Method of Fast Conversion for Solving the Linear Simultaneous Eqautions with Principal Diagonal Coefficients

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Synopsis: By directly reciprocating the equations $ax=h$, which appear in numerical integration, structural analysis, etc., the far better first approximate solution $x'=(2I-a)\ Z'h$ than those, ever obtained, are rationally found. The ruling retroactive errors in x' are then recurrently corrected in series, enabled to converge as fast as possible, in a colum type table. Conventional methods of computation, such as iteration methods, cannot be used generally because considerable errors are actually introduced into the solutions, especially when the coefficients, not situated in the principal diagonal, are large compared to the diagonal ones. Thus, the equations, practically never been solved by the existing methods other than those of elimination, determinants, matrices, etc.[5] up to the present, can perfectly be solved.

Introduction. From the rapid development of the theories and practices in science and engineering in the recent days, the linear simultaneous equations with principal diagonal coefficients must have very frequently been solved, especially last ten years.[6)7)] Neverthless, the methods of solving them have practically been limitted up-to-date to the methods of elimination, determinants, iteration, matrix-postmultipliers, submatrices, etc.[6)7)13)] The method of elimination is the standard one but extremely tedious. That of determinants is rather symbolical, and practically, the order is higher, the computation becomes harder. That of iteration can not often be brought to the end, when the coefficients, not situated in the principal diagonal, are large in comparison to those in the principal diagonal, because of the errors from lack of the mathematical fundaments. The method of post-multipliers is nothing other than that of elimination, and is too complicated to be used generally, as that of sub-matrices is.[6)]

The method, explained in the present paper, is one of those, found by the author, and ideally so constructed that the following items can be fulfilled. That is, (1) the far better first approximations of the solutions than those, ever obtained by the other methods up-to-date, must readily be determined, and (2) the larger corrections prior to the smaller, as well as the predominant-recurrent ones must preferentially be corrected and the subsequent corrections must be enabled to converge as rapidly as possible.

This metnod can be utilized to solve not only the linear simultaneous equations, which appear in numerical integration, but also those in structural analysis.

Reciprocation of the Linear Simultaneous Equations with Principal Diagonal Coefficients

If the given linear simultaneous equations with the principal diagonal coefficients:

$$\left.\begin{array}{l} a_{1,1}x_1+a_{1,2}x_2+\cdots\cdots\cdots\cdots+a_{1,n}x_n=h_1, \\ a_{2,1}x_1+a_{2,2}x_2+\cdots\cdots\cdots\cdots+a_{2,n}x_n=h_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n,1}x_1+a_{n,2}x_2+\cdots\cdots\cdots\cdots+a_{n,n}x_n=h_n, \end{array}\right\} \qquad (1)$$

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1), 2),……Refer to Bibliography, given at the end of this paper.

Ex. 1[13]

No.	Makeup or Remarks	1 $E_{\theta A}$	2 $E_{\theta B}$	3 $E_{\theta C}$	4 $E_{\theta D}$	5 E_{p}
1.1	0	+1	+0.125	+0.188		−0.750
.2	0	+0.111	+1		+0.022	−0.667
.3	0	+0.087		+1	+0.087	−0.522
.4	0		+0.098	+0.073	+1	−0.439
.5	h_5 = 177,100 =	−0.200	−0.200	−0.300	−0.300	+1
2.1	$a_{ij}a_{ji},\ i\neq j$		+0.014	+0.016		+0.150
.2		+0.014			+0.022	+0.133
.3		+0.016			+0.006	+0.157
.4			+0.022	+0.006		+0.132
.5		+0.150	+0.133	+0.157	+0.132	
.6	$\Sigma a_{ij}a_{ji},\ i\neq j$	+0.180	+0.169	+0.179	+0.160	+0.572
.7	$1/(1-\Sigma a_{ij}a_{ji})$	+1.180	+1.169	+1.179	+1.160	+1.572
3.1	$I-(a-I)'$	+1	−0.111	−0.087		+0.200
.2	$=2I-a'$ †	−0.125	+1		−0.098	+0.200
.3		−0.188		+1	−0.073	+0.300
.4			−0.222	−0.087	+1	+0.300
.5		+0.750	+0.667	+0.522	+0.439	+1
4.1	$h'_5\times$					+1
.2	+(3.5)×1.000	+0.750	+0.667	+0.522	+0.439	[+1.000]
.3	Diff.					0
.4	+(3.1)×0.800	[+0.800]	−0.089	−0.070		+0.160
.5	Diff.	−0.050				
.6	Sum		+0.578			
.7	+(3.2)×0.600	−0.075	[+0.600]		−0.059	+0.120
.8	Diff.		−0.022			
.9	Sum			+0.452		
.10	+(3.3)×0.500	−0.094		[+0.500]	−0.037	+0.150
.11	Diff.			−0.048		
.12	Sum				+0.343	
.13	+(3.4)×0.300		−0.067	−0.026	[+0.300]	+0.090
.14	Diff.				+0.043	
.15	Sum	−0.219				
.16	−(3.1)×0.200	[−0.200]	+0.022	+0.017		−0.040
.17	Diff.	−0.019				
.18	Sum		−0.067			
.19	−(3.2)×0.070	+0.009	[−0.070]		+0.007	−0.014
.20	Diff.		+0.003			
.21	Sum			−0.057		
.22	−(3.3)×0.060	+0.011		[−0.060]	+0.006	−0.018
.23	Diff.			+0.003		
.24	Sum				+0.056	
.25	+(3.4)×0.060		−0.013	−0.005	[+0.060]	+0.018
.26	Diff.				−0.004	
.27	Sum		−0.010			
.28	−(3.2)×0.010	+0.001	[−0.010]		+0.001	−0.002
.29	Diff.		0			
.30	Sum				−0.003	
.31	−(3.4)×0.003		+0.001	0	[−0.003]	−0.001
.32	Diff.				0	
.33	Sum	+0.002	+0.001	−0.002		+0.464
.34	Completed	[+0.002]	[+0.001]	[−0.002]		
.35	$\frac{0.464}{1-0.464}$					+0.866
.36	Σ[ ]	+0.602	+0.521	+0.438	+0.357	+1.000
.37	+(4.36)×0.866	+0.522	+0.451	+0.379	+0.309	+0.866
.38	Coeff'ts of h'_5	+1.124	+0.972	+0.817	+0.666	+1.866
.39	Check in (1.5) +1 ≈ +1.002 =	−0.225	−0.194	−0.245	−0.200	+1.866
5.1	+(4.38)×177,100 or E	+199,100	+172,100	+144,600	+118,000	+330,800
6.1	E [13]	+200,000	+173,000	+144,000	+117,000	+330,000

† $(a-I)'$ and a' are the transposed matrices of $(a-I)$ and a respectively.

or in the matrix

$$ax=h,\ a=[a_{i,j}],\ x=\{x_i\},\ h=\{h_i\} \qquad (2)$$

are given, first make unity the principal diagonal coefficients, i. e.,

$$a_{i,i}=1 \qquad (3)$$

so as to facilitate the reduction.

Reciprocating and transforming Eq. (2), we get

$$x=a^{-1}h=\{I+(a-I)\}^{-1}h \qquad (4)$$

$$=(2I-a)Zh=Z(2I-a)h \qquad (5)$$

where

$$Z=\{I-(a-I)^2\}^{-1}=I+(a-I)^2+(a-I)^4+\cdots\cdots \qquad (6)$$

Method of Fast Conversion. There are two ways to compute the values of x according to Eq. (5). In one of them, we first calculate Z and $(2I-a)h$ separately according to Eq. (5) and (6), and then multiply them together. It may be speedy, provided the equations should have only three coefficients, which is already publicated by the author.[12]

In another method, called "Method of Fast Conversion" by the author, we firstly appreciate the far better first approximate values of x, than those ever obtained by the other authorities, referring to Eq. (7) and (8), or more practically to Eq. (13) and (14), explained in the following. The errors in them must stepwise be so estimated that the successive corrections make a series, convergent as fast as possible. Substantially, they are nothing but the series, enabled to converge as fat as possible. Accordingly, if the required accuracy is attained at last, the further computation of terms of higher order may be abandoned. The theoretical fundaments are as follows.

If, instead of computing Z by Eq. (6) exactly, we take such that

$$Z'=z'_{i,i}\ \delta_{i,k}, \qquad (7)^*$$

$$z'_{i,i}=(1-\sum_{j,j\neq i}a_{i,j}a_{j,i})^{-1}, \qquad (8)^*$$

where

$$\delta_{ik}=\text{Kronecker's sign}, \qquad (9)$$

then we get $Z\sim Z'$, (10)*

because the matrix-elements, not situated in the principal diagonal in Z, are comparatively smaller than

* The dashes mean merely the approximate values of the corresponding variables, but not the transposed matrices.

those in the principal diagonal. Introducing Eq. (7) into Eq. (5) we find the first approximation of x

$$x'=(2I-a)Z'h, \tag{11}^*$$

which shall give the far better approximate values of x than those, ever estimated by the existing methods.

Now, if we take, referring to Eq. (2) and (11), such that

$$\left.\begin{array}{llll} x= x' + \Delta x, & a\,\Delta x= h-a\;x' = \Delta h, \\ \Delta x=\Delta x'' + \Delta^2 x, & a\,\Delta^2 x= \Delta h-a\,\Delta\,x'' = \Delta^2 h, \\ \vdots & \vdots \\ \Delta^\tau x=\Delta^\tau x^{(\tau+1)}+\Delta^{\tau+1}x, & a\,\Delta^{\tau+1}x=\Delta^\tau h-a\,\Delta^\tau x^{(\tau+1)}=\Delta^{\tau+1}h, \\ \vdots & \vdots \end{array}\right\} \tag{12}^{*\dagger}$$

then we get the relations

$$\left.\begin{array}{l} \Delta\, x'' = (2I-a)Z''\;\Delta\, h, \\ \Delta^2 x''' = (2I-a)Z'''\;\Delta^2 h, \\ \vdots \\ \Delta^\tau x^{(\tau+1)} = (2I-a)Z^{(\tau+1)}\Delta^\tau h, \\ \vdots \end{array}\right\} \tag{13}^{*\dagger}$$

and

$$x=x'+\Delta\, x''+\Delta^2 x'''+\Delta^3 x''''+\cdots\cdots+\Delta^\tau x^{(\tau+1)}+\cdots\cdots, \tag{14}^{*\dagger}$$

because, in practice, it needs not use such values of Z' in each of Eq. (11) and (13), as given by Eq. (7), but is rather better to take such proper values for Z'', Z''',…… arbitrarily as it let simplify the resulting computation as much as possible and converge them as fast as possible in every step of the correction.

Eq. from (11) to (14) clarify the principle of the new method.

Practice of Calculations. The process of numerical calculations are classified into the two ways, that is, (1) that by the individual absolute term respectively, and (2) that by all the absolute terms simultaneously. In order to complete the calculations as fast as possible and as accurate as possible, the process must be executed according to the following laws respectively.

Process 1. The computation by the respective absolute term.

In this process, the individual contribution of each of absolute terms in the solutions, or the coefficients of each of absolute terms in Eq. (5), are respectively determined, whenever they may numerically or algebraically be given.

Law 1) Divide all the coefficients in each of the given equations by that in the principal diagonal respectively, and let $a_{i,i}=1$ according to Eq. (3).

Law 2) Find $z'_{i,i}$ and Z' according to Eq. (8). It is not always neccessary, excepting the coefficients are comparatively large, after Law 1) is executed.

Law 3) Compute the first approximations of x_1', x_2', x_3',…… according to Eq. (11), that is, $x'=(2I-a)Z'h$.

Law 4) Compute $\Delta x''$, $\Delta^2 x'''$, $\Delta^3 x''''$, ……according to Eq. (12), (13), and (14), always especially attending to Law 5) and also the simplicity of computation. The magnitudes of the ruling-error-corrections must be determined so properly that both under- and over-corrections may be avoided as much as possible.

Law 5) Compute the correction in succession, which has the greater influences to the other corrections or solutions or which has the greatest magnitude, prior to the others. On practice, refer to **Ex. 1** and **2.**

† Δ, Δ^2, Δ^3,……mean merely the differences of the corresponding variables, given by Eq. (12).

In **Ex. 1**, Rows from 1.1 to 1.5 are the given equations, in which $a_{ii}=1$ already according to Law 1). Row 2.7 gives z_{ii}', shown by Eq. (8), which is calculated only for reference. Row from 3.1 to 3.5 are $2I-a'$. In Row 4.1, the absolute term of Row 1.5 is taken as $h_5=1$ for convenience, and Row from 4.1 to 4.39 are the computation of the solutions of the given equuations from 1.1 to 1.5.

Conventional notations are generally used as follows. The number, situated just under ____, is the algebraic sum of the numbers, situated above ____ itself in the same column. The number in [ ] gives the approximately appreciated values of x', $\Delta x''$, $\Delta^2 x'''$, ……at each step of computation, referring to Eq. (12) and (13). The number, situated just under [ ], is the difference of the number in [ ] and that between ____, written down just above [ ] in the same column, but it is not always written down when it is zero.

Now the total sum of the retrograde corrections under [+1.000] in Row 4.2, or +0.464, is given in Row 4.33. In this case, [+1.000] corresponds to x_5' in Eq. (12) and + 0.464 to $\Delta x_5''$ in Eq. (13). It is clear that the correction comes back recurrently as $\Delta^2 x_5''' = +(0.464)^2$ and so on. They make such an infinite series that their sum can be estimated all together which Row 4.35, 4.36, 4.37 and 4.38 show. Thus, we get the coefficients of h_5' in $E's$, as shown in Row 4.38, whose correctness is satisfactorily established by the equation, given by Row 1.5, as shown in Row 4.39. At last, we getthe numerical results of solutions of $E's$ by multiplying Row 4.38 by $h_5' = +177,100$. In Row 6.1, the results given in Bibliography 13), are shown for reference.

EX. 2[6] (1)

No.		1	2	3	4	5	6
	Makeup or Remarks	X_1	X_2	X_3	X_4	X_5	X_6
1.1	$H_1/10.472=h_1$	+1	+0.0483		−0.3758	−0.0498	
.2	$H_2/11.016=h_2$	+0.0459	+1	+0.4540	−0.0473	−0.0950	+0.3405
.3	$H_3/26.000=h_3$		+0.1923	+1			−0.0404
.4	$H_4/6.322=h_4$	−0.6224	−0.0824		+1	+0.0848	+0.0562
.5	$H_5/2.737=h_5$	−0.1903	−0.3822		+0.1958	+1	
.6	$H_6/3.811=h_6$		+0.9662	−0.2705	+0.0915		+1
2.1	$a_{ij}a_{ji},\ i\neq j$		+0.0022		+0.2339	+0.0095	
.2		+0.0022		+0.0873	+0.0039	+0.0363	+0.3289
.3			+0.0873				+0.0109
.4		+0.2339	+0.0039			+0.0166	+0.0051
.5		+0.0095	+0.0363		+0.0166		
.6			+0.3289	+0.0109	+0.0051		
.7	$\Sigma a_{ij}a_{ji};\ i\neq j$	+0.2456	+0.4587	+0.0982	+0.2595	+0.0624	+0.3450
3.1	$I-(a-I)' = 2I-a'$ †	+1	−0.0459		+0.6224	+0.1903	
.2		−0.0483	+1	−0.1924	+0.0824	+0.3822	−0.9663
.3			−0.4540	+1			+0.2705
.4		+0.3758	+0.0473		+1	−0.1958	−0.0915
.5		+0.0498	+0.0950		−0.0848	+1	
.6			−0.3405	+0.0404	−0.0562		+1
4.1	$h_6 = H_6\times$						+0.2576
.2	+(3.6)×0.35		−0.1192	+0.0141	−0.0197		[+0.35]
.3	Diff.						−0.0923
.4	−(3.2)×0.12	+0.0058	[−0.12]	+0.0234	−0.0099	−0.0459	+0.1160
.5	Diff.		+0.0008				
.6	−(3.5)×0.046	−0.0023	−0.0044		+0.0039	[−0.0460]	
.7	Diff.					+0.0001	
.8	Sum			+0.0372			
.9	+(3.3)×0.037		−0.0168	[+0.037]			+0.0100
.10	Diff.			+0.0002			
.11	Sum		−0.0204		−0.0257		
.12	−(3.2)×0.020	+0.0010	[−0.020]	+0.0038	−0.0016	−0.0076	+0.0193
.13	Diff.		−0.0004				
.14	Sum				−0.0273		
.15	−(3.4)×0.027	−0.0101	−0.0013		[−0.027]	+0.0057	+0.0025
.16	Diff.				−0.0003		
.17	Sum	−0.0056					
.18	−(3.1)×0.0056	[−0.0056]	+0.0003		−0.0035	−0.0011	
.19	Sum			+0.0040			
.20	+(3.3)×0.0040		−0.0018	[+0.0040]			+0.0011
.21	Sum		−0.0032		−0.0038		
.22	−(3.4)×0.0032	−0.0014	−0.0002		[−0.0038]	+0.0007	+0.0003
.23	Sum		−0.0034				
.24	−(3.2)×0.0034	+0.0002	[−0.0034]	+0.0007	−0.0003	−0.0013	+0.0033
.25	Sum					−0.0039	
.26	−(3.5)×0.0039	−0.0002	−0.0004		+0.0003	[−0.0039]	
.27	Sum	−0.0014					
.28	−(3.1)×0.0014	[−0.0014]	+0.0001		−0.0008	−0.0003	
.29	Sum				−0.0008		
.30	−(3.4)×0.0008	−0.0003	0		[−0.0008]	+0.0002	+0.0001
.31	+(3.3)×0.0007		−0.0003	[+0.0007]			+0.0002
.32	Sum		−0.0006				
.33	−(3.2)×0.0006	0	[−0.0006]	+0.0001	0	−0.0002	+0.0006
.34	Sum	−0.0003					
.35	−(3.1)×0.0003	[−0.0003]	0		−0.0002	−0.0001	
.36	Sum					−0.0004	
.37	−(3.5)×0.0004	0	0		0	[−0.0004]	
.38	−(3.4)×0.0002	−0.0001	0		[−0.0002]	0	0
.39	Completed	[−0.0001]		+0.0001			
.40	Sum						+0.0611
.41	$\frac{0.0611}{0.2576-0.0611}$						+0.311
.42	Σ[ ]	−0.0074	−0.1440	+0.0418	−0.0318	−0.0503	+0.3500
.43	+(4.42)×0.311	−0.0023	−0.0448	+0.0130	−0.0099	−0.0157	+0.1088
.44	Coefft's of H_6	−0.0097	−0.1888	+0.0548	−0.0417	−0.0660	+0.4588
.45	Check in (1.6) 0.2576≈0.2578=		−0.1824	−0.0148	−0.0038		+0.4588
5.1	$h_5 = H_5\times$					+0.3654	
.2	+(3.5)×0.3654	+0.0182	+0.0347		−0.0310	[+0.3654]	
.3	+(3.2)×0.0347	−0.0017	[+0.0347]	−0.0067	+0.0029	+0.0133	−0.0335
.4	Sum				−0.0281		
.5	−(3.4)×0.0281	−0.0106	−0.0013		[−0.0281]	+0.0055	+0.0026
.6	Sum	+0.0059					
.7	−(3.3)×0.0067		+0.0030	[−0.0067]			−0.0018
.8	+(3.1)×0.0059	[+0.0059]	−0.0003		+0.0037	+0.0011	
.9	+(3.4)×0.0037	+0.0014	+0.0002		[+0.0037]	−0.0007	−0.0003
.10	+(3.1)×0.0014	[+0.0014]	−0.0001		+0.0009	+0.0003	
.11	Sum		+0.0015				
.12	+(3.2)×0.0015	−0.0001	[+0.0015]	−0.0003	+0.0001	+0.0006	−0.0014
.13	Sum				+0.0010		
.14	+(3.4)×0.0010	+0.0004	0		[+0.0010]	−0.0002	−0.0001
.15	Sum	+0.0003					
.16	+(3.1)×0.0003	[+0.0003]	0		+0.0002	+0.0001	
.17	−(3.3)×0.0003		+0.0001	[−0.0003]			−0.0001
.18	+(3.4)×0.0002	+0.0001	0		[+0.0002]	0	0
.19	+(3.1)×0.0001	[+0.0001]			+0.0001		
.20	completed		[+0.0001]		[+0.0001]		
.21	Sum					+0.0200	−0.0346
.22	$\frac{0.0200}{0.3654-0.0200}$					+0.0579	
.23	Σ[ ]	+0.0077	+0.0363	−0.0070	−0.0231	+0.3654	(−0.0346)
.24	+(5.23)×0.0579	+0.0004	+0.0021	−0.0004	−0.0013	+0.0216	(−0.0020)
.25	Sum						(−0.0366)
.26	$-(4.44)\frac{0.0366}{0.2576}$	+0.0014	+0.0263	−0.0078	+0.0059	+0.0094	−0.0652
.27	Coefft's of H_5	+0.0095	+0.0652	−0.0152	−0.0185	+0.3960	−0.0652
.28	Check in (1.5) 0.3654≈0.3657=	−0.0018	−0.0249		−0.0036	+0.3960	

In **Ex. 2**, the same process as used in **Ex. 1**, is preferred. At first, the coefficients of H_6 in $x's$ are computed, and then those of H_5, H_4, ……, and H_1 in order, as they are all given in the table. On the way of calculations, we can use the coefficients of H_6 in $x's$ to determine those of H_5, those of H_6 and H_5 to determine those of H_4, and similarly so on. Therefore, to calculate the coefficients of H_1 in $x's$, the coefficients of H_6, H_5, ……, and H_2 already gotten, are utilized as shown in Row from 9.1 to 9.8. Thus, all the results obtained are tabulated in Row from 10.1 to 10.6.

The facts, that they fairly agree with the exact results, given in Row from 11.1 to 11.6, and the checks, shown in Row 4.45 and 9.9, are all right, establish the present method.

Process 2. The simultaneous computation, when all the absolute terms are numerically given.

For computing x', $\Delta x''$, $\Delta^2 x'''$, $\Delta^3 x''''$, ……, we must especially attend to Law 5) and 3), because the convergency of the corrections can predominantly be improved and the computation extremely simplified by them.

Law 1) Divide all the coefficients in each of the given equations by that in the principal diagonal respectively, and let $a_{ii}=1$ according to Eq. (3).

Law 2) Find z'_{ii} and Z' accoding to Eq. (8). It is not always neccessary, excepting the coefficients are comparatively large, after Law 1) is executed.

Ex. 2[6] **(2)**

No.		1	2	3	4	5	6
	Makeup or Remarks	X_1	X_2	X_3	X_4	X_5	X_6
6.1	$h_4 = H_4 \times$				+0.1581		
.2	+(3.4)×0.1581	+0.0594	+0.0075		[+0.1581]	−0.0310	−0.0145
.3	+(3.1)×0.0594	[+0.0594]	−0.0027		+0.0370	+0.0113	
.4	Sum		+0.0048				
.5	+(3.2)×0.0048	−0.0002	[+0.0048]	−0.0009	+0.0004	+0.0018	−0.0046
6	−(3.3)×0.0009		+0.0004	[−0.0009]			−0.0002
.7	+(3.2)×0.0004	0	[+0.0004]	−0.0001	0	+0.0002	−0.0004
.8	−(3.1)×0.0002	[−0.0002]	0		−0.0001	0	
.9	Completed			[−0.0001]			
.10	Sum				+0.0373	−0.0177	−0.0197
.11	$\frac{0.0373}{0.1581-0.0373}$				+0.3088		
.12	Σ[ ]	+0.0592	+0.0052	−0.0010	+0.1581	(−0.0177)	(−0.0197)
.13	+(6.12)×0.3088	+0.0183	+0.0016	−0.0003	+0.0488	(−0.0055)	(−0.0061)
.14	Sum					(−0.0232)	(−0.0258)
.15	$-(5.27)\frac{0.0232}{0.3654}$	−0.0006	−0.0041	+0.0010	+0.0012	−0.0251	+0.0041
.16	$-(4.44)\frac{0.0258}{0.2576}$	+0.0010	+0.0188	−0.0055	+0.0042	+0.0066	−0.0459
.17	Coeff't's of H_4	+0.0779	+0.0215	−0.0058	+0.2123	−0.0185	−0.0418
.18	Check in (1.4)						
	0.1581=0.1581=	−0.0485	−0.0018		+0.2123	−0.0016	−0.0023
7.1	$h_3 = H_3 \times$			+0.0385			
.2	+(3.3)×0.0385		−0.0175	[+0.0385]			+0.0104
.3	−(3.2)×0.0175	+0.0008	[−0.0175]	+0.0034	−0.0014	−0.0067	+0.0169
.4	+(3.1)×0.0008	[+0.0008]	0		+0.0005	+0.0002	
.5	Sum			+0.0034	−0.0009	−0.0065	+0.0273
.6	$\frac{0.0034}{0.0385-0.0034}$			+0.0969			
.7	Σ[ ]	+0.0008	−0.0175	+0.0385	(−0.0009)	(−0.0065)	(+0.0273)
.8	+(7.7)×0.0969	+0.0001	−0.0017	+0.0037	(−0.0001)	(−0.0006)	(+0.0026)
.9	Sum				(−0.0010)	(−0.0071)	(+0.0299)
.10	$-(6.17)\times\frac{0.0010}{0.1581}$	−0.0005	−0.0001	0	−0.0013	+0.0001	+0.0003
.11	$-(5.27)\frac{0.0071}{0.3654}$	−0.0002	−0.0013	+0.0003	+0.0004	−0.0077	+0.0013
.12	$-(4.44)\frac{0.0299}{0.2576}$	−0.0011	−0.0219	+0.0064	−0.0048	−0.0077	+0.0533
.13	Coeff't's of H_3	−0.0009	−0.0425	+0.0489	−0.0057	−0.0153	+0.0549
.14	Check in (1.3)						
	0.0385=0.0385=		−0.0082	+0.0489			−0.0022
8.1	$h_2 = H_2 \times$		+0.0908				
.2	+(3.2)×0.0908	−0.0044	[+0.0908]	−0.0175	+0.0075	+0.0347	−0.0877
.3	−(3.1)×0.0044	[−0.0044]	+0.0002		−0.0027	−0.0008	
.4	Sum		+0.0002	−0.0175	+0.0048	+0.0339	−0.0877
.5	$\frac{0.0002}{0.0908-0.0002}$		+0.0022				
.6	Σ[ ]	−0.0044	+0.0908	(−0.0175)	(+0.0048)	(+0.0339)	(−0.0877)
.7	+(8.6)×0.0022	0	+0.0002	(0)	(0)	(+0.0001)	(−0.0002)
.8	Sum			(−0.0175)	(+0.0048)	(+0.0340)	(−0.0879)
.9	$-(7.13)\frac{0.0175}{0.0385}$	+0.0004	+0.0192	−0.0221	+0.0026	+0.0069	−0.0248
.10	$+(6.17)\frac{0.0048}{0.1581}$	+0.0024	+0.0007	−0.0002	+0.0064	−0.0007	−0.0013
.11	$+(5.27)\frac{0.0340}{0.3654}$	+0.0009	+0.0061	−0.0014	−0.0017	+0.0368	−0.0061
.12	$-(4.44)\frac{0.0877}{0.2576}$	+0.0033	+0.0644	−0.0187	+0.0142	+0.0225	−0.1565
.13	Coeff't's of H_2	+0.0026	+0.1814	−0.0424	+0.0215	+0.0655	−0.1887
.14	Check in (1.2)						
	0.0908=0.0908=	+0.0001	+0.1814	−0.0193	−0.0010	−0.0062	−0.0642
9.1	$h_1 = H_1 \times$	+0.0955					
.2	+(3.1)×0.0955	[+0.0955]	−0.0044		+0.0594	+0.0182	
.3	Sum		(−0.0044)		(+0.0594)	(+0.0182)	
.4	Σ[ ]	+0.0955					
5	$-(8.13)\frac{0.0044}{0.0908}$	−0.0001	−0.0088	+0.0021	−0.0010	−0.0032	+0.0091
.6	$+(6.17)\frac{0.0594}{0.1581}$	+0.0293	+0.0081	−0.0022	+0.0798	−0.0070	−0.0157
.7	$+(5.27)\frac{0.0182}{0.3654}$	+0.0005	+0.0032	−0.0008	−0.0009	+0.0197	−0.0032
.8	Coeff't's of H_1	+0.1252	+0.0025	−0.0009	+0.0779	+0.0095	−0.0098
.9	Check in (1.1)						
	0.0955=0.0955=	+0.1252	+0.0001		−0.0293	−0.0005	
No.	X	H_1	H_2	H_3	H_4	H_5	H_6
10.1	$X_1 =$ ⊕	+0.1252	+0.0026	−0.0009	+0.0779	+0.0095	−0.0097
.2	$X_2 =$	+0.0025	+0.1814	−0.0425	+0.0215	+0.0652	−0.1009
.3	$X_3 =$	−0.0009	−0.0424	+0.0489	−0.0058	−0.0152	+0.0548
.4	$X_4 =$	+0.0779	+0.0215	−0.0057	+0.2123	−0.0185	−0.0417
5	$X_5 =$	+0.0095	+0.0655	−0.0153	−0.0185	+0.3460	−0.0660
.6	$X_6 =$	−0.0098	−0.1887	+0.0549	−0.0418	−0.0652	+0.4588
11.1	$X_1 =$ ※	+0.12508	+0.00262	−0.00090	+0.07782	+0.00957	−0.00990
.2	$X_2 =$	+0.00262	+0.18151	−0.04253	+0.02163	+0.06563	−0.18887
.3	$X_3 =$	−0.00090	−0.04253	+0.04886	−0.00585	−0.01528	+0.05485
.4	$X_4 =$	+0.07782	+0.02163	−0.00585	+0.21232	−0.01850	−0.04191
.5	$X_5 =$	+0.00957	+0.06563	−0.01528	−0.01850	+0.39589	−0.06586
.6	$X_6 =$	−0.00990	−0.18887	+0.05485	−0.04191	−0.06586	+0.45883

† $(a-I)'$ and a' are the transformed matrices of $(a-I)$ and a respectively.
⊕ Gotten by the new method.
※ The exact results.

Law 3) Evaluate preferently the first approximate solutions x_{i_1}', x_{i_2}', ……, which correspond to the absolute terms h_{i_1} h_{i_2}, ……, which are not zero, according to Eq. (11), and take for the first approximate values x_{j_1}', x_{j_2}', ……, zero, which correspond to the zero absolute terms h_{j_1}, h_{j_2}, ……$=0$.

Here, we must remember that the first approximate value x_r' is predominantly influenced by the magnitude of the corresponding absolute term h_r and slightly by those of the others h_s.

Law 4) Compute $\Delta x''$, $\Delta^2 x'''$, $\Delta^3 x''''$, ……according to Eq. (12), (13), and (14), always especially attending to Law 5) and also the simplicity of computation. The magnitudes of the ruling-error-corrections must be determined so properly that both under- and over-corrections may be avoided as much as possible.

Law 5) Compute the correction in succession, which has the greater influences to the other corrections or solutions, or which has the greatest magnitude, prior to the others. Evaluate the corrections, timely utilizing the proper multipliers.

On practice, refer to **Ex. 3**. In Row from 1.1 to 1.12, the given equations, treated after Law 1), are given. In Row 2.1 to 2.14, z' is calculated according to Row 2), and Row from 3.1 to 3.12, $2I-a'$,. In Row 4.1, the absolute terms of the given equations are shown, from which the first approximations are computed according to Law 3).

Conventional notations are the same as those in **Ex. 1** and **2**. The number, situated just under ____, is the algebraic sum of the num bers, situated above ____ itself in the same column. The number in [ ] gives the approximately appreciated values of x^1, $\Delta x''$, $\Delta^2 x''$, …… at each step of computation, referring to Eq. (12) and (13). The number,

Ex. 3[9)*]

No.		1	2	3	4	5	6	7	8	9	10	11	12
	Makeup or Remarks	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}
1.1	0 =	+1	+0.140	+0.160						+0.200	+0.160		
.2	0 =	+0.074	+1		+0.084					+0.105	+0.084		
.3	0 =	+0.174		+1	+0.152	+0.174					+0.174	+0.174	
.4	0 =		+0.088	+0.077	+1		+0.088				+0.088	+0.088	
.5	0 =			+0.190		+1	+0.143	+0.167				+0.190	+0.167
.6	0 =				+0.103	+0.077	+1		+0.090			+0.103	+0.090
.7	0 =					+0.269		+1	+0.231				+0.269
.8	0 =						+0.113	+0.097	+1				+0.113
.9	2.286 =	+0.752	+0.752							+1			
.10	3.142 =	+0.748	+0.748	+0.748	+0.748						+1		
.11	1.682 =			+0.748	+0.748	+0.748	+0.748					+1	
.12	0.860 =					+0.753	+0.753	+0.753	+0.753				+1
2.1	$a_{ij}a_{ji}$, $i \neq j$		+0.01	+0.03						+0.15	+0.12		
.2		+0.01			+0.01					+0.08	+0.06		
.3		+0.03			+0.01	+0.03					+0.13	+0.13	
.4			+0.01	+0.01			+0.01				+0.07	+0.07	
.5				+0.03			+0.01	+0.05				+0.14	+0.13
.6					+0.01	+0.01			+0.01			+0.08	+0.07
.7						+0.05			+0.02				+0.20
.8							+0.01	+0.02					+0.09
.9		+0.15	+0.08										
.10		+0.12	+0.06	+0.13	+0.07								
.11				+0.13	+0.07	+0.14	+0.08						
.12						+0.13	+0.07	+0.20	+0.09				
.13	$\Sigma a_{ij}a_{ji}$	+0.31	+0.16	+0.33	+0.17	+0.36	+0.18	+0.27	+0.12	+0.23	+0.38	+0.42	+0.49
.14	$z_{ii}=1/(1-\Sigma a_{ij}a_{ji})$	+1.45	+1.19	+1.49	+1.20	+1.56	+1.22	+1.37	+1.14	+1.30	+1.61	+1.72	+1.96
3.1	$I-(a-I)'$ ⊕ $=2I-a'$	+1	-0.074	-0.174						-0.752	-0.748		
.2		-0.140	+1		-0.088					-0.752	-0.748		
.3		-0.160		+1	-0.077	-0.190					-0.748	-0.748	
.4			-0.084	-0.152	+1		-0.103				-0.748	-0.748	
.5				-0.174		+1	-0.077	-0.269				-0.748	-0.753
.6					-0.088	-0.143	+1		-0.113			-0.748	-0.753
.7						-0.167		+1	-0.097				-0.753
.8							-0.090	-0.231	+1				-0.753
.9		-0.200	-0.105							+1			
.10		-0.160	-0.084	-0.174	-0.088						+1		
.11				-0.174	-0.088	-0.190	-0.103					+1	
.12						-0.167	-0.090	-0.269	-0.113				+1
4.1	k_i	0	0	0	0	0	0	0	0	-2.286	-3.141	-1.682	-0.860
.2	-(3.10)×5.0	+0.800	+0.420	+0.870	+0.440						[-5.0]		
.3	Diff.										+1.359		
.4	-(3.11)×3.0			+0.522	+0.264	+0.570	+0.309					[-3.0]	
.5	Diff.											+1.318	
.6	-(3.9)×3.0	+0.600	+0.315							[-3.0]			
.7	Diff.									+0.714			
.8	-(3.12)×2.0					+0.334	+0.180	+0.538	+0.226				[-2.0]
.9	Diff.												+1.140
.10	Sum	+1.400	+0.735	+1.392	+0.704	+0.904	+0.489	+0.538	+0.226	+0.714	+1.359	+1.318	+1.140
.11	+(3.1)×1.4	[+1.400]	-0.104	-0.244						-1.053	-1.047		
.12	Diff.	0											
.13	+(3.2)×0.7	-0.098	[+0.7]		-0.062					-0.527	-0.524		
.14	Diff.		+0.035										
.15	+(3.3)×1.4	-0.224		[+1.4]	-0.108	-0.266					-1.047	-1.047	
.16	Diff.			-0.003									
.17	+(3.4)×0.7		-0.059	-0.105	[+0.7]		-0.072				-0.524	-0.524	
.18	Diff.				+0.004								
.19	+(3.5)×0.9			-0.157		[+0.9]	-0.069	-0.242				-0.673	-0.678
.20	Diff.					+0.004							
.21	+(3.6)×0.5				-0.044	-0.072	[+0.5]		-0.057			-0.372	-0.377
.22	Diff.						-0.011						
.23	+(3.7)×0.5					-0.084		[+0.5]	-0.049				-0.377
.24	Diff.							+0.038					
.25	+(3.8)×0.2						-0.018	-0.046	[+0.2]				-0.151
.26	Diff.								+0.026				
.27	Sum	-0.322	-0.128	-0.515	-0.210	-0.418	-0.170	-0.250	-0.080	-0.866	-1.283	-1.298	-0.443
.28	-(3.9)×0.9	+0.180	+0.095							[-0.9]			
.29	Diff.									+0.034			
.30	-(3.10)×1.3	+0.208	+0.109	+0.226	+0.114						[-1.3]		
.31	Diff.										+0.019		
.32	-(3.11)×1.3			+0.226	+0.114	+0.247	+0.134					[-1.3]	
.33	Diff.											+0.002	
.34	-(3.12)×0.4					+0.067	+0.036	+0.108	+0.045				[-0.4]
.35	Diff.												-0.043
.36	Sum	+0.066	+0.076	-0.065	+0.018	-0.104	0	-0.142	-0.035				
.37	+(3.1)×0.07	[+0.07]	-0.005	-0.012						-0.053	-0.052		
.38	Diff.	-0.004											
.39	+(3.2)×0.08	-0.011	[+0.08]		-0.007					-0.060	-0.060		
.40	Diff.		-0.004										
.41	-(3.3)×0.06	+0.010		[-0.06]	+0.005	+0.011					+0.045	+0.045	
.42	Diff.			-0.003									
.43	+(3.4)×0.02		-0.002	-0.003	[+0.02]		-0.002				-0.015	-0.015	
.44	Diff.				-0.002								
.45	-(3.5)×0.10			+0.017		[-0.100]	+0.008	+0.027				+0.075	+0.075
.46	Diff.					-0.004							
.47	-(3.7)×0.14					+0.023		[-0.14]	+0.014				+0.105
.48	Diff.							-0.002					
.49	-(3.8)×0.04						+0.004	+0.009	[-0.04]				+0.030
.50	Diff.								+0.005				
.51	Sum									-0.079	-0.065	+0.107	+0.167
.52	-(3.9)×0.08	+0.016	+0.008							[-0.08]			
.53	Diff.									+0.001			
.54	-(3.10)×0.07	+0.011	+0.006	+0.012	+0.006						[-0.07]		
.55	Diff.										+0.005		
.56	+(3.11)×0.11			-0.019	-0.010	-0.021	-0.011					[+0.11]	
.57	Diff.											-0.003	
.58	+(3.12)×0.17					-0.028	-0.016	-0.046	-0.019				[+0.17]
.59	Diff.												-0.003
.60	Sum	+0.022	+0.003	-0.008	-0.008	-0.019	-0.017	-0.012	0				
.61	+(3.1)×0.022	[+0.022]	-0.002	-0.004						-0.017	-0.016		
.62	+(3.2)×0.003		[+0.003]							-0.002	-0.002		
.63	-(3.3)×0.008	+0.001		[-0.008]	+0.001	+0.002					+0.006	+0.006	
.64	-(3.4)×0.008		+0.001	+0.001	[-0.008]		+0.001				+0.006	+0.006	
.65	-(3.5)×0.019			+0.003		[-0.019]	+0.002	+0.005				+0.014	+0.014
.66	-(3.6)×0.017				+0.002	+0.002	[-0.017]		+0.002			+0.013	+0.013
.67	-(3.7)×0.012					+0.002		[-0.012]	+0.001				+0.002
.68	Sum									-0.018	-0.001	+0.036	+0.033
.69	-(3.9)×0.018	+0.004	+0.002							[-0.018]			
.70	-(3.10)×0.002	0	0	0	0						[-0.001]		
.71	+(3.11)×0.036			-0.006	-0.003	-0.007	-0.014					[+0.036]	
.72	+(3.12)×0.033					-0.006	-0.003	-0.009	-0.004				[+0.033]
.73	Sum	+0.005	+0.001	-0.006	-0.001	-0.007	-0.005	-0.004	-0.001				
.74	+(3.1)×0.005	[+0.005]	0	-0.001						-0.004	-0.003		
.75	+(3.2)×0.001	0	[+0.001]		0					-0.001	-0.001		
.76	-(3.3)×0.006	+0.001		[-0.006]	0	+0.001					+0.004	+0.004	
.77	-(3.4)×0.001				[-0.001]						+0.001	+0.001	
.78	-(3.5)×0.007			+0.001		[-0.007]	+0.001	+0.002				+0.005	+0.005
.79	-(3.6)×0.005				0	+0.001	[-0.005]					+0.004	+0.004
.80	-(3.7)×0.004					+0.001		[-0.004]					+0.003
.81	-(3.8)×0.001								[-0.001]				+0.001
.82	Sum									-0.005	+0.001	+0.014	+0.013
.83	-(3.9)×0.005	+0.001	+0.001							[-0.005]			
.84	+(3.11)×0.014			-0.002	-0.001	-0.003	-0.001					[+0.014]	
.85	+(3.12)×0.013					-0.002	-0.001	-0.003	-0.001				[+0.013]
.86	Sum	+0.002	+0.001	-0.002	-0.001	-0.002	-0.001	-0.003	-0.001				
.87	Completed	[+0.002]	[+0.001]	[-0.002]	[-0.001]	[-0.002]	[-0.001]	[-0.001]					
88	Σ[ ] or X_i	+1.499	+0.785	+1.324	+0.710	+0.775	+0.476	+0.343	+0.159	-4.003	-6.371	-4.141	-2.184
5.1	X_i 9) †	+1.51	+0.79	+1.31	+0.71	+0.78	+0.48	+0.34	+0.16	-4.01	-6.39	-4.14	-2.18
6.1	X_i ‡	+1.498	+0.785	+1.323	+0.711	+0.774	+0.472	+0.342	+0.159	-4.003	-6.371	-4.135	-2.184

※ Calculated with a Hemmi's 10-in. slide rule.
⊕ $(a-I)'$ and a' are the transposed matrices of $(a-I)$ and a respectively.
† The solutions, given in Bibliography 9).
‡ The solutions, gotten by the author by the same method in another way.

situated just under [ ], is the difference of the number in [ ] and that between ___, written down just above [ ] in the same column, but it is not always written down when it is zero.

In the present example, the approximate values x', $\Delta x''$, $\Delta^2 x'''$, ... are computed by turns as shown in Row 4.1 to 4.9, from 4.10 to 4.26, from 4.27 to 4.35, etc. In Row 4.87, "completed" means the completion of calculation and the last approximations are given in [ ]. In Row 4.88, the results of solution, that is, x_i are given according to Eq. (14). In Row 5.1, the solutions, given in Bibliography 9) are shwn, and in Row 6.1, those, gotten by the same method in another way, both of which establish the reliability of the present method.

Conclusion. It is satisfactorily verified that the linear simultaneous equations with relatively large coefficients in comparison to the principal diagonal coefficients can readily be solved by the new method, though it has practically never been realized by simple methods other than that of elimination up to the present.

It is fundamentally indebted to the finding of the new formulas for the solutions and the first approximation, that is, Eqs. from (5) to (14).

Therefore, z_{ii}' must not be taken as unity as in the ordinary method of iteration, prevailed up to date, if the coefficients except those in the principal diagonal are relatively large, because its inertness is clearly based upon it.

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