

# SENSITIVITIES OF PARAMETERS DUE TO MODEL ERRORS AND MEASUREMENT NOISES IN STRUCTURAL IDENTIFICATION PROBLEMS

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This paper deals with the development of sensitivities of structural parameter estimates with respect to model errors and measurement noises.

Structural identification algorithm is normally developed based on the concept of least square method. Then the determination of parameters is treated as an optimization problem and the parameter estimates are an optimum solution of the problem. Hence the sensitivities of the parameters with respect to errors and noises can be formulated in the manner similar to sensitivities of optimum solutions.

Numerical examples are presented to demonstrate an effectiveness of the method proposed.

*Keywords* : structural identification, sensitivity, model error, input noise, response noise

## 1. INTRODUCTION

In order to verify the seismic safety of structures or to obtain informations for aseismic design, attempts have been made to measure the dynamic behavior of existing structures and to estimate their dynamic characteristics by observing microtremors and earthquakes continuously or by conducting strong vibration experiments<sup>1)~5)</sup>.

Using the concept of the non-linear least square method, Hanada *et al.*<sup>6)</sup> identified the stiffness and the damping coefficient of a given structure on the condition that the residual of the characteristic equation of the system be minimized. In a study aimed at identifying system parameters, Distefano and Rath<sup>7)</sup> introduced a method of identification on the basis of the non-linear filter and invariant imbedding, while Simonian<sup>8), 9)</sup> adopted dynamic programming for the same purpose. The methods of invariant imbedding and dynamic programming, although they differ in a conception, lead to identical governing equations<sup>10)</sup>. Shah and Udwadia<sup>11)</sup> identified the system constants based on the Gauss-Newton method together with the concept of point matching and further referred to the optimum sensor locations. Hoshiya and Saito<sup>12), 13)</sup>, and Maruyama *et al.*<sup>14)</sup>, on the other hand, used the Kalman filter to estimate the system constants and dynamic characteristics. Applying the Gauss-Newton method, Matsui and Kurita<sup>15)</sup> derived a method for estimating the system constants using the data within an arbitrary time interval extracted from acceleration records while simultaneously estimating the initial displacement and velocity of the system at the same interval.

Although a number of methods for estimating the system constants and dynamic characteristics of

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structures have been reported on, as explained above, few studies have been made on the effects of the noise in the measured data and of the error in structural model on the estimation of unknown parameters.

This paper introduces a method of obtaining the sensitivities of parameters unknown or to be identified with respect to the noise and error. Since the estimated values in structural identification are a solution of optimization problem, the sensitivity of the parameters namely corresponds to the sensitivity of the optimum solution<sup>16)</sup>. Calculated sensitivity values make it possible to determine the degree of accuracy required of prescribed constants in a structural model, as well as the accuracy required of the measured data. On the other hand, if the sensitivity of unknown parameters with respect to a noise is too large at a measuring point, it may be said that the point is inappropriate location to place a sensor on. Hereunder, the sensitivity formulas are theoretically developed and simple examples are presented for their verification.

## 2. EVALUATION FUNCTION IN THE IDENTIFICATION PROBLEM

When a structure consisting of  $N$  degrees of freedom is excited by dynamic external force, its equation of motion is given by

$$M\ddot{z} + C\dot{z} + Kz = Q(t) \quad \dots\dots\dots (1)$$

where the symbols signify respectively,

- $M : N \times N$  mass matrix
- $C : N \times N$  damping matrix
- $K : N \times N$  stiffness matrix
- $Q(t) : N \times 1$  dynamic external vector
- $\ddot{z} : N \times 1$  acceleration response vector
- $\dot{z} : N \times 1$  velocity response vector
- $z : N \times 1$  displacement response vector

The initial conditions in Eq. (1) are given at  $t=t_0$  as,

$$z(t_0) = a, \quad \dot{z}(t_0) = b \quad \dots\dots\dots (2)$$

which represent the initial displacement and the initial velocity respectively. Let the unknown parameters in Eq. (1) be  $X = \{X_j\}$  ( $j=1, \dots, M$ ) and the known constants be  $Y = \{Y_l\}$  ( $l=1, \dots, L$ ). Then, the response vectors  $\ddot{z}$ ,  $\dot{z}$  and  $z$  are not only functions of time but are also dependent on  $X$  and  $Y$ . When the measured acceleration record at an observation point  $i$  is given by  $\ddot{u}_i$  and the corresponding analyzed acceleration by  $\ddot{z}_i$ , considering noise  $\varepsilon_i(t)$  in  $\ddot{u}_i$ , the following relationship holds.

$$\ddot{u}_i(t) = \ddot{z}_i(t) + \varepsilon_i(t) \quad i \in A \quad \dots\dots\dots (3)$$

where  $A$  is a set of observation points. By employing the concept of the least square method, the evaluation function  $J$  may be defined as,

$$J(X, Y) = \frac{1}{2} \int_{t_0}^n \sum_{i \in A} w_i (\ddot{u}_i - \ddot{z}_i)^2 dt \quad \dots\dots\dots (4)$$

where  $w_i$  is the weight coefficient. Since there is no appropriate means of selecting  $w_i$  at present,  $w_i=1$  ( $i \in A$ ) is assumed in this paper.

Let the true value of the constant vector of  $Y$  be  $\overline{Y}$ . The unknown parameter  $X$  can be determined from  $\partial J / \partial X_j = 0$  ( $j=1, \dots, M$ ), which are necessary conditions for minimizing the value of Eq. (4). Let values of  $X$  and  $J$  be  $\overline{X}$  and  $J_{\min}$  respectively, when Eq. (4) takes a minimum value. Then, it follows,

$$J_{\min} = J(\overline{X}, \overline{Y}) = \frac{1}{2} \int_{t_0}^n \sum_{i \in A} w_i (\ddot{u}_i - \ddot{z}_i)^2 dt \quad \dots\dots\dots (5)$$

where  $\ddot{u}_i$  is the measured response from which a noise is completely removed by some means. When all these conditions are satisfied,  $J_{\min}$  assumes zero. Also, to distinguish noise contaminated input wave from noiseless wave, the former is denoted by  $\ddot{y}_0$  and the latter by  $\ddot{y}_0$ .

In the evaluation of sensitivity,  $\partial \ddot{z} / \partial X_j$ ,  $\partial \dot{z} / \partial X_j$  and  $\partial z / \partial X_j$  ( $j=1, \dots, M$ ) are required, which are

partial differentials of response vectors with respect to each unknown parameter  $X_j$ . After differentiating Eq. (1) by  $X_j$ , the following equation will be obtained<sup>(7)</sup>.

$$M \frac{\partial \ddot{z}}{\partial X_j} + C \frac{\partial \dot{z}}{\partial X_j} + K \frac{\partial z}{\partial X_j} = \frac{\partial Q}{\partial X_j} - \frac{\partial M}{\partial X_j} \ddot{z} - \frac{\partial C}{\partial X_j} \dot{z} - \frac{\partial K}{\partial X_j} z \quad (j=1, \dots, M) \quad (6)$$

Initial conditions for the above equation are

i) when  $X_j$  is neither initial displacement nor initial velocity :

$$\frac{\partial z}{\partial X_j}(t_0)=\{0\}, \quad \frac{\partial \dot{z}}{\partial X_j}(t_0)=\{0\} \quad (7)$$

ii) when  $X_j$  is initial displacement :

$$\frac{\partial z}{\partial X_j}(t_0)=\{1\}, \quad \frac{\partial \dot{z}}{\partial X_j}(t_0)=\{0\} \quad (8)$$

iii) when  $X_j$  is initial velocity :

$$\frac{\partial z}{\partial X_j}(t_0)=\{0\}, \quad \frac{\partial \dot{z}}{\partial X_j}(t_0)=\{1\} \quad (9)$$

Eqs. (8) and (9) are necessary to solve Eq. (6) when either the initial displacement or the initial velocity is unknown. Eqs. (7) denote the initial conditions when both initial conditions are prescribed. In this case, both initial displacement and velocity are considered to be included in  $Y$ .  $\{1\}$  refers to an  $N \times 1$  vector in which one element corresponding to unknown parameter is 1 and the rests are zero.

### 3. EFFECTS OF MODEL ERRORS

#### (1) Derivation of formulas

In the structural identification problem, some parameters among masses and stiffnesses are treated as known and constant and all other unknown parameters are determined by minimizing the discrepancies between measured and calculated values. In case there are errors in the parameters provided as known, the value of  $X$ , the result of identification, inevitably reflects the errors. If the sensitivity of  $X$  with respect to  $Y_i$  is obtained and the amount of error in  $Y_i$  is known, the effect of the error on  $X$  can be estimated. Conversely, if a certain degree of accuracy is required for the estimation of  $X$ , the accuracy necessary for the known parameters can be evaluated. When an error  $\Delta Y_i$  exists in  $Y_i$ , for which  $\bar{Y}_i$  is a true value, its effect on  $\bar{X}$ , may be written as  $(\partial X_j / \partial Y_i) \Delta Y_i$ . Namely

$$J = \frac{1}{2} \int_{t_0}^{t_1} \sum_{i \in A} w_i \left\{ \ddot{u}_i - \ddot{z}_i(\bar{X}) - \sum_{j=1}^M \frac{\partial \ddot{z}_i}{\partial X_j} \frac{\partial X_j}{\partial Y_i} \Delta Y_i - \frac{\partial \ddot{z}_i}{\partial Y_i} \Delta Y_i \right\}^2 dt \quad (10)$$

where  $\ddot{u}_i$  is the measured but noiseless data referred to in the previous section. If  $\Delta Y_i$  is small enough in magnitude, its effect on  $X$  will also be small. Hence, Eq. (10) will be written in the first order approximation as

$$J = \frac{1}{2} \int_{t_0}^{t_1} \sum_{i \in A} w_i \left\{ \ddot{u}_i - \ddot{z}_i(\bar{X}) - \sum_{j=1}^M \frac{\partial \ddot{z}_i}{\partial X_j} \frac{\partial X_j}{\partial Y_i} \Delta Y_i - \frac{\partial \ddot{z}_i}{\partial Y_i} \Delta Y_i \right\}^2 dt \quad (11)$$

Since  $\ddot{u}_i$  agrees with  $\ddot{z}_i(\bar{X})$ , the first two terms in  $\{ \}$  disappears.  $\partial X_j / \partial Y_i$  is the sensitivity of unknown parameters with respect to the model error, which implies that the estimated values of the unknown parameters are larger than their true values by  $\partial X_j / \partial Y_i$  when there is a unit error in  $Y_i$ . Letting  $\lambda_{ji} = \partial X_j / \partial Y_i$ , Eq. (11) can be rewritten as

$$J = \frac{1}{2} \int_{t_0}^{t_1} \sum_{i \in A} w_i \left\{ - \sum_{j=1}^M \frac{\partial \ddot{z}_i}{\partial X_j} \lambda_{ji} - \frac{\partial \ddot{z}_i}{\partial Y_i} \right\}^2 (\Delta Y_i)^2 dt \quad (12)$$

Since  $(\Delta Y_i)^2$ , the square of the error, in Eq. (12) is arbitrary, the necessary condition to minimize Eq. (12) irrespective of the value of  $(\Delta Y_i)^2$  is  $\partial J / \partial \lambda_{ki} = 0$ ; that is

$$\frac{\partial J}{\partial \lambda_{ki}} = (\Delta Y_i)^2 \int_{t_0}^{t_1} \sum_{i \in A} w_i \left\{ - \sum_{j=1}^M \frac{\partial \ddot{z}_i}{\partial X_j} \lambda_{ji} - \frac{\partial \ddot{z}_i}{\partial Y_i} \right\} \left( - \frac{\partial \ddot{z}_i}{\partial X_k} \right) dt = 0 \quad (13)$$

After some manipulations, Eq. (13) reduces to

$$\sum_{j=1}^M \left\{ \int_{t_0}^{t_1} \sum_{i \in A} w_i \frac{\partial \ddot{z}_i}{\partial X_j} \frac{\partial \ddot{z}_i}{\partial X_k} dt \right\} \lambda_{ji} = - \int_{t_0}^{t_1} \sum_{i \in A} w_i \frac{\partial \ddot{z}_i}{\partial Y_i} \frac{\partial \ddot{z}_i}{\partial X_k} dt \quad (k=1, \dots, M) \dots\dots\dots (14)$$

The above formula gives a set of linear simultaneous equations for  $\lambda_{ji}$ , which can be solved easily.

## (2) An example

To verify the theory introduced in the foregoing section, a two-degrees-of-freedom system, as shown in Fig. 1, is adopted, to which the El Centro wave (El Centro 1940 NS Imperial Valley Earthquake),  $\ddot{y}_0(t)$ , was applied as the input wave with a time increment  $\Delta t = 0.02$  seconds. The dynamic responses of the masses 1 and 2 are calculated by Newmark's  $\beta$  method for a time period from 0 to 20 seconds, the effect of the errors in these masses on damping coefficients  $c_1$  and  $c_2$ , and stiffnesses  $k_1$  and  $k_2$ , is examined. The results are given in Table 1, where  $\partial c_1 / \partial m_1 = 1.8453$ , for example, implies that the estimated value of  $c_1$  becomes 1.8453 tf·sec/m larger if the value of  $m_1$  used in the identification is 1 tf·sec<sup>2</sup>/m larger. The table shows that an increase of  $m_1$  will make the estimates of  $c_1$ ,  $k_1$  and  $k_2$  greater than the respective true values, while it makes the estimate of  $c_2$  smaller than its true value. The table also shows that an increase of  $m_2$ , on the other hand, makes all the estimates of  $c_1$ ,  $c_2$ ,  $k_1$  and  $k_2$  larger than their respective true values. The effect on  $c_1$  is very small, however. To study the accuracy of sensitivities in Table 1, the results from the sensitivity analyses are compared with those obtained by the method of identification in reference 15), assuming that the latter is correct. Also assuming that either of  $m_1$  and  $m_2$  contains 5 %, 10 % and 20 % error, the values of  $c_1$ ,  $c_2$ ,  $k_1$  and  $k_2$  are computed by the method of sensitivity which is developed in the previous section and the method described in reference 15). The results are presented in Table 2. 'Estimated' in Table 2 implies that the values are obtained using the sensitivities in Table 1 and 'Identified' means the results calculated by the method in reference 15). Since the difference between the values from two methods is relatively small, it can be stated that the method based on sensitivity analysis provide good estimates to the effect of error involved.

The effect of errors existing in the initial values will be examined next. Dynamic identification, in general, is made for an arbitrary time interval chosen from the measured response acceleration, but neither the value of the initial displacement nor the value of the initial velocity is known. Table 3 gives the sensitivities of identified parameters with respect to errors in the initial values, which are calculated by the proposed method. It shows that the effect of the error in the initial velocity is very small, while the error in the initial displacement produces comparatively large effects on the estimated values of parameters. Assuming three different magnitudes of errors on initial conditions, which are 5 %, 10 % and 20 % of respective absolute maximums of relative velocity and displacement, the parameters are estimated

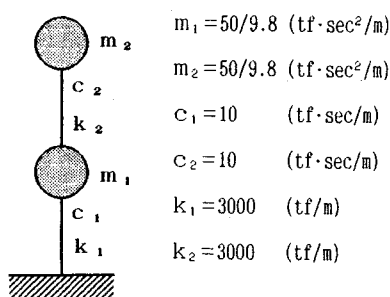


Fig. 1 Two degrees-of-freedom model.

Table 1 Sensitivities with respect to error in masses.

$Y_0$	$\frac{\partial c_1}{\partial Y_0}$	$\frac{\partial c_2}{\partial Y_0}$	$\frac{\partial k_1}{\partial Y_0}$	$\frac{\partial k_2}{\partial Y_0}$
$m_1$	1.8453	-1.6455	75.895	390.08
$m_2$	0.11466	3.6055	512.10	197.92

units:  $\frac{\partial c_i}{\partial m_i}$  (1/sec),  $\frac{\partial k_i}{\partial m_i}$  (1/sec<sup>2</sup>)

Table 2 Estimates based on sensitivities and identified results.

		5 %		10 %		20 %	
		Estimated	Identified	Estimated	Identified	Estimated	Identified
Error in $m_1$	$c_1$	10.471	10.403	10.941	10.716	11.883	11.213
	$c_2$	9.5802	9.6928	9.1605	9.5401	8.3209	9.4666
	$k_1$	3019.4	3020.8	3038.7	3045.0	3077.4	3103.7
	$k_2$	3099.5	3100.9	3199.0	3202.2	3398.0	3398.7
Error in $m_2$	$c_1$	9.9708	9.9316	9.9415	9.5996	9.8830	7.7830
	$c_2$	10.920	11.081	11.840	12.595	13.679	17.628
	$k_1$	3130.6	3131.3	3261.3	3261.5	3522.6	3501.4
	$k_2$	3050.5	3053.6	3101.0	3117.1	3202.0	3296.1

units:  $c_i$  (tf·sec/m),  $k_i$  (tf/m)

Table 3 Sensitivities with respect to error in initial values.

$Y_i$	$\frac{\partial C_1}{\partial Y_0}$	$\frac{\partial C_2}{\partial Y_0}$	$\frac{\partial k_1}{\partial Y_0}$	$\frac{\partial k_2}{\partial Y_0}$	
$a_1$	-24.416	124.30	9056.7	-24141	$a_1$ : initial displacement of mass 1
$a_2$	100.06	-107.14	-5207.2	12671	$a_2$ : initial displacement of mass 2
$b_1$	-2.4514	9.7793	113.80	-218.12	$b_1$ : initial velocity of mass 1
$b_2$	4.3591	-5.1787	25.568	64.596	$b_2$ : initial velocity of mass 2

units:  $\frac{\partial C_i}{\partial a_j}$  (tf·sec/m<sup>2</sup>),  $\frac{\partial k_i}{\partial a_j}$  (tf/m<sup>2</sup>),  $\frac{\partial C_i}{\partial b_j}$  (tf·sec<sup>2</sup>/m<sup>2</sup>),  $\frac{\partial k_i}{\partial b_j}$  (tf·sec/m<sup>2</sup>)

Table 4 Estimates from the sensitivities and identified results.

		5 %		10 %		20 %	
		Estimated	Identified	Estimated	Identified	Estimated	Identified
Error in $a_1$	$C_1$	9.9753	9.6311	9.9505	8.5217	9.9011	4.3799
	$C_2$	10.126	11.003	10.252	13.773	10.504	23.771
	$k_1$	3009.2	3010.8	3018.3	3022.9	3036.7	3029.5
	$k_2$	2975.6	2971.3	2951.1	2938.3	2902.2	2895.0
Error in $a_2$	$C_1$	10.160	9.8433	10.321	9.0184	10.642	5.1636
	$C_2$	9.8282	10.687	9.6565	13.169	9.3130	23.695
	$k_1$	2991.7	2995.4	2983.3	2997.1	2966.6	3004.3
	$k_2$	3020.3	3010.5	3040.6	3003.1	3081.3	2958.9
Error in $b_1$	$C_1$	9.9649	9.9170	9.9298	9.7390	9.8596	9.0923
	$C_2$	10.140	10.267	10.280	10.785	10.560	12.570
	$k_1$	3001.6	3001.8	3003.2	3003.9	3006.5	3008.4
	$k_2$	2996.9	2996.4	2993.8	2992.0	2987.5	2981.6
Error in $b_2$	$C_1$	10.098	10.065	10.196	10.063	10.393	9.8360
	$C_2$	9.8833	9.9867	9.7667	10.187	9.5334	11.265
	$k_1$	3000.6	3000.8	3001.2	3001.9	3002.3	3004.9
	$k_2$	3001.5	3000.9	3002.9	3000.7	3005.8	2998.3

units:  $C_i$  (tf·sec/m),  $k_i$  (tf/m)

by the sensitivity approach and the method of identification. Both results are tabulated in Table 4. The table shows that the approach presented here is applicable when the errors in the initial displacement and velocity are comparatively small, but the accuracy worsens as the errors become larger, presumably because the effects of the initial displacement and velocity attenuate quickly during the time period of 20 seconds and thus have little influence on the evaluation function. In reference 15), the initial displacement and the initial velocity are also identified more precisely when the shorter duration of response records is used. This also endorses the discussion presented above.

#### 4. EFFECTS OF THE NOISE IN THE MEASURED RESPONSE

##### (1) Derivation of formulas

It is assumed that there are no errors in the structural model  $\bar{Y}$ , and no noises in the measured input value  $\bar{y}_0$  and in measured response values  $\bar{u}_i(t)$  ( $i \in A$ ,  $i \neq l$ ) except the response at the measuring point  $l$ . Though measured response accelerations are, in general, the absolute accelerations, the relative accelerations, subtracting the input acceleration from the absolute response accelerations, is used for theoretical development, since the measured input values are considered to be the true values without noise here. When the measured response at measuring point  $l$  is denoted by  $\bar{u}_l$ , the response  $\bar{u}_l$  can be expressed by the sum of true response  $\bar{u}_l$  and noise  $\Delta \bar{u}_l = \varepsilon_l \eta_l(t)$ ; that is  $\bar{u}_l = \bar{u}_l + \Delta \bar{u}_l$ .

$\eta_l(t)$  is a normalized noise having a maximum (absolute) value of 1, and  $\varepsilon_l$  is a scalar which represents the magnitude of the maximum value. When the noise  $\Delta \bar{u}_l$  in measurement exists only at the point  $l$ , the identified parameter may be written as  $\bar{X} + \Delta X^{(l)}$ , where  $\Delta X^{(l)}$  is defined as the subtraction of the true value  $\bar{X}$  from the identified result  $X$  which reflects the noise  $\Delta \bar{u}_l$ . Then, Eq. (4) can be prescribed as,

$$J = \int_{t_0}^n \left[ \frac{1}{2} \sum_{i \in A} w_i \{\ddot{u}_i - \ddot{z}_i(\bar{X} + \Delta X^{(t)})\}^2 + \frac{1}{2} w_i \{\ddot{u}_i + \Delta \ddot{u}_i - \ddot{z}_i(\bar{X} + \Delta X^{(t)})\}^2 \right] dt \dots\dots\dots (15)$$

It is reasonable to assume that  $X$  varies continuously with  $\varepsilon_i$  in  $\Delta \ddot{u}_i$  and that the magnitude of  $\Delta X^{(t)}$  is small when  $\varepsilon_i$  is small. Therefore,

$$\begin{aligned} \ddot{z}_i(\bar{X} + \Delta X^{(t)}) &= \ddot{z}_i(\bar{X}) + \sum_{j=1}^M \frac{\partial \ddot{z}_i}{\partial X_j} \frac{\partial X_j}{\partial \ddot{u}_i} \Delta \ddot{u}_i \\ &= \ddot{z}_i(\bar{X}) + \sum_{j=1}^M \frac{\partial \ddot{z}_i}{\partial X_j} \frac{\partial X_j}{\partial \varepsilon_i} \frac{\partial \varepsilon_i}{\partial \ddot{u}_i} \Delta \ddot{u}_i \\ &= \ddot{z}_i(\bar{X}) + \sum_{j=1}^M \frac{\partial \ddot{z}_i}{\partial X_j} \Gamma_{ji} \varepsilon_i \dots\dots\dots (16) \end{aligned}$$

where  $\Gamma_{ji} = \partial X_j / \partial \varepsilon_i$ . Substituting Eq. (16) into Eq. (15) and using the relationship  $\ddot{z}_i(\bar{X}) = \ddot{u}_i$ , the following equation is obtained after rearranging terms.

$$J = \frac{\varepsilon_i^2}{2} \int_{t_0}^n \sum_{i \in A} w_i \left\{ - \sum_{j=1}^M \frac{\partial \ddot{z}_i}{\partial X_j} \Gamma_{ji} \right\}^2 dt + \frac{\varepsilon_i^2}{2} \int_{t_0}^n w_e \left\{ - \sum_{j=1}^M \frac{\partial \ddot{z}_i}{\partial X_j} \Gamma_{ji} + \eta_i(t) \right\}^2 dt \dots\dots\dots (17)$$

From the necessary condition for Eq. (17) to be minimum irrespective of the value of  $\varepsilon_i^2$ , with respect to the unknown variable  $\Gamma_{ji}$ , one obtains

$$\begin{aligned} \frac{\partial J}{\partial \Gamma_{kl}} &= \varepsilon_i^2 \int_{t_0}^n \sum_{j=1}^M \left( \sum_{i \in A} w_i \frac{\partial \ddot{z}_i}{\partial X_j} \frac{\partial \ddot{z}_i}{\partial X_k} \right) \Gamma_{ji} dt - \varepsilon_i^2 \int_{t_0}^n w_e \eta_i(t) \frac{\partial \ddot{z}_i}{\partial X_k} dt = 0 \\ \therefore \sum_{j=1}^M \left[ \int_{t_0}^n \left( \sum_{i \in A} w_i \frac{\partial \ddot{z}_i}{\partial X_j} \frac{\partial \ddot{z}_i}{\partial X_k} \right) dt \right] \Gamma_{je} &= \int_{t_0}^n w_e \eta_i(t) \frac{\partial \ddot{z}_i}{\partial X_k} dt \quad (k=1, \dots, M) \dots\dots\dots (18) \end{aligned}$$

where  $\Gamma_{ji}$  is the sensitivity with respect to the noise in measured response  $\ddot{u}_i$ . The above equation gives a set of linear simultaneous equations with respect to the unknown variable  $\Gamma_{ji}$ , which can be solved easily.

## (2) An example

To verify the method introduced in the above section, the model in Fig. 1 is used together with a normalized noise shown in Fig. 2, which is a band-limited white noise in the range of 0.1 Hz to 25 Hz and is adjusted to have an absolute maximum value 1. Fig. 3 shows the response accelerations of masses 1 and 2 due to  $\ddot{y}_0$  (El Centro wave treated as true excitation) when the noise in Fig. 2 is superimposed on these responses, the identified parameters are expected to differ from their true values. Sensitivities of the parameters with respect to the noise are tabulated in Table 5. Table 6 compares damping coefficients and

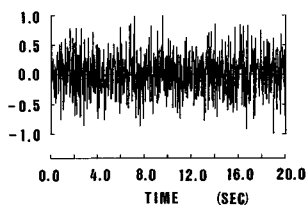


Fig. 2 Band-limited white noise.

Table 5 Sensitivities with respect to noise in measured response.

$\varepsilon_x$	$\frac{\partial C_1}{\partial \varepsilon_x}$	$\frac{\partial C_2}{\partial \varepsilon_x}$	$\frac{\partial K_1}{\partial \varepsilon_x}$	$\frac{\partial K_2}{\partial \varepsilon_x}$
$\varepsilon_1$	-0.00824	0.13871	4.8032	-11.393
$\varepsilon_2$	-0.13882	0.05514	7.0382	-16.385

units:  $\frac{\partial C}{\partial \varepsilon_x}$  (tf·sec<sup>3</sup>/m<sup>2</sup>),  $\frac{\partial K}{\partial \varepsilon_x}$  (tf·sec<sup>2</sup>/m<sup>2</sup>)

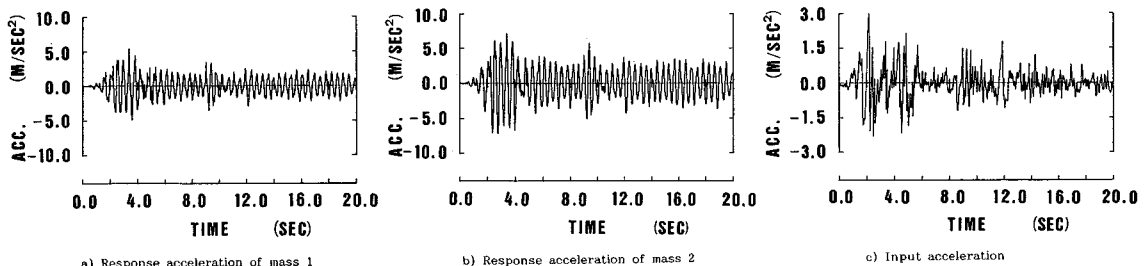


Fig. 3 Response and input acceleration waves.

Table 6 Estimates from the sensitivities and the identified results.

		5 %		10 %		20 %	
		Estimated	Identified	Estimated	Identified	Estimated	Identified
Error in $\ddot{u}_1$	C <sub>1</sub>	9.9977	9.9904	9.9955	9.9808	9.9910	9.9615
	C <sub>2</sub>	10.038	10.048	10.076	10.097	10.152	10.193
	K <sub>1</sub>	3001.3	3001.0	3002.6	3001.9	3005.3	3003.9
	K <sub>2</sub>	2996.9	2997.9	2993.8	2995.9	2987.5	2991.5
Error in $\ddot{u}_2$	C <sub>1</sub>	9.9503	9.9060	9.9006	9.8117	9.8013	9.6219
	C <sub>2</sub>	10.020	10.128	10.039	10.255	10.079	10.508
	K <sub>1</sub>	3002.5	3001.8	3005.0	3003.6	3010.1	3007.5
	K <sub>2</sub>	2994.1	2995.5	2988.3	2990.9	2976.5	2981.5

units: c<sub>i</sub>(tf-sec/m), k<sub>i</sub>(tf/m)

stiffnesses by using the sensitivities and those by the identification method in reference 15) for three different magnitudes of errors which are chosen to be 5 %, 10 % and 20 % of maximum of corresponding responses. It may be said that the effect of noise is very small, as long as it is given by white noise.

## 5. EFFECT OF INPUT NOISE

### (1) Derivation of formulas

An earthquake acceleration record with  $\Delta t = 0.02$  sec is adopted as the input. The true value of the earthquake acceleration  $\ddot{y}_0(t)$  is designated by  $\ddot{y}_0$ , and the measurement noise by  $\Delta \ddot{y}_0(t) = \varepsilon_0 \eta_0(t)$ , a noise contaminated input acceleration is expressed as  $\ddot{y}_i(t) = \ddot{y}_0 + \Delta \ddot{y}_0(t)$ .  $\eta_0(t)$  is a normalized noise whose absolute maximum value is 1. It is assumed that no error exists in the model or no noise is involved in the measured responses. Since analysis is performed by using the input acceleration, computed response reflects this noise and differs from the true response. Therefore, the identified values of unknown parameters  $X$  cannot be free from errors either. Thus,  $X$  can be written as  $X = \bar{X} + \Delta X^{(0)}$ , and the analyzed acceleration is denoted as  $\ddot{z}_i(\ddot{y}_0 + \Delta \ddot{y}_0, \bar{X} + \Delta X^{(0)})$ . Thus, the evaluation function, Eq. (4) becomes

$$J = \int_{t_0}^n \left[ \frac{1}{2} \sum_{i \in A} w_i \{ \ddot{u}_i - \ddot{z}_i(\ddot{y}_0 + \Delta \ddot{y}_0, \bar{X} + \Delta X^{(0)}) \}^2 \right] dt \quad (19)$$

Since  $\Delta \ddot{y}_0$  is noise, its effect will be presumed small. Hence Taylor series expansion of  $\ddot{z}_i$  in Eq. (19) leads to

$$\begin{aligned} \ddot{z}_i(\ddot{y}_0 + \Delta \ddot{y}_0, \bar{X} + \Delta X^{(0)}) &\doteq \ddot{z}_i(\ddot{y}_0, \bar{X}) + \frac{\partial \ddot{z}_i}{\partial \ddot{y}_0} \Delta \ddot{y}_0 + \sum_{j=1}^M \frac{\partial \ddot{z}_i}{\partial X_j} \frac{\partial X_j}{\partial \ddot{y}_0} \Delta \ddot{y}_0 \\ &= \ddot{z}_i(\ddot{y}_0, \bar{X}) + \frac{\partial \ddot{z}_i}{\partial \varepsilon_0} \frac{\partial \varepsilon_0}{\partial \ddot{y}_0} \varepsilon_0 \eta_0 + \sum_{j=1}^M \frac{\partial \ddot{z}_i}{\partial X_j} \frac{\partial X_j}{\partial \varepsilon_0} \frac{\partial \varepsilon_0}{\partial \ddot{y}_0} \varepsilon_0 \eta_0 \\ &= \ddot{z}_i(\ddot{y}_0, \bar{X}) + \frac{\partial \ddot{z}_i}{\partial \varepsilon_0} \varepsilon_0 + \sum_{j=1}^M \frac{\partial \ddot{z}_i}{\partial X_j} \Gamma_{j0} \varepsilon_0 \quad (20) \end{aligned}$$

where  $\Gamma_{j0} = \partial X_j / \partial \varepsilon_0$  is the sensitivity with respect to the input noise.

Substitution of Eq. (20) into Eq. (19) results in

$$\begin{aligned} J &= \int_{t_0}^n \left[ \frac{1}{2} \sum_{i \in A} w_i \left\{ \ddot{u}_i - \ddot{z}_i(\ddot{y}_0, \bar{X}) - \frac{\partial \ddot{z}_i}{\partial \varepsilon_0} \varepsilon_0 - \sum_{j=1}^M \frac{\partial \ddot{z}_i}{\partial X_j} \Gamma_{j0} \varepsilon_0 \right\}^2 \right] dt \\ &= \int_{t_0}^n \left[ \frac{1}{2} \sum_{i \in A} w_i \left\{ -\frac{\partial \ddot{z}_i}{\partial \varepsilon_0} \varepsilon_0 - \sum_{j=1}^M \frac{\partial \ddot{z}_i}{\partial X_j} \Gamma_{j0} \varepsilon_0 \right\}^2 \right] dt \quad (21) \end{aligned}$$

In Eq. (21),  $\varepsilon_0^2$ , the square of the maximum value of the noise, is positive and arbitrary. Therefore, as minimizing Eq. (21) is equivalent to minimizing Eq. (19),  $\Gamma_{k0}$  can be determined from the necessary condition for Eq. (21) to be minimum. By taking differentiation of Eq. (21) with respect to  $\Gamma_{k0}$ , the necessary condition is given as

Table 7 Sensitivities with respect to noise in input acceleration.

$\frac{\partial C_1}{\partial \varepsilon_0}$	$\frac{\partial C_2}{\partial \varepsilon_0}$	$\frac{\partial K_1}{\partial \varepsilon_0}$	$\frac{\partial K_2}{\partial \varepsilon_0}$
-1.7749	0.59407	-66.060	64.423

units:  $\frac{\partial C_i}{\partial \varepsilon_0}$  (tf·sec<sup>3</sup>/m<sup>2</sup>),  $\frac{\partial K_i}{\partial \varepsilon_0}$  (tf·sec<sup>2</sup>/m<sup>2</sup>)

Table 8 Estimates from the sensitivities and the identified results.

		5 %		10 %		20 %	
		Estimated	Identified	Estimated	Identified	Estimated	Identified
Error in $\dot{y}_0$	$C_1$	9.7338	9.7453	9.4675	9.5195	8.9351	9.1720
	$C_2$	10.089	10.134	10.178	10.351	10.356	10.997
	$K_1$	2990.1	2989.9	2980.2	2979.5	2960.4	2957.7
	$K_2$	3009.7	3009.7	3019.3	3019.4	3038.7	3038.0

units:  $C_i$  (tf·sec/m),  $K_i$  (tf/m)

$$\frac{\partial J}{\partial \Gamma_{k0}} = \varepsilon_0^2 \int_{t_0}^{t_1} \sum_{i \in A} w_i \left\{ -\frac{\partial \ddot{z}_i}{\partial \varepsilon_0} - \sum_{j=1}^M \frac{\partial \ddot{z}_i}{\partial X_j} \Gamma_{j0} \right\} \left( -\frac{\partial \ddot{z}_i}{\partial X_k} \right) dt = 0 \quad (22)$$

Rearrangement of the above equation yields

$$\sum_{j=1}^M \left[ \int_{t_0}^{t_1} \left( \sum_{i \in A} w_i \frac{\partial \ddot{z}_i}{\partial X_j} \frac{\partial \ddot{z}_i}{\partial X_k} \right) dt \right] \Gamma_{j0} = - \int_{t_0}^{t_1} \left( \sum_{i \in A} w_i \frac{\partial \ddot{z}_i}{\partial \varepsilon_0} \frac{\partial \ddot{z}_i}{\partial X_k} \right) dt \quad (k=1, \dots, M) \quad (23)$$

$\partial \ddot{z}_i / \partial \varepsilon_0$  in the right hand side of Eq. (23) can be computed in the following manner. The response of the structure due to the noise contaminated input  $\ddot{y}_0 = \ddot{y}_0 + \varepsilon_0 \eta_0(t)$  is

$$M\ddot{z} + C\dot{z} + Kz = f(\ddot{y}_0 + \varepsilon_0 \eta_0) \quad (24)$$

where  $f$  is an  $N \times 1$  vector. When compared with Eq. (1),  $Q(t) = f(\ddot{y}_0 + \varepsilon_0 \eta_0)$ . Partial differentiation of Eq. (24) with respect to  $\varepsilon_0$  gives

$$M \frac{\partial \ddot{z}}{\partial \varepsilon_0} + C \frac{\partial \dot{z}}{\partial \varepsilon_0} + K \frac{\partial z}{\partial \varepsilon_0} = f \eta_0 \quad (25)$$

By solving the above equation,  $\partial \ddot{z}_i / \partial \varepsilon_0$  on the right side of Eq. (23) can be obtained. Eq. (23) is a set of linear simultaneous equations with respect to the unknown variable  $\Gamma_{j0}$ , which can be solved without so much difficulty.

## (2) An example

To verify the theory induced in the preceding section, the model in Fig. 1 is used again together with the normalized band-limited white noise  $\eta_0(t)$  in Fig. 2. Table 7 gives the sensitivity of the identified parameters with respect to the noise in the input wave. The El Centro wave is superimposed by three different levels of white noise to generate noise contaminated data. The magnitude of noise  $\varepsilon_0$  are 5 %, 10 %, and 20 % of maximum of the El Centro wave. Table 8 compares the estimated values of  $X$  obtained by the sensitivity analysis and those by the identification method in reference 15). Comparatively good agreements between these data suggest that the sensitivity of unknown parameters with respect to the input noise is highly precise. Identified values which are very close to the true values also imply that the effect of the white noise on the identification is very small.

## 6. CONCLUSIONS

The formulas have been developed, which determine the effects (sensitivities) of model error and measurement noise on the parameter estimation in structural identification problems. This problem is a sort of optimization, in which the sensitivity of a parameter with respect to error or noise is fundamentally the same as sensitivity of the optimum solution.

From the example problems, one can conclude the following.

(1) Compared with noise in measurements, model error has larger effect on the estimated value of a parameter. Using the sensitivity, the effect of an error on the estimated value of the parameter can be calculated with considerable accuracy.

(2) Error in the initial velocity has little effect on the estimated value of a parameter, while the effect of an error in the initial displacement is not necessarily small. Though the same calculation is made on different models, the same tendency is also observed. Given the fact that both the initial displacement and the initial velocity attenuate in a short period of time due to the damping characteristic of the system in the



response analysis, the reason for the difference between the effects of the displacement and velocity is not clear.

(3) Noise involved in the response and input measurements has little effect on the estimated value of a parameter. It may be due to the reason that the white noise does not have dominant frequency characteristics.

(4) In all examples, the existence of an error or noise is assumed at only one point. If there are model errors and noises at more than one place, the estimated values may deviate more from the true values due to the interaction of these errors. There are also a few contradictory cases where an estimated value from sensitivity analysis is larger than the true value while the corresponding identified result is smaller than it. This contradiction occurs, when the sensitivity is extremely small, presumably as a result of calculation error including the effect of trapezoidal integration. The reason that an estimate of the damping coefficient is less accurate, is among others, due to the substantially smaller order of the parameter compared with that of the stiffness. This is expected to improve, however, if magnitudes of parameters are balanced or the technique of variable transformation is introduced, as is considered in structural optimization<sup>16)</sup>.

It is also confirmed that the accuracy of estimation improves when an identification is performed with the measured and calculated displacement in stead of using measured and calculated accelerations. Only one example of white noise is used in this paper, but there are no conspicuous differences even if the sensitivity is calculated using other forms of white noise as long as the system has two degrees of freedom. Though no discussion is made on noise with dominant frequency characteristics, the methods proposed here are applicable whether the noise has a particular feature or not. It has been further confirmed that the effect of an white noise on the identification is small enough from the engineering point of view, even if it has an SN ratio of 100 %. The methods developed above can provide a good estimate of the effect with considerable accuracy.

In this study, the response  $\tilde{z}_i$  is expanded around, the true value of the unknown parameter, to obtain its sensitivity with respect to noise or error. Though the true value is usually unknown, all the theoretical formulas introduced above can be still utilized by adopting the estimated value  $\bar{X}$  based on data containing error or noise as the true value.

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