coordinate systems. They are coincident with the solutions of Cerruti's and Boussinesq's problems when the three components of the force vector are specialized.

Key Words : *elasticity, singular problem, elastic half-space, Cerruti's problem, Boussinesq's problem*

1. INTRODUCTION

As some of three-dimensional problems of elasticity, there are singular boundary-value problems for an infinite elastic medium and an elastic half-space. Kelvin's, Cerruti's and Boussinesq's problems[1),2)] are well known. Kelvin's problem is one where one point in the infinite elastic medium is subjected to a concentrated force. Cerruti's and Boussinesq's problems are ones where one point at the surface of the elastic half-space is subjected to a tangential force and a normal force, respectively. The problem where one interior point of the elastic half-space is subjected to a vertical or a horizontal force has been solved by Mindlin[3)]. His solution is applied to some boundary-value problems for finite solids by integrating the solution.

For Cerruti's problem among these problems, Saada[1)] has briefly stated the process of induction which used the superposition of the Galerkin vectors. On the other hand, Lur'e[2)] has stated the process of induction which used potentials of simple layers with densities equal to given forces. Their methods of solution seem, however, to slightly lack the details and the simplicity of the analysis as far as they are concerned with methods of solution to boundary-value problems. Judging from their methods, Cerruti's problem seems to be a little more complicated than Kelvin's or Boussinesq's problem.

This paper is concerned with a method of solution to a singular boundary-value problem where one point at the surface of the elastic half-space is subjected to a force vector with three components, i.e., two tangential and one normal forces. The problem reduces to Boussinesq's problem when two tangential forces are neglected and to Cerruti's problem when one normal and one tangential forces are neglected. Therefore, the problem is considered to be the generalization of Cerruti's or Boussinesq's problem. Although the solutions of Cerruti's and Boussinesq's problems have been represented in specific coordinate systems so far, the solutions in this paper are represented in the expressions for a displacement vector and a stress tensor which are applicable to arbitrary coordinate systems belonging to orthogonal curvilinear coordinates. From the viewpoint of practice, the solutions are applied to rectangular Cartesian, cylindrical and spherical coordinates and demonstrate the concrete expressions for displacement and stress components in these coordinate systems.

2. BASIC AND SINGULAR SOLUTIONS

We consider a singular boundary-value problem where the surface of an elastic half-space is subjected to a force vector S at one point, as shown in **Fig.1**. In this case, the force vector S is given in

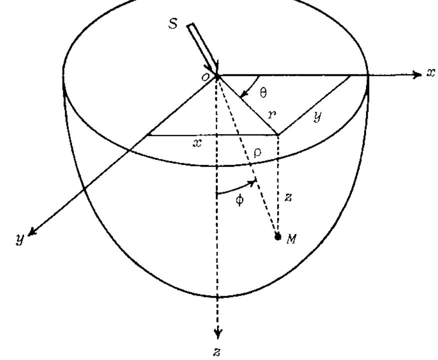

Fig.1 Coordinate system of elastic half-space.

the form

$$\boldsymbol{S}=S_x\boldsymbol{i}+S_y\boldsymbol{j}+S_z\boldsymbol{k}=[S_x,\ S_y,\ S_z] \tag{1}$$

in which $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ denote unit vectors in rectangular Cartesian coordinates.

(1) The Generalized Boussinesq Solution

Using the generalized Boussinesq solution[4] for a three-dimensional elasticity solution, it is expressed in orthogonal curvilinear coordinates as

$$2G\boldsymbol{u}=\mathrm{grad}(\lambda_0+\boldsymbol{r}\cdot\boldsymbol{\lambda})-4(1-\nu)\boldsymbol{\lambda}+2\mathrm{rot}\boldsymbol{\vartheta} \tag{2}$$

in which

$$\nabla^2\lambda_0=0,\quad \nabla^2\boldsymbol{\lambda}=\boldsymbol{0},\quad \nabla^2\boldsymbol{\vartheta}=\boldsymbol{0} \tag{3a-c}$$

and G and ν denote the shear modulus and Poisson's ratio, respectively, and $\boldsymbol{u}$ and $\boldsymbol{r}$ denote a displacement vector and a position vector, respectively. If we let $\boldsymbol{t}_n$ denote a stress vector, it is expressed as[2]

$$\boldsymbol{t}_n=2G\left[\frac{\nu}{1-2\nu}\boldsymbol{n}\mathrm{div}\boldsymbol{u}+(\boldsymbol{n}\cdot\mathrm{grad})\boldsymbol{u}+\frac{1}{2}(\boldsymbol{n}\times\mathrm{rot}\boldsymbol{u})\right] \tag{4}$$

in which $\boldsymbol{n}$ denotes a unit normal vector. Furthermore, if we let σ_{nl} denote a stress tensor, it is expressed as

$$\sigma_{nl}=\boldsymbol{t}_n\cdot\boldsymbol{l} \tag{5}$$

in which $\boldsymbol{l}$ denotes a unit tangential vector or a unit normal vector. From Eqs.(3a-c), the particular solutions to scalar and vector potentials are obtained as

$$\lambda_0=B\log(\rho+z)+\frac{\boldsymbol{D}\cdot\boldsymbol{r}}{\rho+z} \tag{6}$$

$$\boldsymbol{\lambda}=\frac{\boldsymbol{P}}{8\pi(1-\nu)\rho} \tag{7}$$

$$\boldsymbol{\vartheta}=\boldsymbol{C}\log(\rho+z) \tag{8}$$

in which B, $\boldsymbol{D}$, $\boldsymbol{P}$ and $\boldsymbol{C}$ denote an unknown constant scalar and unknown constant vectors, respectively, that is,

$$\boldsymbol{D}=D_1\boldsymbol{i}+D_2\boldsymbol{j}=[D_1,\ D_2,\ 0] \tag{9}$$

$$\boldsymbol{P}=P_1\boldsymbol{i}+P_2\boldsymbol{j}+P_3\boldsymbol{k}=[P_1,\ P_2,\ P_3] \tag{10}$$

$$\boldsymbol{C}=C_1\boldsymbol{i}+C_2\boldsymbol{j}=[C_1,\ C_2,\ 0] \tag{11}$$

If we substitute Eqs.(6), (7) and (8) into Eq.(2), we obtain an expression for the displacement vector. Furthermore, from the expression, Eqs.(4) and (5), we obtain the stress vector and the stress tensor. To make the explanation brief, we express the displacement vector, the stress vector and the stress tensor in the sum of four solutions. Then, we have

$$2G\boldsymbol{u}=2G(\boldsymbol{u}^{(1)}+\boldsymbol{u}^{(2)}+\boldsymbol{u}^{(3)}+\boldsymbol{u}^{(4)}) \tag{12a}$$

$$\boldsymbol{t}_n=\boldsymbol{t}_n^{(1)}+\boldsymbol{t}_n^{(2)}+\boldsymbol{t}_n^{(3)}+\boldsymbol{t}_n^{(4)} \tag{12b}$$

$$\sigma_{nl}=\sigma_{nl}^{(1)}+\sigma_{nl}^{(2)}+\sigma_{nl}^{(3)}+\sigma_{nl}^{(4)} \tag{12c}$$

To induce the four solutions as stated below, we use the following formulae[5] :

$$\mathrm{grad}(\varphi\phi)=\phi\,\mathrm{grad}\varphi+\varphi\,\mathrm{grad}\phi \tag{13a}$$

$$\mathrm{div}(\varphi\boldsymbol{A})=(\mathrm{grad}\varphi)\cdot\boldsymbol{A}+\varphi\,\mathrm{div}\boldsymbol{A} \tag{13b}$$

$$\mathrm{rot}(\varphi\boldsymbol{A})=(\mathrm{grad}\varphi)\times\boldsymbol{A}+\varphi\,\mathrm{rot}\boldsymbol{A} \tag{13c}$$

$$\mathrm{div}(\boldsymbol{A}\times\boldsymbol{B})=\boldsymbol{B}\cdot\mathrm{rot}\boldsymbol{A}-\boldsymbol{A}\cdot\mathrm{rot}\boldsymbol{B} \tag{13d}$$

$$\mathrm{rot}(\boldsymbol{A}\times\boldsymbol{B})=(\boldsymbol{B}\cdot\mathrm{grad})\boldsymbol{A}-(\boldsymbol{A}\cdot\mathrm{grad})\boldsymbol{B}+\boldsymbol{A}\mathrm{div}\boldsymbol{B}-\boldsymbol{B}\mathrm{div}\boldsymbol{A} \tag{13e}$$

$$\mathrm{grad}(\boldsymbol{A}\cdot\boldsymbol{B})=(\boldsymbol{B}\cdot\mathrm{grad})\boldsymbol{A}+(\boldsymbol{A}\cdot\mathrm{grad})\boldsymbol{B}+\boldsymbol{B}\times\mathrm{rot}\boldsymbol{A}+\boldsymbol{A}\times\mathrm{rot}\boldsymbol{B} \tag{13f}$$

$$(\boldsymbol{A}\cdot\mathrm{grad})\boldsymbol{r}=\boldsymbol{A},\quad \mathrm{rot}\boldsymbol{r}=\boldsymbol{0} \tag{13g,h}$$

$$\mathrm{div}\boldsymbol{r}=3,\quad \mathrm{div\,grad}\varphi=\nabla^2\varphi \tag{13i, j}$$

$$\mathrm{rot\,grad}\varphi=\boldsymbol{0},\quad \mathrm{div\,rot}\boldsymbol{A}=0 \tag{13k, l}$$

in which φ, ϕ and $\boldsymbol{A}$, $\boldsymbol{B}$ are scalar fields and vector fields, respectively.

(2) Solution to Double Line of Center of Dilatation

The first term of Eq.(12a) is a solution corresponding to a double line of center of dilatation which is concerned with a double force along the x and y axes with moments about the y and x axes and is as follows :

$$2G\boldsymbol{u}^{(1)}=\mathrm{grad}\frac{\boldsymbol{D}\cdot\boldsymbol{r}}{\rho+z} \tag{14a}$$

Substituting Eq.(14a) into Eq.(4) and performing very complicated vector operations, the stress vector is obtained in the form

$$\boldsymbol{t}_n^{(1)}=-\frac{1}{\rho(\rho+z)^2}\Big\langle\boldsymbol{D}(\boldsymbol{n}\cdot\boldsymbol{r}+\rho\boldsymbol{n}\cdot\boldsymbol{k})+(\boldsymbol{n}\cdot\boldsymbol{D})(\boldsymbol{r}+\rho\boldsymbol{k})+(\boldsymbol{D}\cdot\boldsymbol{r})\boldsymbol{n}-\frac{\boldsymbol{D}\cdot\boldsymbol{r}}{\rho^2(\rho+z)}\{\boldsymbol{r}[(3\rho+z)(\boldsymbol{n}\cdot\boldsymbol{r})+2\rho^2(\boldsymbol{n}\cdot\boldsymbol{k})]+2\rho^2\boldsymbol{k}(\boldsymbol{n}\cdot\boldsymbol{r}+\rho\boldsymbol{n}\cdot\boldsymbol{k})\}\Big\rangle \tag{14b}$$

Then, from Eq.(5), we obtain

$$\sigma_{nl}^{(1)}=-\frac{1}{\rho(\rho+z)^2}\Big\langle (\boldsymbol{D}\cdot\boldsymbol{r})\delta_{nl}+(\boldsymbol{D}\cdot\boldsymbol{l})(\boldsymbol{n}\cdot\boldsymbol{r}+\rho\boldsymbol{n}\cdot\boldsymbol{k})+(\boldsymbol{n}\cdot\boldsymbol{D})(\boldsymbol{r}\cdot\boldsymbol{l}+\rho\boldsymbol{k}\cdot\boldsymbol{l})-\frac{\boldsymbol{D}\cdot\boldsymbol{r}}{\rho^2(\rho+z)}\{(\boldsymbol{r}\cdot\boldsymbol{l})[(3\rho+z)(\boldsymbol{n}\cdot\boldsymbol{r})+2\rho^2(\boldsymbol{n}\cdot\boldsymbol{k})]+2\rho^2(\boldsymbol{k}\cdot\boldsymbol{l})(\boldsymbol{n}\cdot\boldsymbol{r}+\rho\boldsymbol{n}\cdot\boldsymbol{k})\}\Big\rangle \tag{14c}$$

in which δ_{nl} denotes Kronecker's delta :

$$\delta_{nl}=\begin{cases}1 & \text{for} \quad l=n\\ 0 & \text{for} \quad l\neq n\end{cases} \tag{15}$$

(3) Solution of Kelvin's Problem

The second term of Eq.(12a) is the solution of Kelvin's problem where one point in an infinite elastic medium is subjected to a force vector $\boldsymbol{P}$ and is as follows :

$$2G\boldsymbol{u}^{(2)}=-\frac{1}{8\pi(1-\nu)\rho}\left[(3-4\nu)\boldsymbol{P}+\frac{(\boldsymbol{P}\cdot\boldsymbol{r})\boldsymbol{r}}{\rho^2}\right] \tag{16a}$$

Substituting Eq.(16a) into Eq.(4), the stress vector is obtained in the form

$$\boldsymbol{t}_n^{(2)}=-\frac{1-2\nu}{8\pi(1-\nu)\rho^3}\left[(\boldsymbol{P}\cdot\boldsymbol{r})\boldsymbol{n}-(\boldsymbol{n}\cdot\boldsymbol{r})\boldsymbol{P}-(\boldsymbol{n}\cdot\boldsymbol{P})\boldsymbol{r}-\frac{3}{(1-2\nu)\rho^2}(\boldsymbol{P}\cdot\boldsymbol{r})\cdot(\boldsymbol{n}\cdot\boldsymbol{r})\boldsymbol{r}\right] \tag{16b}$$

Then, from Eq.(5), we obtain

$$\sigma_{nl}^{(2)}=-\frac{1-2\nu}{8\pi(1-\nu)\rho^3}\left[(\boldsymbol{P}\cdot\boldsymbol{r})\delta_{nl}-(\boldsymbol{n}\cdot\boldsymbol{r})(\boldsymbol{P}\cdot\boldsymbol{l})-(\boldsymbol{n}\cdot\boldsymbol{P})(\boldsymbol{r}\cdot\boldsymbol{l})-\frac{3}{(1-2\nu)\rho^2}(\boldsymbol{P}\cdot\boldsymbol{r})\cdot(\boldsymbol{n}\cdot\boldsymbol{r})(\boldsymbol{r}\cdot\boldsymbol{l})\right] \tag{16c}$$

(4) Solution to Line of Center of Rotation

The third term of Eq.(12a) is a solution corresponding to a line of center of rotation which is concerned with a double force along the y and x axes with moments about the x and y axes and is as follows :

$$2G\boldsymbol{u}^{(3)}=2\,\mathrm{rot}\,[\boldsymbol{C}\log(\rho+z)] \tag{17a}$$

Substituting Eq.(17a) into Eq.(4), the stress vector is obtained in the form

$$\boldsymbol{t}_n^{(3)}=\frac{2}{\rho^2}\Big\langle(\boldsymbol{C}\times\boldsymbol{k})\frac{(\boldsymbol{n}\cdot\boldsymbol{r})}{\rho}-\frac{\boldsymbol{k}}{\rho+z}\Big[\boldsymbol{r}\cdot(\boldsymbol{C}\times\boldsymbol{k})\Big(\frac{\boldsymbol{n}\cdot\boldsymbol{r}}{\rho}+\frac{\boldsymbol{n}\cdot\boldsymbol{r}+\rho\boldsymbol{n}\cdot\boldsymbol{k}}{\rho+z}\Big)-\rho\boldsymbol{n}\cdot(\boldsymbol{C}\times\boldsymbol{k})\Big]-\frac{1}{2(\rho+z)}\Big\{(\boldsymbol{r}\cdot\boldsymbol{C})\cdot\Big[\frac{\boldsymbol{n}\times\boldsymbol{r}}{\rho}+\frac{1}{\rho+z}(\boldsymbol{n}\times\boldsymbol{r}+\rho\boldsymbol{n}\times\boldsymbol{k})\Big]-\rho\boldsymbol{n}\times\boldsymbol{C}\Big\}\Big\rangle \tag{17b}$$

Then, from Eq.(5), we obtain

$$\sigma_{nl}^{(3)}=\frac{2}{\rho^2}\Big\{(\boldsymbol{C}\times\boldsymbol{k})\cdot\boldsymbol{l}\frac{\boldsymbol{n}\cdot\boldsymbol{r}}{\rho}-\frac{\boldsymbol{k}\cdot\boldsymbol{l}}{\rho+z}\cdot\Big[\boldsymbol{r}\cdot(\boldsymbol{C}\times\boldsymbol{k})\Big(\frac{\boldsymbol{n}\cdot\boldsymbol{r}}{\rho}+\frac{\boldsymbol{n}\cdot\boldsymbol{r}+\rho\boldsymbol{n}\cdot\boldsymbol{k}}{\rho+z}\Big)-\rho\boldsymbol{n}\cdot(\boldsymbol{C}\times\boldsymbol{k})\Big]+\frac{\boldsymbol{n}\times\boldsymbol{l}}{2(\rho+z)}\cdot\Big[(\boldsymbol{r}\cdot\boldsymbol{C})\Big(\frac{\boldsymbol{r}}{\rho}+\frac{\boldsymbol{r}+\rho\boldsymbol{k}}{\rho+z}\Big)-\rho\boldsymbol{C}\Big]\Big\} \tag{17c}$$

(5) Solution to Line of Center of Dilatation

The fourth term of Eq.(12a) is a solution corresponding to a line of center of dilatation which is concerned with a single force along the z axis without moments and is as follows :

$$2G\boldsymbol{u}^{(4)}=B\,\mathrm{grad}\log(\rho+z) \tag{18a}$$

Substituting Eq.(18a) into Eq.(4), the stress vector is obtained in the form

$$\boldsymbol{t}_n^{(4)}=\frac{B}{\rho(\rho+z)}\Big\{\boldsymbol{n}-\frac{\boldsymbol{k}}{\rho+z}(\boldsymbol{n}\cdot\boldsymbol{r}+\rho\boldsymbol{n}\cdot\boldsymbol{k})-\frac{\boldsymbol{r}}{\rho^2(\rho+z)}[(2\rho+z)\boldsymbol{n}\cdot\boldsymbol{r}+\rho^2\boldsymbol{n}\cdot\boldsymbol{k}]\Big\} \tag{18b}$$

Then, from Eq.(5), we obtain

$$\sigma_{nl}^{(4)}=\frac{B}{\rho(\rho+z)}\Big\{\delta_{nl}-\frac{\boldsymbol{k}\cdot\boldsymbol{l}}{\rho+z}(\boldsymbol{n}\cdot\boldsymbol{r}+\rho\boldsymbol{n}\cdot\boldsymbol{k})-\frac{\boldsymbol{r}\cdot\boldsymbol{l}}{\rho^2(\rho+z)}[(2\rho+z)\boldsymbol{n}\cdot\boldsymbol{r}+\rho^2\boldsymbol{n}\cdot\boldsymbol{k}]\Big\} \tag{18c}$$

3. BOUNDARY AND EQUILIBRIUM CONDITIONS

Boundary conditions at the surface except the origin of the elastic half-space become

$$\text{at } z=0,\quad \sigma_{zz}=0,\quad \sigma_{zx}=0,\quad \sigma_{zy}=0 \tag{19a-c}$$

in which σ_{zz}, σ_{zx} and σ_{zy} denote stress components, and equilibrium conditions become

$$\boldsymbol{R}+\boldsymbol{S}=\boldsymbol{0},\ \boldsymbol{M}=\boldsymbol{0} \tag{20a, b}$$

in which

$$\boldsymbol{R}=\iint_{\Sigma}\boldsymbol{t}_n do=[R_x, R_y, R_z] \tag{21a}$$

$$M=\int\int_{\Sigma}(\boldsymbol{r}\times \boldsymbol{t}_n)do=[M_x, M_y, M_z] \tag{21b}$$

and Σ and do denote the surface of a hemisphere with radius ρ and the infinitesimal area on the surface, respectively. Introducing spherical coordinates (ρ, ϕ, θ), boundary conditions (19a-c) become

$$(\sigma_{\phi\phi})_{\phi=\pi/2}=0, \quad (\sigma_{\rho\phi})_{\phi=\pi/2}=0 \tag{22a, b}$$

$$(\sigma_{\phi\theta})_{\phi=\pi/2}=0 \tag{22c}$$

and resultant force (21a) and resultant moment (21b) are expressed as[6]

$$R_x=\rho^2\int_0^{2\pi}\int_0^{\pi/2}[(\sigma_{\rho\rho}\sin\phi+\sigma_{\rho\phi}\cos\phi)\cos\theta - \sigma_{\rho\theta}\sin\theta]\sin\phi d\phi d\theta \tag{23a}$$

$$R_y=\rho^2\int_0^{2\pi}\int_0^{\pi/2}[(\sigma_{\rho\rho}\sin\phi+\sigma_{\rho\phi}\cos\phi)\sin\theta + \sigma_{\rho\theta}\cos\theta]\sin\phi d\phi d\theta \tag{23b}$$

$$R_z=\rho^2\int_0^{2\pi}\int_0^{\pi/2}(\sigma_{\rho\rho}\cos\phi-\sigma_{\rho\phi}\sin\phi)\cdot\sin\phi d\phi d\theta \tag{23c}$$

$$M_x=-\rho^3\int_0^{2\pi}\int_0^{\pi/2}(\sigma_{\rho\phi}\sin\theta+\sigma_{\rho\theta}\cos\phi\cos\theta)\cdot\sin\phi d\phi d\theta \tag{24a}$$

$$M_y=\rho^3\int_0^{2\pi}\int_0^{\pi/2}(\sigma_{\rho\phi}\cos\theta-\sigma_{\rho\theta}\cos\phi\sin\theta)\cdot\sin\phi d\phi d\theta \tag{24b}$$

$$M_z=\rho^3\int_0^{2\pi}\int_0^{\pi/2}\sigma_{\rho\theta}\sin\phi\sin\phi d\phi d\theta \tag{24c}$$

As seen in Eqs.(22a) to (24c), the stress components in the spherical coordinates are needed to satisfy the boundary and equilibrium conditions. They are directly written from Eqs.(14c), (16c), (17c) and (18c) by making use of the following relationships :

$$\boldsymbol{r}=[\rho, 0, 0], \quad z=\rho\cos\phi \tag{25a, b}$$

$$\boldsymbol{i}=[\sin\phi\cos\theta, \cos\phi\cos\theta, -\sin\theta] \tag{25c}$$

$$\boldsymbol{j}=[\sin\phi\sin\theta, \cos\phi\sin\theta, \cos\theta] \tag{25d}$$

$$\boldsymbol{k}=[\cos\phi, -\sin\phi, 0] \tag{25e}$$

From Eq.(14c), we obtain

$$\sigma_{\rho\rho}^{(1)}=0;$$

$$\sigma_{\phi\phi}^{(1)}=\frac{1-\cos\phi}{\rho^2(1+\cos\phi)\sin\phi}(D_1\cos\theta+D_2\sin\theta);$$

$$\sigma_{\rho\phi}^{(1)}=-\frac{1}{\rho^2(1+\cos\phi)}(D_1\cos\theta+D_2\sin\theta);$$

$$\sigma_{\phi\theta}^{(1)}=-\frac{1-\cos\phi}{\rho^2(1+\cos\phi)\sin\phi}(D_1\sin\theta-D_2\cos\theta);$$

$$\sigma_{\rho\theta}^{(1)}=\frac{1}{\rho^2(1+\cos\phi)}(D_1\sin\theta-D_2\cos\theta) \tag{26a-e}$$

From Eq.(16c), we obtain

$$\sigma_{\rho\rho}^{(2)}=\frac{2-\nu}{4\pi(1-\nu)\rho^2}[(P_1\cos\theta+P_2\sin\theta)\sin\phi + P_3\cos\phi];$$

$$\sigma_{\phi\phi}^{(2)}=-\frac{1-2\nu}{8\pi(1-\nu)\rho^2}[(P_1\cos\theta+P_2\sin\theta)\sin\phi + P_3\cos\phi];$$

$$\sigma_{\rho\phi}^{(2)}=\frac{1-2\nu}{8\pi(1-\nu)\rho^2}[(P_1\cos\theta+P_2\sin\theta)\cos\phi - P_3\sin\phi];$$

$$\sigma_{\phi\theta}^{(2)}=0;$$

$$\sigma_{\rho\theta}^{(2)}=-\frac{1-2\nu}{8\pi(1-\nu)\rho^2}(P_1\sin\theta-P_2\cos\theta) \tag{27a-e}$$

From Eq.(17c), we obtain

$$\sigma_{\rho\rho}^{(3)}=-\frac{2\sin\phi}{\rho^2(1+\cos\phi)}(C_1\sin\theta-C_2\cos\theta);$$

$$\sigma_{\phi\phi}^{(3)}=\frac{2\sin\phi}{\rho^2(1+\cos\phi)}(C_1\sin\theta-C_2\cos\theta);$$

$$\sigma_{\rho\phi}^{(3)}=-\frac{1}{\rho^2}\left(2-\frac{1}{1+\cos\phi}\right)(C_1\sin\theta-C_2\cos\theta);$$

$$\sigma_{\phi\theta}^{(3)}=\frac{\sin\phi}{\rho^2(1+\cos\phi)}(C_1\cos\theta+C_2\sin\theta);$$

$$\sigma_{\rho\theta}^{(3)}=-\frac{1}{\rho^2}\left(2-\frac{1}{1+\cos\phi}\right)(C_1\cos\theta+C_2\sin\theta); \tag{28a-e}$$

From Eq.(18c), we obtain

$$\sigma_{\rho\rho}^{(4)}=-\frac{B}{\rho^2}, \quad \sigma_{\phi\phi}^{(4)}=\frac{B\cos\phi}{\rho^2(1+\cos\phi)};$$

$$\sigma_{\rho\phi}^{(4)}=\frac{B\sin\phi}{\rho^2(1+\cos\phi)}, \quad \sigma_{\phi\theta}^{(4)}=0;$$

$$\sigma_{\rho\theta}^{(4)}=0 \tag{29a-e}$$

The required stress components are each sum of four solutions with superscripts (1), (2), (3) and (4) and are expressed in the form

$$\sigma_{\rho\rho}=\sigma_{\rho\rho}^{(1)}+\sigma_{\rho\rho}^{(2)}+\sigma_{\rho\rho}^{(3)}+\sigma_{\rho\rho}^{(4)}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \tag{30}$$

$$\sigma_{\rho\theta}=\sigma_{\rho\theta}^{(1)}+\sigma_{\rho\theta}^{(2)}+\sigma_{\rho\theta}^{(3)}+\sigma_{\rho\theta}^{(4)}$$

From Eq.(30), stress components $\sigma_{\phi\phi}$, $\sigma_{\rho\phi}$ and $\sigma_{\phi\theta}$ at $\phi=\pi/2$ become

$$(\sigma_{\phi\phi})_{\phi=\pi/2}=\frac{\cos\theta}{\rho^2}\left[D_1-\frac{1-2\nu}{8\pi(1-\nu)}P_1-2C_2\right] + \frac{\sin\theta}{\rho^2}\left[D_2-\frac{1-2\nu}{8\pi(1-\nu)}P_2+2C_1\right] \tag{31a}$$

$$(\sigma_{\rho\phi})_{\phi=\pi/2}=-\frac{\cos\theta}{\rho^2}(D_1-C_2)-\frac{\sin\theta}{\rho^2}(D_2+C_1)$$

$$-\frac{1}{\rho^2}\left[\frac{1-2\nu}{8\pi(1-\nu)}P_3-B\right] \tag{31b}$$

$$(\sigma_{\phi\theta})_{\phi=\pi/2}=-\frac{\sin\theta}{\rho^2}(D_1-C_2)+\frac{\cos\theta}{\rho^2}\cdot(D_2+C_1) \tag{31c}$$

Substituting $\sigma_{\rho\rho}$, $\sigma_{\rho\phi}$ and $\sigma_{\rho\theta}$ in Eq.(30) into Eqs.(23a-c) and paying attention to

$$\int_0^{\pi/2}\frac{\sin\phi}{1+\cos\phi}d\phi=\log 2 \tag{32a}$$

$$\int_0^{\pi/2}\frac{\cos\phi\sin\phi}{1+\cos\phi}d\phi=1-\log 2 \tag{32b}$$

we obtain the resultant forces as

$$R_x=-\pi\left(D_1-\frac{P_1}{2\pi}-3C_2\right) \tag{33a}$$

$$R_y=-\pi\left(D_2-\frac{P_2}{2\pi}+3C_1\right) \tag{33b}$$

$$R_z=\frac{P_3}{2}-2\pi B \tag{33c}$$

Furthermore, substituting $\sigma_{\rho\phi}$ and $\sigma_{\rho\theta}$ in Eq.(30) into Eqs.(24a-c), we obtain the resultant moments as

$$M_x=\pi\rho\left[D_2-\frac{P_2(1-2\nu)}{8\pi(1-\nu)}+2C_1\right] \tag{34a}$$

$$M_y=-\pi\rho\left[D_1-\frac{P_1(1-2\nu)}{8\pi(1-\nu)}-2C_2\right] \tag{34b}$$

$$M_z=0 \tag{34c}$$

Substituting Eqs.(31a-c) into boundary conditions (22a-c), we obtain

$$D_1-\frac{1-2\nu}{8\pi(1-\nu)}P_1-2C_2=0 \tag{35a}$$

$$D_2-\frac{1-2\nu}{8\pi(1-\nu)}P_2+2C_1=0 \tag{35b}$$

$$D_1-C_2=0,\quad D_2+C_1=0 \tag{35c, d}$$

$$\frac{1-2\nu}{8\pi(1-\nu)}P_3-B=0 \tag{35e}$$

Next, substituting Eqs.(33a-c) into equilibrium condition (20a), we obtain

$$-\pi\left(D_1-\frac{P_1}{2\pi}-3C_2\right)+S_x=0 \tag{36a}$$

$$-\pi\left(D_2-\frac{P_2}{2\pi}+3C_1\right)+S_y=0 \tag{36b}$$

$$\frac{P_3}{2}-2\pi B+S_z=0 \tag{36c}$$

Last, substituting Eqs.(34a-c) into equilibrium condition (20b), we obtain

$$D_2-\frac{1-2\nu}{8\pi(1-\nu)}P_2+2C_1=0 \tag{37a}$$

$$D_1-\frac{1-2\nu}{8\pi(1-\nu)}P_1-2C_2=0 \tag{37b}$$

If Eqs.(35a, b) are satisfied, Eqs.(37a, b) are automatically satisfied. Therefore, solving the system of linear algebraic equations (35a-e) and (36a-c) with D_1, D_2, P_1, P_2, P_3, C_1, C_2 and B, the unknown constants are determined as

$$\begin{aligned}&D_1=\frac{1-2\nu}{2\pi}S_x,\quad D_2=\frac{1-2\nu}{2\pi}S_y;\\&P_1=-4(1-\nu)S_x,\quad P_2=-4(1-\nu)S_y;\\&P_3=-4(1-\nu)S_z,\quad C_1=-\frac{1-2\nu}{2\pi}S_y;\\&C_2=\frac{1-2\nu}{2\pi}S_x,\quad B=-\frac{1-2\nu}{2\pi}S_z\end{aligned} \tag{38a-h}$$

That is

$$\boldsymbol{D}=\frac{1-2\nu}{2\pi}[S_x, S_y, 0] \tag{39a}$$

$$\boldsymbol{P}=-4(1-\nu)[S_x, S_y, S_z] \tag{39b}$$

$$C=\frac{1-2\nu}{2\pi}[-S_y, S_x, 0] \tag{39c}$$

$$B=-\frac{1-2\nu}{2\pi}S_z \tag{39d}$$

Thus, the present problem was completely solved.

4. SOLUTIONS OF THE GENERALIZED CERRUTI PROBLEM

If we substitute the known constants obtained in the foregoing chapter into Eqs.(14a, c), (16a, c), (17a, c) and (18a, c) and find the sum of four results, we obtain the solutions to the displacement vector and the stress tensor.

Substituting known constant vector (39a) into Eqs. (14a) and (14c), we obtain the following solution :

$$2G\boldsymbol{u}^{(1)}=\frac{1-2\nu}{2\pi}\left(\mathrm{grad}\frac{\boldsymbol{S}\cdot\boldsymbol{r}}{\rho+z}-\boldsymbol{S}\cdot\boldsymbol{k}\,\mathrm{grad}\frac{z}{\rho+z}\right) \tag{40a}$$

$$\begin{aligned}\sigma_{nl}^{(1)}=&-\frac{1-2\nu}{2\pi}\frac{1}{\rho(\rho+z)^2}\Big\langle \boldsymbol{S}\cdot\boldsymbol{r}\delta_{nl}+\boldsymbol{S}\cdot\boldsymbol{l}(\boldsymbol{n}\cdot\boldsymbol{r}\\&+\rho\boldsymbol{n}\cdot\boldsymbol{k})+\boldsymbol{S}\cdot\boldsymbol{n}(\boldsymbol{r}\cdot\boldsymbol{l}+\rho\boldsymbol{k}\cdot\boldsymbol{l})\\&-\frac{\boldsymbol{S}\cdot\boldsymbol{r}}{\rho^2(\rho+z)}\{\boldsymbol{r}\cdot\boldsymbol{l}[(3\rho+z)\boldsymbol{n}\cdot\boldsymbol{r}\\&+2\rho^2\boldsymbol{n}\cdot\boldsymbol{k}]+2\rho^2\boldsymbol{k}\cdot\boldsymbol{l}(\boldsymbol{n}\cdot\boldsymbol{r}+\rho\boldsymbol{n}\cdot\boldsymbol{k})\}\\&-(\boldsymbol{S}\cdot\boldsymbol{k})\Big\{\boldsymbol{k}\cdot\boldsymbol{r}\delta_{nl}+\frac{\boldsymbol{k}\cdot\boldsymbol{l}}{\rho+z}[(\rho-z)\boldsymbol{n}\cdot\boldsymbol{r}\\&+2\rho^2\boldsymbol{n}\cdot\boldsymbol{k}]+\frac{\boldsymbol{r}\cdot\boldsymbol{l}}{\rho+z}\Big[(\rho-z)\boldsymbol{k}\cdot\boldsymbol{n}\\&-\frac{\boldsymbol{k}\cdot\boldsymbol{r}}{\rho^2}(3\rho+z)(\boldsymbol{n}\cdot\boldsymbol{r})\Big]\Big\}\Big\rangle\end{aligned} \tag{40b}$$

Substituting known constant vector (39b) into Eqs.(16a) and (16c), we obtain the following solution :

$$2G\boldsymbol{u}^{(2)}=\frac{1}{2\pi\rho}\left[(3-4\nu)\boldsymbol{S}+\frac{(\boldsymbol{S}\cdot\boldsymbol{r})\boldsymbol{r}}{\rho^2}\right] \quad (41a)$$

$$\sigma_{nl}^{(2)}=\frac{1-2\nu}{2\pi\rho^3}\left[\boldsymbol{S}\cdot\boldsymbol{r}\delta_{nl}-\boldsymbol{S}\cdot\boldsymbol{l}(\boldsymbol{n}\cdot\boldsymbol{r})-\boldsymbol{S}\cdot\boldsymbol{n}(\boldsymbol{r}\cdot\boldsymbol{l})\right.$$
$$\left.-\frac{3\boldsymbol{S}\cdot\boldsymbol{r}}{(1-2\nu)\rho^2}(\boldsymbol{n}\cdot\boldsymbol{r})(\boldsymbol{r}\cdot\boldsymbol{l})\right] \quad (41b)$$

Substituting known constant vector (39c) into Eqs.(17a) and (17c), we obtain the following solution :

$$2G\boldsymbol{u}^{(3)}=\frac{1-2\nu}{\pi}(\boldsymbol{S}\times\boldsymbol{k})\times\mathrm{grad}\log(\rho+z) \quad (42a)$$

$$\sigma_{nl}^{(3)}=\frac{1-2\nu}{\pi\rho^2}\left\{(\boldsymbol{S}\cdot\boldsymbol{l})\frac{\boldsymbol{n}\cdot\boldsymbol{r}}{\rho}-\frac{\boldsymbol{k}\cdot\boldsymbol{l}}{\rho+z}\left[(\boldsymbol{S}\cdot\boldsymbol{r})\left(\frac{\boldsymbol{n}\cdot\boldsymbol{r}}{\rho}\right.\right.\right.$$
$$\left.\left.+\frac{\boldsymbol{n}\cdot\boldsymbol{r}+\rho\boldsymbol{n}\cdot\boldsymbol{k}}{\rho+z}\right)-\rho\boldsymbol{S}\cdot\boldsymbol{n}\right]-\frac{\boldsymbol{n}\times\boldsymbol{l}}{2(\rho+z)}$$
$$\cdot\left[(\boldsymbol{S}\times\boldsymbol{k})\cdot\boldsymbol{r}\left(\frac{\boldsymbol{r}}{\rho}+\frac{\boldsymbol{r}+\rho\boldsymbol{k}}{\rho+z}\right)-\rho(\boldsymbol{S}\times\boldsymbol{k})\right]$$
$$\left.-\frac{\rho\boldsymbol{S}\cdot\boldsymbol{k}}{(\rho+z)^2}(\boldsymbol{k}\cdot\boldsymbol{l})(\boldsymbol{n}\cdot\boldsymbol{r}+\rho\boldsymbol{n}\cdot\boldsymbol{k})\right\} \quad (42b)$$

Substituting known constant (39d) into Eqs.(18a) and (18c), we obtain the following solution :

$$2G\boldsymbol{u}^{(4)}=-\frac{1-2\nu}{2\pi}(\boldsymbol{S}\cdot\boldsymbol{k})\mathrm{grad}\log(\rho+z) \quad (43a)$$

$$\sigma_{nl}^{(4)}=-\frac{1-2\nu}{2\pi\rho(\rho+z)}\boldsymbol{S}\cdot\boldsymbol{k}\left\{\delta_{nl}-\frac{\boldsymbol{k}\cdot\boldsymbol{l}}{\rho+z}(\boldsymbol{n}\cdot\boldsymbol{r}\right.$$
$$+\rho\boldsymbol{n}\cdot\boldsymbol{k})-\frac{\boldsymbol{r}\cdot\boldsymbol{l}}{\rho^2(\rho+z)}[(2\rho+z)\boldsymbol{n}\cdot\boldsymbol{r}$$
$$\left.+\rho^2\boldsymbol{n}\cdot\boldsymbol{k}]\right\} \quad (43b)$$

As a result, substituting solutions (40a), (41a), (42a) and (43a) into Eq.(12a) and putting in order, the displacement vector is expressed in the form

$$2G\boldsymbol{u}=\frac{1-2\nu}{2\pi}\left\{\frac{\boldsymbol{S}}{\rho+z}+(\boldsymbol{S}\cdot\boldsymbol{r})\mathrm{grad}\frac{1}{\rho+z}\right.$$
$$+\frac{1}{\rho(1-2\nu)}\left[(3-4\nu)\boldsymbol{S}+\frac{(\boldsymbol{S}\cdot\boldsymbol{r})\boldsymbol{r}}{\rho^2}\right]$$
$$+2(\boldsymbol{S}\times\boldsymbol{k})\times\mathrm{grad}\log(\rho+z)$$
$$\left.-(\boldsymbol{S}\cdot\boldsymbol{k})\mathrm{grad}\left[\frac{z}{\rho+z}+\log(\rho+z)\right]\right\} \quad (44)$$

and substituting solutions (40b), (41b), (42b) and (43b) into Eq.(12c) and putting in order, the stress tensor is expressed in the form

$$\sigma_{nl}=\frac{1-2\nu}{2\pi\rho}\left\langle\frac{2\rho+z}{\rho^2(\rho+z)^2}\boldsymbol{S}\cdot\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{k})\delta_{nl}\right.$$
$$+\frac{\boldsymbol{S}\cdot\boldsymbol{l}}{(\rho+z)^2}\left[\frac{2\rho+z}{\rho^2}\boldsymbol{r}\cdot\boldsymbol{k}(\boldsymbol{n}\cdot\boldsymbol{r})-\rho\boldsymbol{n}\cdot\boldsymbol{k}\right]$$
$$-\frac{\boldsymbol{S}\cdot\boldsymbol{n}}{(\rho+z)^2}\left\{\boldsymbol{r}\cdot\boldsymbol{l}\left[1+\frac{(\rho+z)^2}{\rho^2}\right]\right.$$
$$\left.-(\rho+2z)\boldsymbol{k}\cdot\boldsymbol{l}\right\}-\boldsymbol{S}\cdot\boldsymbol{r}\left\{\frac{2\boldsymbol{k}\cdot\boldsymbol{l}}{\rho(\rho+z)}\left[\frac{\boldsymbol{n}\cdot\boldsymbol{r}}{\rho}\right.\right.$$
$$\left.+\frac{\boldsymbol{r}\cdot\boldsymbol{k}}{(\rho+z)^2}(\boldsymbol{n}\cdot\boldsymbol{r}+\rho\boldsymbol{n}\cdot\boldsymbol{k})\right]-\frac{\boldsymbol{r}\cdot\boldsymbol{l}}{\rho^2}$$
$$\cdot\left[\frac{1}{(\rho+z)^3}\{(3\rho+z)\boldsymbol{n}\cdot\boldsymbol{r}+2\rho^2\boldsymbol{n}\cdot\boldsymbol{k}\}\right.$$
$$\left.\left.-\frac{3\boldsymbol{n}\cdot\boldsymbol{r}}{(1-2\nu)\rho^2}\right]\right\}-\frac{\boldsymbol{n}\times\boldsymbol{l}}{\rho(\rho+z)}\cdot\left[(\boldsymbol{S}\times\boldsymbol{r})\right.$$
$$\left.\cdot\boldsymbol{r}\left(\frac{\boldsymbol{r}}{\rho}+\frac{\boldsymbol{r}+\rho\boldsymbol{k}}{\rho+z}\right)-\rho(\boldsymbol{S}\times\boldsymbol{k})\right]-\frac{\rho\boldsymbol{S}\cdot\boldsymbol{k}}{(\rho+z)^2}$$
$$\cdot\left\{\delta_{nl}-\frac{2\boldsymbol{r}\cdot\boldsymbol{l}}{\rho(\rho+z)}(\boldsymbol{n}\cdot\boldsymbol{r}+\rho\boldsymbol{n}\cdot\boldsymbol{k})+\frac{\boldsymbol{k}\cdot\boldsymbol{l}}{\rho(\rho+z)}\right.$$
$$\left.\left.\cdot[2\boldsymbol{n}\cdot\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{k})-\rho(\rho-z)\boldsymbol{n}\cdot\boldsymbol{k}]\right\}\right\rangle \quad (45)$$

Solutions (44) and (45) to the displacement vector and the stress tensor can be used for arbitrary coordinate systems belonging to the orthogonal curvilinear coordinates.

5. EXPRESSIONS FOR DISPLACEMENT AND STRESS COMPONENTS IN SPECIFIC COORDINATE SYSTEMS

In this chapter, we consider solutions (44) and (45) in specific coordinate systems, for instance, rectangular Cartesian, cylindrical and spherical coordinates.

(1) Rectangular Cartesian Coordinates

In rectangular Cartesian coordinates (x, y, z), we have the following relationships :

$$\boldsymbol{r}=[x, y, z],\quad \rho=(x^2+y^2+z^2)^{\frac{1}{2}} \quad (46a, b)$$

$$\boldsymbol{S}=[S_x, S_y, S_z] \quad (46c)$$

From solution (44) and relationships (46a-c), the displacement components u_x, u_y and u_z are expressed as

$$2Gu_x=\frac{S_x}{2\pi\rho}\left[1+\frac{x^2}{\rho^2}+\frac{1-2\nu}{\rho+z}\left(\rho-\frac{x^2}{\rho+z}\right)\right]$$
$$+\frac{S_yxy}{2\pi\rho}\left[\frac{1}{\rho^2}-\frac{1-2\nu}{(\rho+z)^2}\right]$$
$$+\frac{S_zx}{2\pi\rho}\left(\frac{z}{\rho^2}-\frac{1-2\nu}{\rho+z}\right) \quad (47a)$$

$$2Gu_y=\frac{S_xxy}{2\pi\rho}\left[\frac{1}{\rho^2}-\frac{1-2\nu}{(\rho+z)^2}\right]$$
$$+\frac{S_y}{2\pi\rho}\left[1+\frac{y^2}{\rho^2}+\frac{1-2\nu}{\rho+z}\left(\rho-\frac{y^2}{\rho+z}\right)\right]$$
$$+\frac{S_zy}{2\pi\rho}\left(\frac{z}{\rho^2}-\frac{1-2\nu}{\rho+z}\right) \quad (47b)$$

$$2Gu_z=\frac{S_x x}{2\pi\rho}\left(\frac{z}{\rho^2}+\frac{1-2\nu}{\rho+z}\right)+\frac{S_y y}{2\pi\rho}\left(\frac{z}{\rho^2}+\frac{1-2\nu}{\rho+z}\right)+\frac{S_z}{2\pi\rho}\left[2(1-\nu)+\frac{z^2}{\rho^2}\right] \tag{47c}$$

Furthermore, from solution (45) and relationships (46a-c), the stress components σ_{xx}, $\cdots$, σ_{xy} are expressed as

$$\sigma_{xx}=-\frac{S_x x}{2\pi\rho^3}\left[\frac{3x^2}{\rho^2}-\frac{1-2\nu}{(\rho+z)^2}\left(\rho^2-y^2-\frac{2\rho y^2}{\rho+z}\right)\right]-\frac{S_y y}{2\pi\rho^3}\left[\frac{3x^2}{\rho^2}-\frac{1-2\nu}{(\rho+z)^2}\left(3\rho^2-y^2-\frac{2\rho y^2}{\rho+z}\right)\right]-\frac{S_z}{2\pi\rho}\left\{\frac{3x^2 z}{\rho^4}-\frac{1-2\nu}{\rho+z}\left[1-\frac{y^2(2\rho+z)}{\rho^2(\rho+z)}\right]\right\} \tag{48a}$$

$$\sigma_{yy}=-\frac{S_x x}{2\pi\rho^3}\left[\frac{3y^2}{\rho^2}-\frac{1-2\nu}{(\rho+z)^2}\left(3\rho^2-x^2-\frac{2\rho x^2}{\rho+z}\right)\right]-\frac{S_y y}{2\pi\rho^3}\left[\frac{3y^2}{\rho^2}-\frac{1-2\nu}{(\rho+z)^2}\left(\rho^2-x^2-\frac{2\rho x^2}{\rho+z}\right)\right]-\frac{S_z}{2\pi\rho}\left\{\frac{3y^2 z}{\rho^4}-\frac{1-2\nu}{\rho+z}\left[1-\frac{x^2(2\rho+z)}{\rho^2(\rho+z)}\right]\right\} \tag{48b}$$

$$\sigma_{zz}=-\frac{3S_x xz^2}{2\pi\rho^5}-\frac{3S_y yz^2}{2\pi\rho^5}-\frac{3S_z z^3}{2\pi\rho^5} \tag{48c}$$

$$\sigma_{yz}=-\frac{3S_x xyz}{2\pi\rho^5}-\frac{3S_y y^2 z}{2\pi\rho^5}-\frac{3S_z yz^2}{2\pi\rho^5} \tag{48d}$$

$$\sigma_{zx}=-\frac{3S_x x^2 z}{2\pi\rho^5}-\frac{3S_y xyz}{2\pi\rho^5}-\frac{3S_z xz^2}{2\pi\rho^5} \tag{48e}$$

$$\sigma_{xy}=-\frac{S_x y}{2\pi\rho^3}\left[\frac{3x^2}{\rho^2}+\frac{1-2\nu}{(\rho+z)^2}\left(\rho^2-x^2-\frac{2\rho x^2}{\rho+z}\right)\right]-\frac{S_y x}{2\pi\rho^3}\left[\frac{3y^2}{\rho^2}+\frac{1-2\nu}{(\rho+z)^2}\left(\rho^2-y^2-\frac{2\rho y^2}{\rho+z}\right)\right]-\frac{S_z xy}{2\pi\rho^3}\left[\frac{3z}{\rho^2}-\frac{(1-2\nu)(2\rho+z)}{(\rho+z)^2}\right] \tag{48f}$$

If we set that

$$S_x=P,\quad S_y=0,\quad S_z=0 \tag{49a-c}$$

solutions (47a-c) and (48a-f) are coincident with the solutions[1] of Cerruti's problem and Mindlin's solution[3] for the case of $c=0$. Furthermore, if we set that

$$S_x=0,\quad S_y=0,\quad S_z=P \tag{50a-c}$$

solutions (47a-c) and (48a-f) are coincident with the solutions[1] of Boussinesq's problem and Mindlin's solution[3] for the case of $c=0$.

(2) Cylindrical Coordinates

In cylindrical coordinates (r, θ, z), we have the following relationships :

$$\boldsymbol{r}=[r,0,z],\quad \rho=(r^2+z^2)^{\frac{1}{2}} \tag{51a, b}$$

$$\boldsymbol{S}=[S_r,S_\theta,S_z],\quad \boldsymbol{k}=\boldsymbol{e}_z \tag{51c, d}$$

$$S_r=S_x\cos\theta+S_y\sin\theta \tag{51e}$$

$$S_\theta=-S_x\sin\theta+S_y\cos\theta,\quad S_z=S_z \tag{51f, g}$$

in which $\boldsymbol{e}_z$ denotes a unit vector in the cylindrical coordinates. From solution (44) and relationships (51a-g), the displacement components u_r, u_θ and u_z are expressed as

$$2Gu_r=\frac{1}{2\pi\rho}(S_x\cos\theta+S_y\sin\theta)\left[1+\frac{r^2}{\rho^2}+(1-2\nu)\frac{z}{\rho+z}\right]+\frac{S_z r}{2\pi\rho}\left(\frac{z}{\rho^2}-\frac{1-2\nu}{\rho+z}\right) \tag{52a}$$

$$2Gu_\theta=-\frac{1}{2\pi\rho}(S_x\sin\theta-S_y\cos\theta)\cdot\left[1+(1-2\nu)\frac{\rho}{\rho+z}\right] \tag{52b}$$

$$2Gu_z=\frac{r}{2\pi\rho}(S_x\cos\theta+S_y\sin\theta)\left(\frac{z}{\rho^2}+\frac{1-2\nu}{\rho+z}\right)+\frac{S_z}{2\pi\rho}\left[\frac{z^2}{\rho^2}+2(1-\nu)\right] \tag{52c}$$

Furthermore, from solution (45) and relationships (51a-g), the stress components σ_{rr}, $\cdots$, $\sigma_{r\theta}$ are expressed as

$$\sigma_{rr}=-\frac{r}{2\pi\rho^3}(S_x\cos\theta+S_y\sin\theta)\left[\frac{3r^2}{\rho^2}-(1-2\nu)\frac{\rho^2}{(\rho+z)^2}\right]-\frac{S_z}{2\pi\rho^2}\left[\frac{3r^2 z}{\rho^3}-(1-2\nu)\frac{\rho}{\rho+z}\right] \tag{53a}$$

$$\sigma_{\theta\theta}=\frac{1-2\nu}{2\pi}(S_x\cos\theta+S_y\sin\theta)\frac{rz(2\rho+z)}{\rho^3(\rho+z)^3}+\frac{S_z(1-2\nu)}{2\pi\rho^2}\left(\frac{z}{\rho}-\frac{\rho}{\rho+z}\right) \tag{53b}$$

$$\sigma_{zz}=-\frac{3rz^2}{2\pi\rho^5}(S_x\cos\theta+S_y\sin\theta)-\frac{3S_z z^3}{2\pi\rho^5} \tag{53c}$$

$$\sigma_{\theta z}=0 \tag{53d}$$

$$\sigma_{zr}=-\frac{3r^2 z}{2\pi\rho^5}(S_x\cos\theta+S_y\sin\theta)-\frac{3S_z rz^2}{2\pi\rho^5} \tag{53e}$$

$$\sigma_{r\theta}=\frac{(1-2\nu)r}{2\pi\rho(\rho+z)^2}(S_x\sin\theta-S_y\cos\theta) \tag{53f}$$

If we substitute Eqs.(49a-c) into solutions (52a-c) and (53a-f), we obtain the representations in the cylindrical coordinates of the solutions of Cerruti's problem. Furthermore, by the use of Eqs.(50a-c), we obtain the representations in the cylindrical coordinates of the solutions of Boussinesq's problem, which are coincident with the representations in Saada's book[1].

(3) Spherical Coordinates

In spherical coordinates (ρ, ϕ, θ), we have the following relationships :

$$\boldsymbol{r}=[\rho, 0, 0], \quad \boldsymbol{S}=[S_\rho, S_\phi, S_\theta] \tag{54a, b}$$

$$z=\rho\cos\phi, \quad \boldsymbol{k}=\cos\phi\boldsymbol{e}_\rho-\sin\phi\boldsymbol{e}_\phi \tag{54c, d}$$

$$S_\rho=(S_x\cos\theta+S_y\sin\theta)\sin\phi+S_z\cos\phi \tag{54e}$$

$$S_\phi=(S_x\cos\theta+S_y\sin\theta)\cos\phi-S_z\sin\phi \tag{54f}$$

$$S_\theta=-S_x\sin\theta+S_y\cos\theta \tag{54g}$$

in which $\boldsymbol{e}_\rho$ and $\boldsymbol{e}_\phi$ denote unit vectors in the spherical coordinates. From solution (44) and relationships (54a-g), the displacement components u_ρ, u_ϕ and u_θ are expressed as

$$2Gu_\rho=\frac{\sin\phi}{\pi\rho}(S_x\cos\theta+S_y\sin\theta)\left[2(1-\nu)-\frac{1-2\nu}{1+\cos\phi}\right]-\frac{S_z}{2\pi\rho}[1-2\nu-4(1-\nu)\cos\phi] \tag{55a}$$

$$2Gu_\phi=\frac{1}{2\pi\rho}(S_x\cos\theta+S_y\sin\theta)\left[(1-2\nu)\cdot\left(\frac{1}{1+\cos\phi}-2\right)+(3-4\nu)\cos\phi\right]-\frac{S_z}{2\pi\rho}\frac{\sin\phi}{1+\cos\phi}[2(1-\nu)+(3-4\nu)\cos\phi] \tag{55b}$$

$$2Gu_\theta=-\frac{1}{2\pi\rho}(S_x\sin\theta-S_y\cos\theta)\cdot\left(1+\frac{1-2\nu}{1+\cos\phi}\right) \tag{55c}$$

Furthermore, from solution (45) and relationships (54a-g), the stress components $\sigma_{\rho\rho}$, $\cdots$, $\sigma_{\rho\theta}$ are expressed as

$$\sigma_{\rho\rho}=-\frac{\sin\phi}{\pi\rho^2}(S_x\cos\theta+S_y\sin\theta)\cdot\left(2-\nu-\frac{1-2\nu}{1+\cos\phi}\right)+\frac{S_z}{2\pi\rho^2}[1-2\nu-2(2-\nu)\cos\phi] \tag{56a}$$

$$\sigma_{\phi\phi}=\frac{1-2\nu}{2\pi\rho^2}(S_x\cos\theta+S_y\sin\theta)\cdot\frac{\cot\phi\cos\phi}{1+\cos\phi}(1-\cos\phi)+\frac{S_z(1-2\nu)}{2\pi\rho^2}\frac{\cos^2\phi}{1+\cos\phi} \tag{56b}$$

$$\sigma_{\theta\theta}=\frac{1-2\nu}{2\pi\rho^2}(S_x\cos\theta+S_y\sin\theta)\cdot\frac{\cot\phi(1-\cos\phi)}{1+\cos\phi}(2+\cos\phi)+\frac{S_z(1-2\nu)}{2\pi\rho^2}\frac{\cos\phi-\sin^2\phi}{1+\cos\phi} \tag{56c}$$

$$\sigma_{\rho\phi}=\frac{1-2\nu}{2\pi\rho^2}(S_x\cos\theta+S_y\sin\theta)\cdot\frac{\cos\phi(1-\cos\phi)}{1+\cos\phi}+\frac{S_z(1-2\nu)}{2\pi\rho^2}\frac{\sin\phi\cos\phi}{1+\cos\phi} \tag{56d}$$

$$\sigma_{\phi\theta}=\frac{1-2\nu}{2\pi\rho^2}(S_x\sin\theta-S_y\cos\theta)\cdot\frac{\cot\phi(1-\cos\phi)}{1+\cos\phi} \tag{56e}$$

$$\sigma_{\rho\theta}=\frac{1-2\nu}{2\pi\rho^2}(S_x\sin\theta-S_y\cos\theta)\cdot\frac{1-\cos\phi}{1+\cos\phi} \tag{56f}$$

Thus, the expressions for the displacement and stress components in three coordinate systems, i.e., rectangular Cartesian, cylindrical and spherical coordinates were demonstrated.

6. NUMERICAL EXAMPLES

In this chapter, we consider numerical examples of solutions (47a-c) and (48a-f) under a force vector in the form

$$\boldsymbol{S}=P\left[\frac{1}{2},\ \frac{1}{2},\ \frac{1}{\sqrt{2}}\right]$$

Numerical calculations were made for an elastic half-space with Poisson's ratio $\nu=0.3$ and a standard distance c. The distributions of σ_{xx}, σ_{yy} and σ_{zz} at $y=0$ are shown in **Figs.2-4**. **Fig.2** shows that the value of σ_{xx} at the surface ($z=0$) becomes infinite at $x=0$ and that the decay along the x and z directions is rapid. **Fig.3** shows that the value of σ_{yy} at the surface becomes infinite at $x=0$ and that the distribution of σ_{yy} at the surface is symmetric with respect to $x=0$. **Fig.4** shows that the value of σ_{zz} in the vicinity of the surface becomes maximum at $x=0$ and that the decay along the x direction is very rapid.

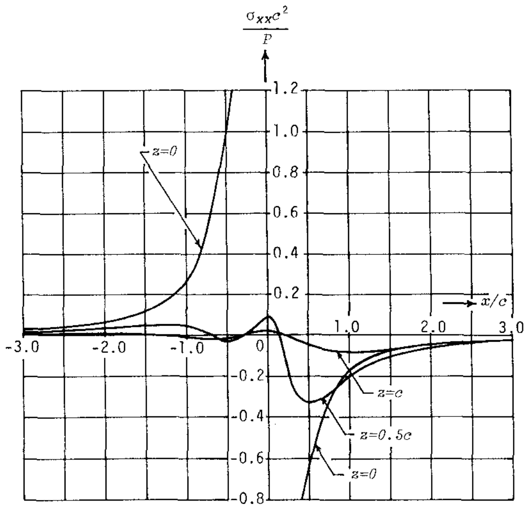

Fig.2 Distribution of σ_{xx}.
(y=0, ν=0.3)

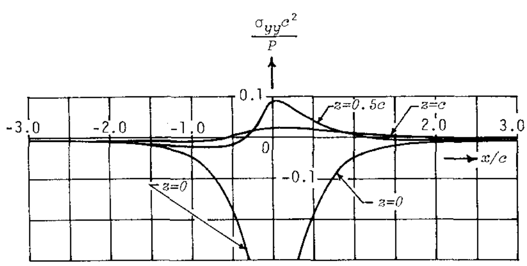

Fig.3 Distribution of σ_{yy}.
(y=0, ν=0.3)

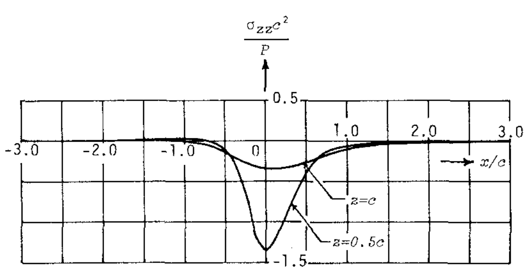

Fig.4 Distribution of σ_{zz}.
(y=0, ν=0.3)

7. CONCLUSION

A singular boundary-value problem of an elastic half-space subjected to a force vector with three components, i.e., two tangential and one normal forces, at one point of the surface was solved, and Cerruti's problem was generalized. The use of the generalized Boussinesq solution is convenient to a method of solution to the problem, because the solution of a rotation type is necessary to the analysis of the tangential forces. On constructing solutions, it is the point to need deliberation that the particular solutions to a harmonic function and harmonic vectors included in the generalized Boussinesq solution are successfully determined. Although the calculation of the stress vector in Eq.(4) needs very complicated vector operations, it is carried out by combining formulae (13a-l) well. Since solutions (44) and (45) obtained as a result can be used for arbitrary coordinate systems belonging to orthogonal curvilinear coordinates, they may have considerable generality. From the viewpoint of practice, solutions (44) and (45) were applied to rectangular Cartesian, cylindrical and spherical coordinates and demonstrated the concrete expressions for the displacement and stress components in these coordinate systems. The expressions may have wide applicability, because they include the solutions of Cerruti's and Boussinesq's problems.

For the reasons mentioned above, the author concludes that the solutions presented in this paper should be useful for a singular boundary-value problem of an elastic half-space.

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半無限弾性体における Cerruti 問題の一般化について

奥村　勇

表面の１点に集中力ベクトルを受ける半無限弾性体の特異境界値問題が解析されている．集中力ベクトルは，表面に対する２つの接線力及び１つの垂直力の３成分を持っている．解は，直交曲線座標における変位ベクトル及び応力テンソルの形式で与えられている．その解から，直交曲線座標の一例として，直交座標，円柱座標及び球座標における変位成分及び応力成分が具体的に求められ，これらの座標系における解が，スカラー表示されている．これらの解は，集中力ベクトルの３成分を特殊化した時に，Boussinesq 問題及び Cerruti 問題の解に一致する．