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A GENERAL CONSTITUTIVE MODEL OF CRACKED REINFORCED CONCRETE PLATES

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Cracked reinforced concrete subjected to the non-proportional membrane forces exhibits complex displacement characteristics such as both normal and tangential displacements at cracks of several different orientations. In this paper, a constitutive equation to describe the behavior of these displacement fields are constructed as a mixed model which is a combination of some basic displacement fields using the concept of the damage mechanics. The model can satisfactorily solve the nonlinear behavior of cracked reinforced concrete which includes compressive resistance of concrete parallel to cracks, tension stiffening, and shear resistance of concrete and the dowel action of reinforcement on the crack plane and is formulated so that the material stiffness is directly applicable to nonlinear FEM analysis. Its applicability is illustrated by prediction of some experiments.

Keywords : *aggregate interlock, dowel action, tension stiffening effect, stress reduction and reinforcement tensors, constitutive model, cracked reinforced concrete*

1. INTRODUCTION

Most of the reinforced concrete structures may be in the cracked state under the loaded condition. After crack initiation the following some important factors and the coupling action between these factors result in the nonlinear behavior of cracked reinforced concrete.

(1) Deterioration of compressive stiffness of concrete in a direction parallel to cracks.

(2) Tensile resistance of concrete normal to cracks, so called tension stiffening, which represents the tensile stiffness of concrete between cracks through bond between reinforcement and surrounding concrete.

(3) Shear stiffness of cracked concrete due to aggregate interlocking and the dowel action of reinforcement on the crack plane.

(4) Tensile resistance of reinforcement.

Due to the practical importance for the design of RC structures under applied loads, a large research effort has been made for solving these nonlinear behavior and recent developments have been strongly influenced by the application of the finite element method which is considered to be a very powerful means to simulate nonlinearities of various kinds in calculations. However, it requires a simple, suitable and general constitutive model to represent accurately the behavior of nonlinearities of reinforced concrete structures. In the authors' knowledge, the attention of recent research efforts is paid in the unidirectional behavior of reinforced concrete or simple models in which the structure is assumed only in basic displacement field and the smeared cracks occur only in unidirection. All of these models are not

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satisfactory to predict the behavior of reinforced concrete structures which are in general plane state and the coupling actions between the mentioned four factors of cracked reinforced concrete.

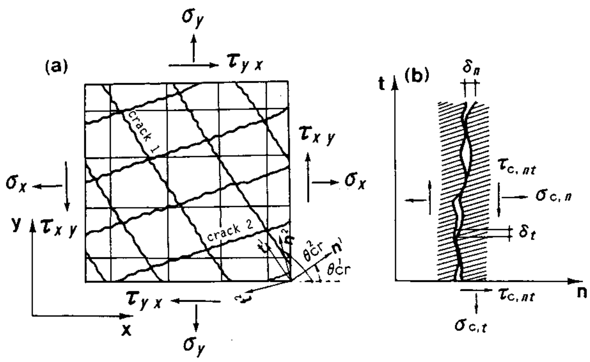

Fig. 1 Cracked reinforced concrete plate.

For the displacement field of the thin plate structures or thin curved structures with cracking as shown in Fig. 1 (a) (for convenience, only two directional cracks are shown), it is recognized that the behavior of cracked reinforced concrete can be divided into two basic cracking modes. The first one is the case that the cracking only results in a displacement field normal to the crack surface without the shear displacement along cracking. Due to the bond action between reinforcement and surrounding concrete, the tension stiffness of concrete is still existent. Another case is that the tangential displacement along cracking is induced simultaneously with normal displacement, and concrete stress is compressed. It can be considered a portion near by the crack plane if the structure is subjected to the nonproportional loading or the orientation of loading are reversed. The shear transfer test specimens such as under push-off, or pull-off tests can be thought to be in the displacement field in this mode. For convenience, we have named the former as the S-mode (frictionless separation mode), the latter as the F-mode (frictional contact mode)[1),2)]. These two cracking modes can be distinguished by the following stress and displacement conditions (Fig. 1 (b)).

the S-mode, $\sigma_{c,n} \geqq 0$, $\tau_{c,nt}=0$, $\delta_t=0$

the F-mode, $\sigma_{c,n}<0$, $\tau_{c,nt} \neq 0$, $\delta_t \neq 0$

where, $\sigma_{c,n}$ and $\tau_{c,nt}$ denote the normal and shear concrete stresses, and δ_t denotes the slip displacement along the cracks.

On the other hand, it is recognized that the displacement field of actual reinforced concrete which exhibits both frictional displacement and non-frictional displacement to the normal direction of cracks may be a combination of these two basic displacement fields. Based on this consideration, the mixed mode formulation is developed for actual reinforced concrete by means of the concept of the damage mechanics in which the total strain is decomposed into elastic and damage strain.

2. DEFINITION OF THE STRESS REDUCTION AND REINFORCEMENT TENSORS

Damage strain has the relation to the damage tensors in nature, which express the rate of damage of material from the intact condition. The damage tensors express material nonlinearity in explicit and simple form which enables the straightforward construction of the nonlinear constitutive equations. Similar to the damage mechanics concept, the stress reduction tensor Ω^D_{ijpq} of the fourth rank may be defined in the following form to write stress reduction from the intact condition.

$$\Delta\sigma_{ij} = -\Omega^D_{ijpq} D^c_{pqmn} \epsilon_{mn} \quad \cdots\cdots (1\cdot a)$$

where $\Delta\sigma_{ij}$ is the reduction of three nominal stress due to damage in the solid from the intact condition, ϵ_{mn} is the strain tensor of second rank, and D^c_{pqmn} is the fourth-order material-stiffness tensor for linearly elastic concrete. As $\Omega_{ijpq}=\Omega_{jipq}=\Omega_{jiqp}=\Omega_{ijqp}$, matrix expression for Eq. (1·a) is written as

$$\{\Delta\sigma\} = -[\Omega]_D [D]_c \{\epsilon\} \quad \cdots\cdots (1\cdot b)$$

in which, $[D]_c$ is the material-stiffness matrix for a linearly elastic isotropic concrete.

Similarly, the reinforcement tensor Ω^R_{ijpq} may be written in terms of stress increase from the intact condition due to the reinforcement in concrete as

$$\Delta\sigma_{ij} = \Omega^R_{ijpq} D^c_{pqmn} \epsilon_{mn} \quad \cdots\cdots (2\cdot a)$$

and, Eq. (2·a) is written in matrix form as

$$\{\Delta\sigma\}=[\Omega]_R[D]_c\{\epsilon\} \quad (2\cdot b)$$

Then the general constitutive equation for reinforced concrete can be derived as

$$\{\sigma\}=([I]-[\Omega]_{D1}-[\Omega]_{D2}-\cdots+[\Omega]_{R1}+[\Omega]_{R2}+\cdots)[D]_c\{\epsilon\} \quad (3)$$

In general, the fundamental assumption of the constitutive model is that the total strain is composed of an elastic strain $\{\epsilon\}_e$, and a damage strain $\{\epsilon\}_D$

$$\{\epsilon\}=\{\epsilon\}_e+\{\epsilon\}_D \quad (4)$$

Furthermore, the damage strain may be decomposed into a crack strain $\{\epsilon\}_{cr}$ due to cracking and a plastic strain $\{\epsilon\}_p$ due to compressive stress.

$$\{\epsilon\}_D=\{\epsilon\}_{cr}+\{\epsilon\}_p \quad (5)$$

For multiple cracking system, the crack strain is considered to be decomposed into several contributions as

$$\{\epsilon\}_{cr}=\{\epsilon\}_{cr}^1+\{\epsilon\}_{cr}^2+\cdots+\{\epsilon\}_{cr}^i+\cdots \quad (6)$$

Stress reduction due to the damage in concrete from the intact condition is written with the damage strain $\{\epsilon\}_D$, in the following form as well,

$$\{\Delta\sigma\}=-[D]_c\{\epsilon\}_D \quad (7)$$

Substitution of Eq. (7) into Eq. (1·b) yields

$$\{\epsilon\}_D=[D]_c^{-1}[\Omega]_D[D]_c\{\epsilon\} \quad (8)$$

Hence, if $\{\epsilon\}_{cr}$ or stress reduction $\{\Delta\sigma\}$ is obtained, the stress reduction tensor $[\Omega]_{D1}$ due to cracking can be derived. In other words, if $\{\epsilon\}_{cr}$ is represented in terms of total strain in such a way that

$$\{\epsilon\}_{cr}=[\Lambda]\{\epsilon\} \quad (9)$$

then the stress reduction tensor due to the cracking strain is obtained as

$$[\Omega]_{D1}=[D]_c[\Lambda][D]_c^{-1} \quad (10)$$

Using the same argument, if we have the relation of $\{\epsilon\}_p=[\varkappa]\{\epsilon\}$, another stress reduction tensor $[\Omega]_{D2}$ due to a plastic strain can be written as

$$[\Omega]_{D2}=[D]_c[\varkappa][D]_c^{-1} \quad (11)$$

Similarly if the stress increase due to reinforcement is given in terms of total strain in such a way that

$$\{\Delta\sigma\}=[D]_s\{\epsilon\} \quad (12)$$

where $[D]_s$ is the material-stiffness matrix for reinforcement.

Then, the reinforcement tensor is obtained as

$$[\Omega]_R=[D]_s[D]_c^{-1} \quad (13)$$

3. REVIEW OF TWO BASIC CONSTITUTIVE MODELS OF DISPLACEMENTS

In the previous works[3),4)], two basic models, which were named as S-mode and F-mode, respectively, have been developed to predict two important factors of so called tension stiffening and interface shear transfer problems. In this section, these two models will be reviewed and the damage due to cracking are derived as stress reduction tensors by the definition based on the concept of damage mechanics.

a) S-mode : In the Reference 3), the crack strain of i-th crack system in the two dimensional stress field has been derived as

$$\{\epsilon\}_{cr}^i=[S]^i\{\sigma\} \quad (14)$$

where $\{\sigma\}$=applied stress vector with three components σ_x, σ_y and τ_{xy}, and $[S]^i$=crack tension stiffening matrix which relates the crack strains and the applied stresses, respectively. Then the following relation for the crack tension stiffening matrix is derived as

$$[S]=\sum_{i=1}^{m}[S]^i \quad (m=\text{number of crack orientations}) \quad (15)$$

Once we obtain the form of Eq. (15), it is possible to construct a stress reduction tensor due to cracking. As applied stress $\{\sigma\}$ equals the sum of $[D]_s\{\epsilon\}$ and $[D]_c\{\epsilon\}_e$,

$$\{\epsilon\}_{cr}=[S][D]_c\{\epsilon\}_e+[S][D]_s\{\epsilon\} \qquad (16)$$

and as crack strain $\{\epsilon\}_{cr}=\{\epsilon\}-\{\epsilon\}_e-\{\epsilon\}_p$, referring to Eqs. (4) and (5),

$$([I]-[S][D]_s-[D]_c^{-1}[\Omega]_{D2}[D]_c)\{\epsilon\}=([S][D]_c+[I])\{\epsilon\}_e \qquad (17)$$

here, substitution of $[\kappa]=[D]_c^{-1}[\Omega]_{D2}[D]_c$, which will be discussed latter, is made according to the Eq. (11). By substituting $\{\epsilon\}_e$ of Eq. (17) to Eq. (16) the matrix $[\Lambda]$ can be derived as follows

$$[\Lambda]=[S]([D]_s+([S]+[D]_c^{-1})^{-1})([I]-[S][D]_s-[D]_c^{-1}[\Omega]_{D2}[D]_c) \qquad (18)$$

Then the stress reduction tensor due to crack strain can be obtained as Eq. (10).

Up to now the constitutive equation of Eq. (3) for S-mode can be rewritten as

$$\{\sigma\}=([I]-[\Omega]_{D1}^S-[\Omega]_{D2}^S+[\Omega]_R^S)[D]_c\{\epsilon\}=([I]-[\Omega]_S)[D]_c\{\epsilon\} \qquad (19)$$

where subscript or superscript S refers to the S-mode.

b) F-mode : When the concrete has differential tangential movement along the two surfaces at the cracks, the shear dilatancy and shear friction give rise to complicated problems, and the relation between crack opening δ_n and crack slip δ_t, versus shear stress $\tau_{c,nt}$, and normal stress $\sigma_{c,n}$, transmitted across the crack due to interlock, are still in argument. In the past, Yoshikawa *et al.*[4),5)] have given the following equation between $(\sigma_{c,n}, \tau_{c,nt})$ and (δ_n, δ_t) in the local coordinate system (n, t) (Fig. 1). Here, we consider an arbitrary i-th crack system without the loss of generality, however, the notation i is neglected in the formulations for convenience.

$$\begin{Bmatrix} d\sigma_{c,n} \\ d\tau_{c,nt} \end{Bmatrix}=k_t\begin{bmatrix} 1/(\mu_f\beta_d) & -1/\mu_f \\ -(1-\rho)/\beta_d & 1 \end{bmatrix}\begin{Bmatrix} d\delta_n \\ d\delta_t \end{Bmatrix} \qquad (20)$$

where $\rho=\mu_f\beta_d k_n/k_t$, and the four coefficients k_t, k_n, β_d and μ_f are the shear stiffness, normal stiffness, dilatancy ratio and frictional coefficient, respectively, which are identified in the Reference 4).

We will now develop the constitutive equation for the frictional displacement field, Eq. (20) is rewritten as the relation of total displacement and total stresses in the following form,

$$\begin{Bmatrix} \sigma_{c,n} \\ \tau_{c,nt} \end{Bmatrix}=2\,l_c\begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix}\begin{Bmatrix} \epsilon_n \\ \gamma_{nt} \end{Bmatrix}_{cr}=[\Delta]\begin{Bmatrix} \epsilon_n \\ \gamma_{nt} \end{Bmatrix}_{cr} \qquad (21\cdot a)$$

where

$$\left.\begin{aligned} \Delta_{11}&=\left[\int k_t/(\mu_f\beta_d)d\delta_n\right]\Big/\delta_n, \quad \Delta_{12}=\left[-\int (k_t/\mu_f)d\delta_t\right]\Big/\delta_t, \\ \Delta_{21}&=\left[\int [k_t(\rho-1)/\beta_d]d\delta_n\right]\Big/\delta_n, \quad \Delta_{22}=\left[\int k_t d\delta_t\right]\Big/\delta_t, \end{aligned}\right\} \qquad (21\cdot b)$$

where $2\,l_c$ is the crack spacing. Making the Eq. (21・a) in inverse form and enlarging into three components, we have

$$\begin{Bmatrix} \epsilon_n \\ \epsilon_t \\ \gamma_{nt} \end{Bmatrix}_{cr}=[\Delta]^{-1}\begin{Bmatrix} \sigma_{c,n} \\ \sigma_{c,t} \\ \tau_{c,nt} \end{Bmatrix} \qquad (21\cdot c)$$

We transform it for the relation in the global coordinate system (x, y) (Fig. 1) using transformation matrices $[T]_1$ and $[T]_2$ for stresses and engineering strains as

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}_{cr}=[T]_2[\Delta]^{-1}[T]_1^{-1}\begin{Bmatrix} \sigma_{c,x} \\ \sigma_{c,y} \\ \tau_{c,xy} \end{Bmatrix} \qquad (21\cdot d)$$

that is

$$\{\epsilon\}_{cr}^i=[F]^i\{\sigma\}_c, \quad [F]^i=[T]_2[\Delta]^{-1}[T]_1^{-1} \qquad (21\cdot e)$$

Then, similar to the case of the S-mode, the shear transfer matrix which relates the crack strain and stress vector of concrete transmitted across the cracks due to interlock in the global coordinates can be derived as

$$\{\epsilon\}_{cr}=\sum_{i=1}^{m}[F]^i\{\sigma\}_c \quad \text{and} \quad [F]=\sum_{i=1}^{m}[F]^i \quad (m=\text{number of crack orientations}) \qquad (22)$$

On the other hand, the concrete portion which does not contain cracks has elastic rigidity and the stress equilibrium at crack system gives the relation,

$$\{\sigma\}_c=[D]_c\{\epsilon\}_e \quad (23)$$

By comparing the Eq. (22) and Eq. (23) and making use of the relation of $\{\epsilon\}_e=\{\epsilon\}-\{\epsilon\}_{cr}-\{\epsilon\}_p$ from Eqs. (4) and (5), we have

$$\{\epsilon\}_{cr}=[F][D]_c(\{\epsilon\}-\{\epsilon\}_{cr}-\{\epsilon\}_p) \quad (24)$$

Then, the relation between crack strain and total strain can be written as

$$\{\epsilon\}_{cr}=([I]+[F][D]_c)^{-1}[F][D]_c([I]-[D]_c^{-1}[\Omega]_{D2}[D]_c)\{\epsilon\} \quad (25)$$

Hence, referring to Eq. (10), the stress reduction tensor due to cracking in the F-mode is derived as

$$[\Omega]_{D1}^F=([D]_c^{-1}+[F])^{-1}[F]([I]-[\Omega]_{D2}) \quad (26)$$

For the F-mode, the constitutive Eq. (3) can be written as

$$\{\sigma\}=([I]-[\Omega]_{D1}^F-[\Omega]_{D2}^F+[\Omega]_R^F)[D]_c\{\epsilon\}=([I]-[\Omega]_F)[D]_c\{\epsilon\} \quad (27)$$

where subscript or superscript F refers to F-mode.

4. COMPRESSIVE BEHAVIOR OF CRACKED REINFORCED CONCRETE

For plain concrete structures, cracking is a localized and directional fracture since the stiffness and strength characteristics in directions parallel to the cracking plane are not directly affected. However, for RC structures in which the bond action exists between the reinforcement and the surrounding concrete, tensile cracks cause damage in concrete which decreases its strength and stiffness with initiating and propagating of the microcracks in concrete. This damage is very complicated due to the complex load type, geometry and boundary conditions, reinforcement, and bond characteristics. In the direction parallel to the cracking, the plastic yielding and the crushing type of fracture can occur. For convenience, the stress reduction tensor $[\Omega]_{D2}$ is used to consider the damage. The concrete compressive stress $\sigma_{c,t}$ is related to compressive strain ϵ_t parallel to cracks may be expressed by a second degree parabola as,

$$\frac{\sigma_{c,t}}{f_c}=2\frac{\epsilon_t}{\epsilon_0}-\frac{1}{\lambda_{con}}\left(\frac{\epsilon_t}{\epsilon_0}\right)^2 \quad (28)$$

in which f_c is the compressive strength of uncracked concrete, ϵ_0 is the strain that corresponds to f_c and λ_{con} is the effectiveness factor to consider the deterioration of compressive strength of concrete. This factor is still in argument, however, from the authors' experience, for the case of shear loading type, Vecchio and Collins' suggestion[6] in which the reduction factor is increasing with the increase of tensile strain ϵ_n in the transverse direction seems to be reasonable, although it should be modified for compression-tension loading type. Collins and Vecchio showed that a suitable formula for λ_{con} is :

$$\lambda_{con}=\frac{1}{0.85-0.27(\epsilon_n/\epsilon_t)} \quad (\leq 1) \quad (29)$$

where both variable ϵ_n and material constant ϵ_0 are given positive values. The stress reduction from the intact condition of concrete parallel to cracks can be written as

$$\Delta\sigma=-\epsilon_t^2/(2\lambda_{con}\epsilon_0)E_c \quad (30\cdot a)$$

where $E_c=2f_c/\epsilon_0$

By transforming Eq. (30·a) into global coordinate system, the matrix expression of the stress reduction can be written in the form of

$$\{\Delta\sigma\}=-[\Phi]\{\epsilon\},\quad [\Phi]=\begin{bmatrix} s^4 & c^2s^2 & -cs^3 \\ s^2c^2 & c^4 & -c^3s \\ -cs^3 & -c^3s & c^2s^2 \end{bmatrix}\frac{E_c\epsilon_t}{2\lambda_{con}\epsilon_0},\quad c=\cos\theta_{cr},\quad s=\sin\theta_{cr} \quad (30\cdot b)$$

Finally, $[\Omega]_{D2}$ can be simply derived as

$$[\Omega]_{D2}=[\Phi][D]_c^{-1} \quad (31)$$

5. TENSILE RESISTANCE AND DOWEL ACTION OF REINFORCEMENT

Although recent researches have pointed that the constitutive equation of reinforcement is different from the one of plain steel due to the bond action between the reinforcement and the surrounding concrete, for convenience, reinforcement is considered elastic, perfectly plastic. After yielding, perfect plasticity is assumed with the tangent modulus set to zero. Having specified elastic moduli of materials, the stiffness matrix for two orthogonal reinforcement can be written as

$$[D]_{st}=\begin{bmatrix} p_x E_s & 0 & 0 \\ 0 & p_y E_s & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdots\cdots (32)$$

where E_s is the reinforcing modulus of elasticity, and p_x and p_y are the reinforcing ratios in x and y directions. The opening of the crack can be controlled by the component of reinforcement that crosses the shear plane at right angles, such bars will also be subjected to shear displacement, hence, a certain amount of shear resistance can be directly transmitted by dowel action.

Based on the observation of experiments, with near zero slip between the crack faces, shear resistance due to dowel action is very small, and most of the shear resistance comes from the direct bearing of aggregate interlocking. As shear slip and separation increases, an increasing amount of shear resistance is provided by dowel action. However, at shear approaching ultimate, the tensile resistance in the reinforcement will approach its yield point, if secondary failure of the concrete near the shear plane is prevented, the dowel action contribution to shear resistance will consequently reduce, at failure, the shear stiffness contributed by dowel action will vanish. Depending on these facts, the shear stiffness due to the dowel action G_d may be described by the following exponential function,

$$G_d = G_{dm}\, e^{-\beta\sigma_t} \cdots\cdots (33)$$

where G_{dm} is maximum shear stiffness due to dowel action and is represented as

$$G_{dm} = \alpha p_n f_y \cdots\cdots (34)$$

and, α and β are material constants, f_y is the yield strength of reinforcement and p_n is an equivalent reinforcement ratio in the direction normal to a crack plane. Here it is assumed to be expressed as

$$p_n = p_x \cos^2\theta_{cr} + p_y \sin^2\theta_{cr} \cdots\cdots (35)$$

Similar to the derivation of Eq. (30), the matrix expression of shear stiffness due to dowel action in global coordinates can be derived as

$$[D]_{sd}=\begin{bmatrix} 4s^2c^2 & -4s^2c^2 & -2sc(c^2-s^2) \\ -4s^2c^2 & 4s^2c^2 & 2sc(c^2-s^2) \\ -2sc(c^2-s^2) & 2sc(c^2-s^2) & (c^2-s^2)^2 \end{bmatrix} G_d \cdots\cdots (36)$$

Then, for F-mode, the matrix $[D]_s$ can be written as the sum of matrices $[D]_{st}$ and $[D]_{sd}$, that is,

$$[D]_s = [D]_{st} + [D]_{sd} \cdots\cdots (37)$$

while existence of dowel action is considered to be negligible in S-mode where there is almost no frictional displacement ($[D]_s=[D]_{st}$).

6. CONSTITUTIVE EQUATIONS OF CRACKED RC ELEMENTS FOR THE MIXED MODE

In stress conditions when the crack spacing is comparatively wide and frictional displacement at cracks occurs, the mixed mode of displacement takes place. In other words, the concrete close to the cracks is stressed in compression to the normal direction to crack surface while the region away from the crack is still stressed in tension. Hence it is considered that the constitutive equation for stress fields of this kind is expressed by the combination of Eq. (19) and Eq. (27) as the mixed mode.

For the cracked reinforced concrete element, the shear forces and normal forces are supposed to be applied. Taking out a representative portion between two cracks as shown in Fig. 2(a), it is possible to

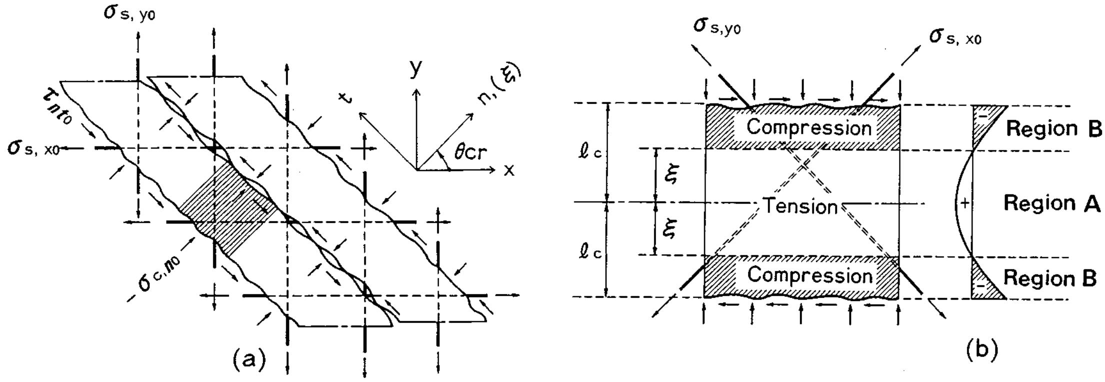

Fig. 2 Stress Condition at a Crack in the Mixed Mode Displacement Condition.

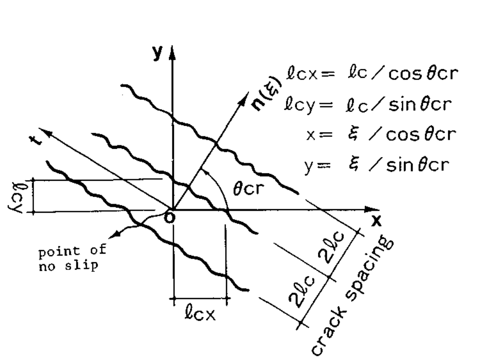

Fig. 3 Crack Spacings of Parallel Cracks.

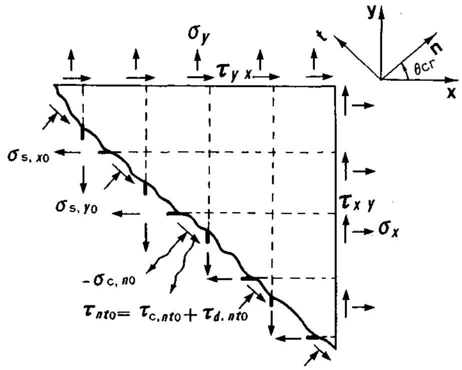

Fig. 4 Free-Body Diagram for Cracked Element.

separate the concrete into two regions, one in which the compressive force is working to the normal direction for crack surfaces and the other region where the tensile stress is working to the normal direction for cracked surfaces as shown in Fig. 2(b) for unidirectional crack system.

The transition point of the normal stress from the minus sign (compression) to the plus sign (tension) can be derived as follows. In the previous work[3], we have derived the distribution of the concrete stresses between two adjacent cracks, which is expressed as

$$\sigma_{c,x}=\frac{\sigma_x}{(1+np_x)}+\left(\frac{np_x\sigma_x}{1+np_x}-\sigma_{s,x0}\right)\frac{\cosh(x/b_{cx})}{\cosh(l_{cx}/b_{cx})} \qquad (38\cdot a)$$

$$\sigma_{c,y}=\frac{\sigma_y}{(1+np_y)}+\left(\frac{np_y\sigma_y}{1+np_y}-\sigma_{s,y0}\right)\frac{\cosh(y/b_{cy})}{\cosh(l_{cy}/b_{cy})} \qquad (38\cdot b)$$

$$\sigma_{c,xy}=\tau_{xy} \qquad (38\cdot c)$$

where the notations $2\,l_c$ and b_c with suffix x, y denote the crack spacings and the bond characteristics coefficients in the x and y directions (shown in Fig. 3), respectively, n is a ratio of Young's modulus of steel to that of concrete, and $\sigma_{s,x0}$ and $\sigma_{s,y0}$ are the stresses in the orthogonal reinforcement at the crack, respectively. By considering the equilibrium equations of the free body shown in Fig. 4, we have

$$\sigma_{s,x0}=\sigma_x+\sigma_{c,n0}+(\tau_{xy}+\tau_{nt0})\tan\theta_{cr} \qquad (39\cdot a)$$

$$\sigma_{s,y0}=\sigma_y+\sigma_{c,n0}+(\tau_{xy}-\tau_{nt0})\cot\theta_{cr} \qquad (39\cdot b)$$

where $\sigma_{c,n0}$ is the normal stress of concrete and τ_{nt0} is the shear stress which is sum of the shear resistance $\tau_{c,nt0}$ due to aggregate interlocking and the shear resistance $\tau_{d,nt0}$ due to the dowel action of reinforcement on the crack plane. These stresses can be obtained by F-mode in the calculation procedure.

Expressing the state of concrete stress in the element in $(n,\ t)$ coordinate system, the concrete stress component normal to the crack plane can be obtained as

$$\sigma_{c,n}=c^2\left\{\frac{\sigma_x}{1+np_x}+\left(\frac{np_x\sigma_x}{1+np_x}-\sigma_{s,x0}\right)\frac{\cosh(x/b_{cx})}{\cosh(l_{cx}/b_{cx})}\right\}+$$
$$s^2\left\{\frac{\sigma_y}{1+np_y}+\left(\frac{np_y\sigma_y}{1+np_y}-\sigma_{s,y0}\right)\frac{\cosh(y/b_{cy})}{\cosh(l_{cy}/b_{cy})}\right\}+2\,sc\tau_{xy} \qquad (40)$$

The location where concrete stress $\sigma_{c,n}$ changes from compression to tension is written as the following equation.

$$c^2\left(\sigma_{s,x0}-\frac{np_x\sigma_x}{1+np_x}\right)\frac{\cosh(\xi/(b_{cx}c))}{\cosh(l_{cx}/b_{cx})}+s^2\left(\sigma_{s,y0}-\frac{np_y\sigma_y}{1+np_y}\right)\frac{\cosh(\xi/(b_{cy}s))}{\cosh(l_{cy}/b_{cy})}$$
$$=\frac{c^2\sigma_x}{1+np_x}+\frac{s^2\sigma_y}{1+np_y}+2\,sc\tau_{xy} \qquad (41\cdot a)$$

where the substitution of $x=\xi/\cos\theta_{cr}$ and $y=\xi/\sin\theta_{cr}$ is made by referring to Fig. 3. For unidirectional reinforcement which is normal to the crack plane, ξ can be analytically derived as

$$\xi=\cosh^{-1}\left[\frac{\sigma_{s,n0}-np_n\sigma_{c,n0}}{\sigma_{s,n0}-\sigma_{c,n0}}\cosh(l_c/b_c)\right] \qquad (41\cdot b)$$

where $\sigma_{s,n0}$ is the stress in the reinforcement at the cracks.

If $\xi\leqq 0$, there exists no tension zone and only the F-mode exists. However, if $0<\xi<l_c$ the stress state is in the mixed mode. Moreover, if $\xi\geqq l_c$ the displacement is in S-mode, and there exists no shear slip on the crack plane.

The portion where concrete is stressed in tension to the ξ direction, has to be treated as having slip between steel and concrete and its situation is exactly the same as the case of the S-mode. We separate the concrete portion along the ξ direction to the region B where $\sigma_{c,n}$ is in compression and, to the regions A where $\sigma_{c,n}$ is in tension. The concrete stress is zero at the boundary between A and B. Hence it can be considered that the constitutive equation of Eq. (19) which predicts the tension stiffness is applicable in the region A.

For the region B, the stress and displacement field is mainly influenced by aggregate interlocking and the dowel action on the crack plane and the tension stiffening effect never occurs because the plane stress field of concrete is stressed in compression so that the constitutive equations developed for the F-mode are considered to be applicable. However, the slipping out of a bar at the region A contributes to the crack opening of the region B, the total crack opening δ_n should be the sum of the contributions from the region A and from the region B. The total crack slip δ_t is also contributed by the two regions.

Based on these consideration, we develop the constitutive equations for the mixed mode. Obviously, at the boundary of two regions, the stress equilibrium should be satisfied and also the stresses at the boundary of two regions should correspond to the applied stresses. Hence $\{\sigma\}_A=\{\sigma\}_B=\{\sigma\}$, and the total elongation of the portion between two cracks is the sum of the elongation of each region of A and B, and the average strain of the total portion $\{\epsilon\}$ is written as

$$\begin{Bmatrix}\epsilon_n\\ \epsilon_t\\ \gamma_{nt}\end{Bmatrix}=\begin{bmatrix}\eta&0&0\\0&\eta&0\\0&0&\eta\end{bmatrix}\begin{Bmatrix}\epsilon_n\\ \epsilon_t\\ \gamma_{nt}\end{Bmatrix}_A+\begin{bmatrix}1-\eta&0&0\\0&1-\eta&0\\0&0&1-\eta\end{bmatrix}\begin{Bmatrix}\epsilon_n\\ \epsilon_t\\ \gamma_{nt}\end{Bmatrix}_B=[\eta]\{\epsilon\}_A+[\zeta]\{\epsilon\}_B \qquad (42)$$

where η is the fraction to the crack spacing of the length of the area where the concrete is in compression along the ξ direction and frictionless mode is predominant. This is written in the linear case as $\eta=\xi/l_c$. For multiple crack systems, the average values of ξ and l_c to each crack of different orientation are adapted. As we already have the constitutive equations (19) and (27) for the regions A and B, Eq. (42) is rewritten as

$$\{\epsilon\}=[\eta][D]_c^{-1}([I]-[\Omega]_S)^{-1}\{\sigma\}+[\zeta][D]_c^{-1}([I]-[\Omega]_F)^{-1}\{\sigma\} \qquad (43\cdot a)$$

Hence, $\{\sigma\}=([I]-[\Omega])[D]_c\{\epsilon\} \qquad (43\cdot b)$

where, $\{\Omega\}=[I]-([\eta]([I]-[\Omega]_S)^{-1}+[\zeta]([I]-[\Omega]_F)^{-1})^{-1} \qquad (43\cdot c)$

The relationship of the F-mode, the S-mode and the mixed mode can be expressed as **Table 1**. It should be noted that we can not have the frictional mode from the beginning since crack initiation is always to the principal tensile direction and the first mode should be the frictionless mode. After a small crack width is formed, then the frictional mode or the mixed mode can exist. At the initiation of the first step of the frictional mode, the stress equilibrium requires that $\{\sigma\}_A=\{\sigma\}_B=\{\sigma\}$ and the constitutive Eq. (43) of the first step must satisfy this condition.

Table 1 Relationship of the S-mode, the F-mode and the mixed mode.

Region	Idealized Mode	Applicable Equation
A	S-mode	Eq. (19)
B	F-mode	Eq. (27)
Whole Element	Mixed Mode	Eq. (43)

7. NUMERICAL EXAMPLES

Fig. 5 shows the test predictions of Mattock (1981)[7] for specimens MM 2 M, ML 2 M and CB 2 M which are considered to be in the displacement fields of the F-mode. Although the published experimental results are only the typical shear-slip curves which is compared with the predicted values by Eq. (27) in **Figs. 5** (a_1), (b_1) and (c_1), the other predictions such as the behavior of crack width and normal stress on the crack plane is also shown with **Figs. 5** (a_2), (a_3), (b_2), (b_3), (c_2) and (c_3). **Figs. 5** (a_1), (b_1) and (c_1) also show the calculated value of shear resistance due to the dowel action which can be found by comparing the solid lines and dashed lines (here, identified values $\alpha=8.6$ and $\beta=5.3$, are adopted for predicting the dowel action, and the same values were used in the succeeding examples).

Experiments corresponding to the mixed mode are very scarce due to its complexity. However, Millard and Johnson[8] carried out this type of experiment using the specimen shown in **Fig. 6** (a). They introduced a crack at the center of a specimen by applying tensile forces at both ends. Then maintaining the tensile

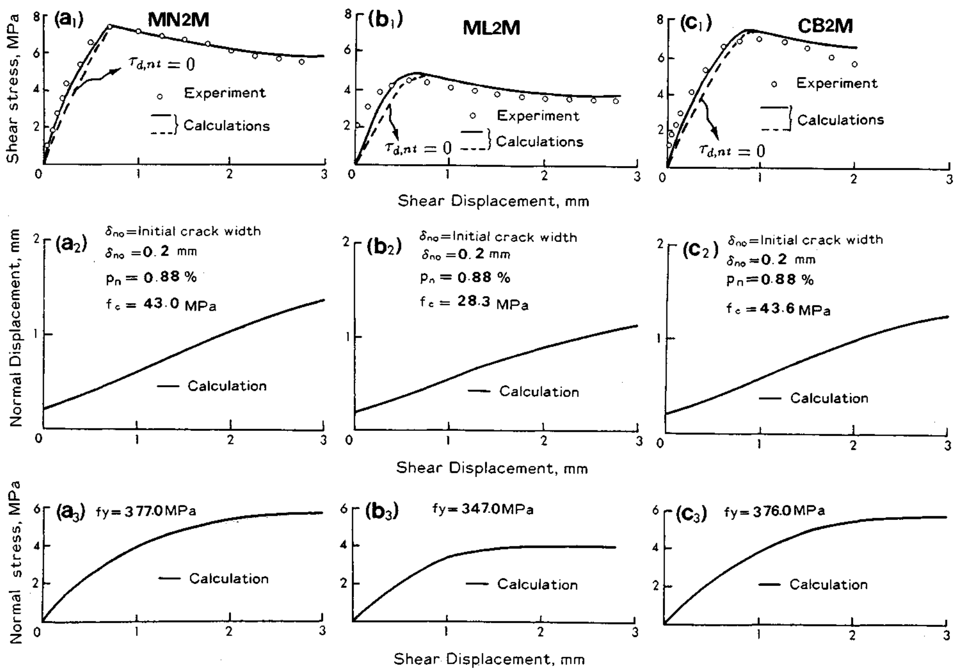

Fig. 5 Comparison of calculated values with experimental data (Mattock 1981).

stress, the shear force was applied at the center. The stress condition of concrete is such that the compressive stress is working at the crack to the normal direction to a crack surfaces while the tensile force is working at the ends to the same direction. The experimental relations between the shear stress and the shear displacement are shown in comparison with the calculated relations in **Figs.** 6 (b_1-b_3) (specimen mark : 1-4-7) and (c_1-c_3) (specimen mark : 2-5-8).

In the calculated procedure, it is found that the shear rigidity is very dependent on the extent of fraction of the region A. In **Fig.** 7, the differences in the shear rigidity due to the extent of the fraction of the region A of the total area, which is assumed as given constant, were shown. The numerical calculations show that the greater the fraction of the region A, the softer is the shear rigidity.

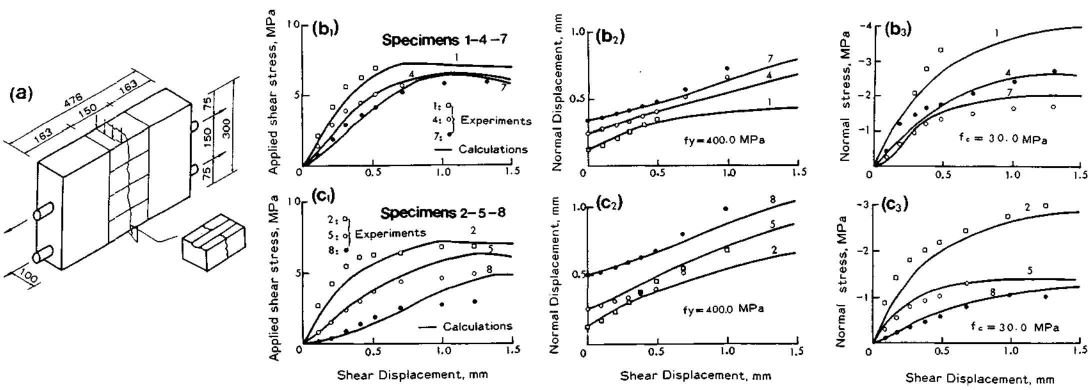

Fig. 6 Prediction of Millard and Johnson's experiment (1985).

Oesterle and Russell[9] have carried out experiments applying biaxial tension to specimens in the first stage, giving rise to substantial crack width to the X and Y direction, then applying shear force up to failure of the specimens, as shown in **Fig.** 8 (a). Out of three specimens, two specimens with reinforcement ratios of 0.022 for the X direction and 0.013 for the Y direction were tested by applying loads monotonically. Both showed very similar behavior. For the specimen MB 3, crack widths observed at the state when biaxial tension of 5.7 MPa for the X direction and 3.5 MPa for the Y direction were applied were 0.48 mm to the X direction and 0.38 mm to the Y direction, while Eq. (19) gives 0.46 mm for the X direction and 0.40 mm for the Y direction. The experiments showed that when the shear force was applied maintaining the last tensile stress level constant, the crack width closed in the Y direction and got wider in the X direction. Eq. (43) also gives the same characteristics. The shear strain and applied shear stress relation during that process is given in **Fig.** 8 (b),

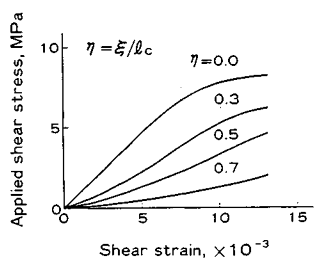

Fig. 7 The Variation of Shear Rigidity.

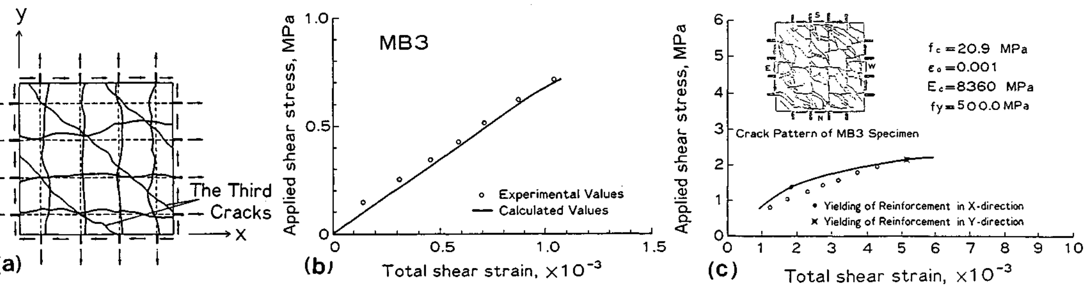

Fig. 8 Shear Rigidity, (b) before the formation of third cracks, (c) after the formation of third cracks.

comparing the observed values. At 0.7 MPa of applied shear stress, the third crack, oriented 135°to the X direction, has occurred, and some variation in the shear rigidity was observed. At this stage, the shear rigidity shown in **Fig.** 8(c) is also predicted by Eq. (43) while the third crack is opening.

8. CONCLUSION

In this paper, firstly, two basic constitutive models of displacement which have been developed to predict successfully two important factors of so called tension stiffening and interface shear transfer are reviewed. Constitutive equations of composite material of concrete and reinforcement in two dimensional stress field are constructed as a mixed model using stress reduction and reinforcement tensors by combining the two basic displacement fields. The constitutive model permits different inelastic processes to occur simultaneously and is able to deal with the cases of unidirectional cracks, orthogonal cracks and multiple, non-orthogonal cracks, etc. in the structure. Several numerical examples are included to demonstrate the accuracy of the theoretical model. It is shown that the present model express rationally the tension stiffening effect, damage of solid concrete, dowel action, and crack stiffness, in which shear dilatancy and shear friction are incorporated, and is formulated so that the material stiffness is directly applicable to nonlinear finite element analysis as a smeared crack model.

Despite of the limitation of the current experimental data, numerical simulations will be carried out by introducing this general constitutive equation in the finite element analysis and developed to predict the more realistic behavior of reinforced concrete plates and shells in engineering practices. This is planned to be given in different publication.

REFERENCES

1) Tanabe, T. and Yoshikawa, H. : Constitutive equation of a cracked reinforced concrete panel, IABSE Colloquium on Computational Mechanics of concrete structures, DELFT, pp. 17-34, 1987.
2) Wu, Z. S., Yoshikawa, H. and Tanabe, T. : Nonlinear aspects of cracked reinforced concrete by the damage mechanics concept, Proceedings of the Second East Asia-Pacific Conference on Structural Engineering and Construction, Chiang Mai, Thailand, pp. 481-486, Jan., 1989.
3) Wu, Z. S., Yoshikawa, H. and Tanabe, T. : Tension stiffness model for cracked reinforced concrete, J. Struct. Div., ASCE, to appear.
4) Yoshikawa, H., Wu, Z. S. and Tanabe, T. : An analytical model for shear slip of cracked concrete, J. Structure Division, ASCE, Vol. 115, No. ST4, pp. 771-788, 1989.
5) Yoshikawa, H. : Analytical models for the mechanical behavior of reinforced concrete members subjected to inplane stresses, Dissertation to University of Tokyo, February, 1987.
6) Vecchio, F. J. and Collins, M. P. : The response of reinforced concrete to in-plane shear and normal stress, Publication No. 82-03, Univ. of Toronto, Toronto, Ontario, Canada, 1982.
7) Mattock, A. H. : Cyclic shear transfer and type of interface, J. Struct. Div., ASCE, Vol. 107, No. ST10, pp. 1945-1963, 1981.
8) Millard, S. G. and Johnson, R. P. : Shear transfer in cracked reinforced concrete, Magazine of Concrete Research, Vol. 37, No. 130, pp. 3-15, 1985.
9) Oesterle, H. G. and Russell, H. G. : Shear transfer in large scale reinforced concrete containment elements, Construction Technology Laboratories, NUREG/CR-1374, 1980.

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