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GENERATION OF RAYLEIGH WAVES DUE TO IMPULSIVE RESPONSE OF A FINITE ELASTIC LAYER ON A RIGID BASEMENT

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It is an old saying that vibration and wave propagation are the same in principle. The saying is applied to vibration of an elastic layer as well as to dispersive Rayleigh wave in the layer. Based on vibration mode analysis for the layer of finite length, the Rayleigh wave characteristics such as dispersion and rotation of particle motion are discussed. Formulation of the impulsive response of the layer leads to familiar relationships between applied force and resulting deflection. The amplitude of the dispersive Rayleigh wave generated by a horizontal and/or vertical impulse is also discussed in reference to Poisson's ratio.

Keywords : Rayleigh wave, dispersion, vibration mode

1. INTRODUCTION

Rayleigh wave generated in layered media shows dispersive characteristics which demand more complicated treatment than Love wave. The complicated treatment of Rayleigh wave is inherent, partly because its dispersive characteristics depend on Poisson's ratio and partly because particle motion at a point in the media takes place in both horizontal and vertical directions.

A theory of surface waves which include Rayleigh wave and Love wave, has traditionally been formulated for elastic layered media of horizontally infinite length[1]. Because of the infinite length, the theory requires sophysticated technique in mathematical formulation like integration over $(-\infty, +\infty)$. What is worse still is in experimental studies in which physical models of finite length are inevitably used in laboratories. To cope with such difficulties, it is worth recalling that the surface waves are defined as solutions of an eigenvalue problem. On this basis, if the solution is first determined for a layered system of finite length, and then modified to be adapted to a system of infinite length, the resulting solution could be identical to a solution traditionally formulated for surface waves. The extension from the finite system to the infinite one is quite similar to the extension process from Fourier series to Fourier transform. As properties of Fourier series serve us to deepen our understanding of those of Fourier transform, an approach from the layered media of finite length will facilitate our understanding of surface waves in the media of infinite length.

From this viewpoint, free vibration of an elastic layer of finite length was studied on the basis of experimental and theoretical results in our previous papers[2)~4)]. In this paper, forced vibration of the layer

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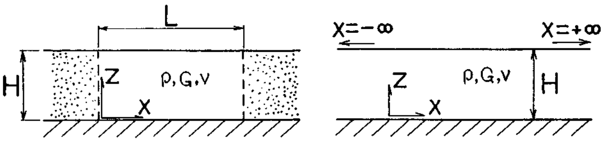
is to be discussed in connection with generation of dispersive Rayleigh wave.

2. RAYLEIGH WAVE CONFINED IN A FINITE LAYER

(1) Fundamental equation

Consider a homogeneous elastic layer resting on a rigid basement, as shown in Fig. 1 (a). Depth, length, mass density, shear modulus and Poisson's ratio of the layer are denoted by H, L, ρ, G and ν, respectively. Boundary conditions for each side end ($x=0$ or L) are free for horizontal displacement, but fixed for vertical displacement. At the bottom, any displacements are not allowed to take place. Regarding two-dimensional motion in the x-z plane, solutions of the free-vibration equations of motion, which is derived from Eqs. (7) and (8) by assigning the applied-force term to zero, are expressed by[3),4)]



(a) Finite Layer (b) Infinite Layer

Fig. 1 Definition Sketch.

$$u=k\left\{C_1(\cosh rz-\cosh sz)+C_2\left(\sinh rz-\frac{sr}{k^2}\sinh sz\right)\right\}\cos kx\exp(i\omega t), \quad (1)$$

$$w=r\left\{C_1\left(\sinh rz-\frac{k^2}{sr}\sinh sz\right)+C_2(\cosh rz-\cosh sz)\right\}\sin kx\exp(i\omega t), \quad (2)$$

where constants C_1 and C_2 are related by

$$C_2=-\frac{(k^2+s^2)\cosh rH-2k^2\cosh sH}{(k^2+s^2)\sinh rH-2sr\sinh sH}C_1. \quad (3)$$

In Eqs. (1) and (2), u and w denote displacement components in the x- and z-directions, respectively, i is the imaginary unit, and ω is natural circular frequency. The constant k is defined by

$$k=\frac{n\pi}{L}. \quad (n=1, 2, 3, \cdots\cdots) \quad (4)$$

The remaining constants s and r are related with k and ω by the following relations,

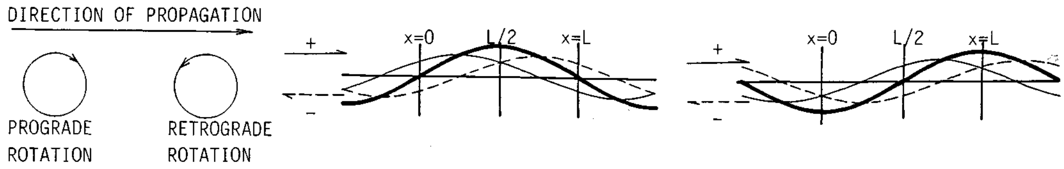
$$r^2-k^2+\frac{\omega^2}{V_p^2}=0, \quad s^2-k^2+\frac{\omega^2}{V_s^2}=0, \quad (5)$$

where V_p and V_s are P- and S-wave velocities, respectively. A characteristic equation which determines natural frequencies of the layer is expressed, as follows;

$$-4k^2(k^2+s^2)+\{4k^4+(k^2+s^2)^2\}\cosh rH\cosh sH-\left\{4srk^2+\frac{(k^2+s^2)^2}{sr}\right\}\sinh sH\sinh rH=0. \quad (6)$$

In the above formulation, k and ω/k may well be called a wave number and phase velocity, respectively, and expression in Eqs. (1), (2), (5), and (6) are identical to those for dispersive Rayleigh wave in the surface layer of infinite length shown in Fig. 1 (b). By definition, however, the wave number and the modal frequency for the finite layer are discrete as in Eq. (4), while those for the infinite layer are continuous. To account for the formal identity as well as the difference in definition, it is useful to consider a relationship between steady state vibration of the finite layer and wave propagation in the infinite layer. The relationship has been called the equivalence between vibration and wave propagation. In the present case, it states that the vibration modes given by Eqs. (1)～(6) are equivalent with Rayleigh wave which is confined in the finite layer.

When the Rayleigh wave is confined, a wave component will travel back and forth in the layer and change its amplitude due to the interaction with other components. After superposition of reciprocating motions, the components satisfying the boundary conditions only remain in the layer, yielding the discrete modes of vibration in the layer. In the limiting case where the layer length is extended to infinity, the wave number in Eq. (4) results in continuous variable without the violation of the definition.



(a) Direction of Rotation (b) Vertical Motion (c) Horizontal Motion

Fig. 2 Particle Motion of Stationary Vibration and Traveling Rayleigh Wave.

(2) Particle orbit

Due to the boundary conditions of the finite layer, the vibration shape of the fundamental mode ($n=1$ in Eq. (4)) become like those shown by bold lines in Figs. 2 (b) and (c). Apparently, the wave length pertaining to this mode is $2\ L$ along the surface. A particle on the free surface draws a linear orbit during the vibration motion, as can be seen from Eqs. (1) and (2).

The vibration mode shape shown by the bold lines can be divided into two Rayleigh wave components of the same wave length and amplitude, as shown in Fig. 2 (b) and (c). One indicated by fine solid lines is traveling in the $+x$ direction, and the other indicated by fine broken lines is traveling in the $-x$ direction. Note in Fig. 2 that vertical motion of a wave component traveling in either direction lags behind its horizontal motion by $\pi/2$ in phase. Thus, particle motion associated with each wave component is found to be elliptical and retrograde (see Fig. 2 (a)).

It is obvious from the above discussion that natural modes of vibration of a finite layer can be obtained by simply adding the two Rayleigh wave components.

(3) Phase and group velocities

In the previous paper[3], phase velocities c $(=\omega/k)$ were determined from laboratory experiments, in which steady state vibration and associated frequencies were observed. A good agreement in the phase velocities was obtained between the experiments and numerical calculations based on Eq. (6). Although the following discussion is limited to the lowest few modes and wave length $\lambda/H>1.0$, it is sufficient for the present purpose.

Dispersive Rayleigh waves consist of two major branches. One is M_1, and the other is M_2. Both M_1 and M_2 have subordinate branches called modes M_{11}, M_{12}……, M_{21}, M_{22}……, and so on. According to the calculation of phase velocity, Poisson's ratio has the significant effect on dispersive characteristics of the mode M_{21} but has little effect on those of the mode M_{11}.

From the phase velocity of each mode, the associated group velocity can be calculated by $d\omega/dk$. Fig. 3 shows an example calculated for the layer shown in Fig. 1 (b). The group velocity of the modes M_{11} and M_{12} is positive everywhere, while the modes M_{21} and M_{22} possess negative group velocities in the range above a certain wave length.

The negative group velocity has been a matter of controversy[5], and leaves some questions still unsolved. In the authors' opinion, reformulation of the group velocity by means of a vibrational approach could make a contribution toward the settlement of the controversy.

(4) Amplitude ratio at free surface

From Eqs. (1) and (2), a ratio of amplitudes of u and w at the free surface was calculated and denoted by the ratio u_0/w_0 in Fig. 4 along with the direction of particle motion at the free surface. Fig. 4 (b) indicates that the modes M_{11} and M_{21} interchange their nature with one another at wave length $4.62\ H$[6),7)], and the modes M_{12} and M_{22} behave similarly at wave length $1.54\ H$. Although the ratio u_0/w_0 of their modes look discontinuous at each wave length, the interchange takes place continuously as the wave length changes. Such apparent discontinuity is also found in the group velocity dispersion curve shown in Fig. 3.

Regardless of Poisson's ratio, the amplitude ratio u_0/w_0 of the mode M_{11} becomes infinite as wave length increases to infinity. At the limiting state, motion in the mode M_{11} is equivalent to shear vibration of the

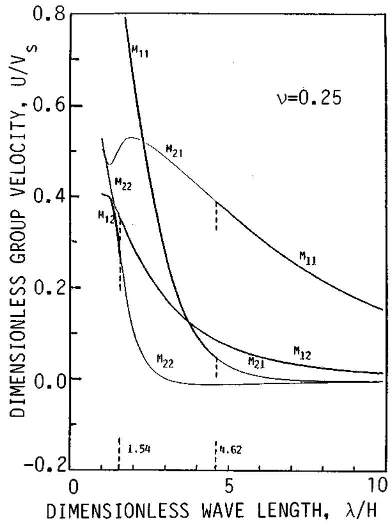


Fig. 3 Group Velocity.

layer, because the ratio u_0/w_0 becomes infinite at the period $T=4\ H/V_S$. The ratio u_0/w_0 of higher modes at infinite wave length is either zero or infinity, depending upon Poisson's ratio. In a case where the ratio u_0/w_0 at the infinite wave length is zero, motion related to the Rayleigh wave can be regarded as vertical vibration of the layer.

To classify types of motion of the Rayleigh wave at infinite wave length, asymptotic periods of the lowest few modes are summarized in Table 1. When the asymptotic period is associated with the so-called quarter-wave length law of S- (or P-) wave, the limiting state corresponds to horizontal (or vertical) vibration.

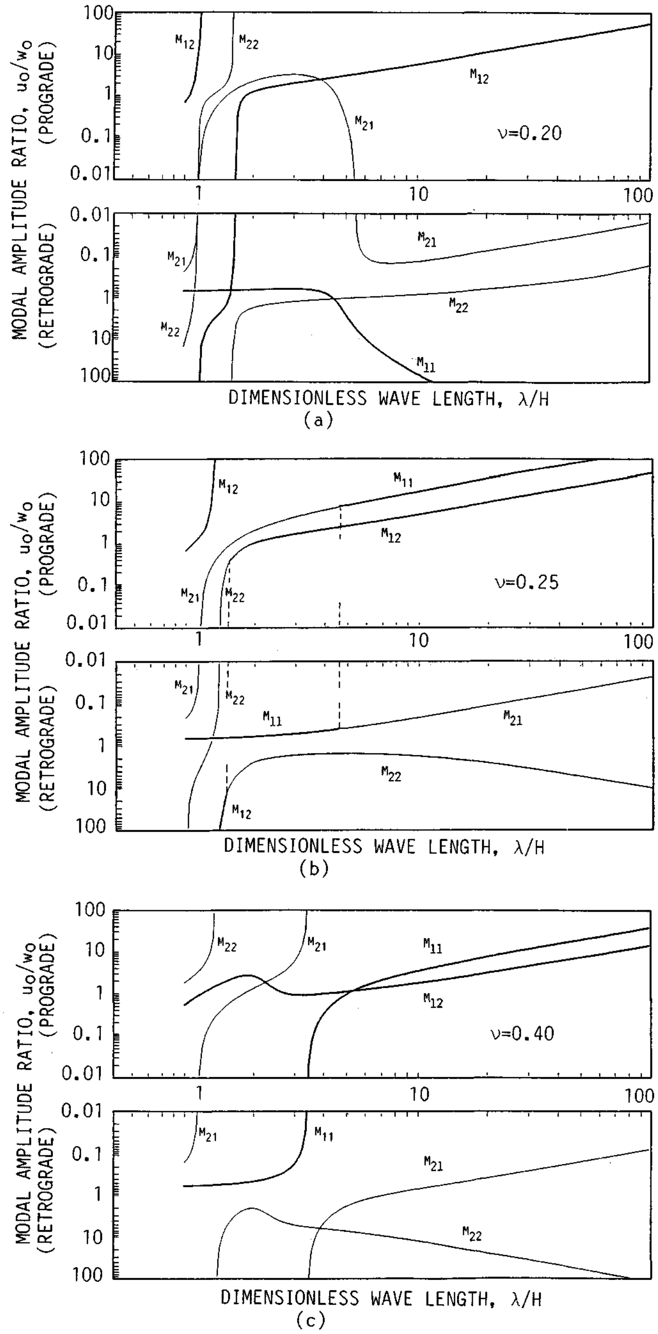


Fig. 4 Modal Amplitude Ratio at the Free Surface Calculated for Three Values of Poisson's Ratio.

3. IMPULSIVE RESPONSE

(1) Formulation

When point force F is applied at (x, z), the equations of motion of the layer become

$$\rho\frac{\partial^2 u}{\partial t^2}=\frac{G}{1-2\nu}\frac{\partial\Delta}{\partial x}+G\nabla^2 u+f_x(x,\ z,\ t), \quad\cdots\cdots (7)$$

$$\rho\frac{\partial^2 w}{\partial t^2}=\frac{G}{1-2\nu}\frac{\partial\Delta}{\partial z}+G\nabla^2 w+f_z(x,\ z,\ t), \quad\cdots\cdots (8)$$

where f_x and f_z denote the x- and z-components of F, respectively, Δ $(=\partial u/\partial x+\partial w/\partial z)$ means the

Table 1 Asymptotic Periods at Infinite Wave Length.

Poisson's Ratio		0.20	0.25	0.30	0.40	0.49
Vp to Vs Ratio		1.63	1.73	1.87	2.45	7.14
Mode	M11	4H/Vs	4H/Vs	4H/Vs	4H/Vs	4H/Vs
	M21	4H/Vp	4H/Vp	4H/Vp	4H/Vp	4H/3Vs
	M12	4H/3Vs	4H/3Vs	4H/3Vs	4H/3Vs	4H/5Vs
	M22	4H/3Vp	4H/5Vs	4H/5Vs	4H/5Vs	4H/7Vs

volumetric strain, and ∇^2 is Laplacian.

According to the mode-superposition method, horizontal and vertical components of the response are expressed, as follow;

$$u(x, z, t)=\sum_m \sum_n U_m(z) \cos k_n x\, Q_{mn}(t), \quad (9)$$

$$w(x, z, t)=\sum_m \sum_n W_m(z) \sin k_n x\, Q_{mn}(t), \quad (10)$$

where

$$k_n=\frac{n\pi}{L} \qquad (n=1, 2, 3, \cdots\cdots), \quad (11)$$

$$U_m(z)=k_n\left\{(\cosh rz-\cosh sz)+C\left(\sinh rz-\frac{sr}{k_n^2}\sinh sz\right)\right\}, \quad (12)$$

$$W_m(z)=r\left\{\left(\sinh rz-\frac{k_n^2}{sr}\sinh sz\right)+C(\cosh rz-\cosh sz)\right\}, \quad (13)$$

and

$$C=-\frac{(k_n^2+s^2)\cosh rH-2k_n^2\cosh sH}{(k_n^2+s^2)\sinh rH-2sr\sinh sH}. \quad (14)$$

Substituting Eqs. (12) and (13) into Eqs. (9) and (10), Eqs. (7) and (8) lead to

$$\sum_n \sum_m U_m \cos k_n x\, \ddot{Q}_{mn}=\sum_n \sum_m \{-\omega_{mn}^2 U_m \cos k_n x\, Q_{mn}\}+f_x/\rho, \quad (15)$$

$$\sum_n \sum_m W_m \sin k_n x\, \ddot{Q}_{mn}=\sum_n \sum_m \{-\omega_{mn}^2 W_m \sin k_n x\, Q_{mn}\}+f_z/\rho, \quad (16)$$

where a dot denotes differenciation with respect to time, and ω_{mn} is the natural circular frequency of the mode $m-n$. Besides a well-known orthogonality of trigonometric functions, an orthogonal relationship exists between the functions $U_m(z)$ and $W_m(z)$, which is expressed in the present case as,

$$\int_0^H \{U_i(z)\, U_j(z)+W_i(z)\, W_j(z)\}\, dz=0. \qquad (i\neq j) \quad (17)$$

Because of these orthogonal relationships, a series of mathematical arrangement, which includes multiplying each term of Eq. (15) by $U_m(z)\cos k_n x$ and Eq. (16) by $W_m(z)\sin k_n x$, adding both sides of resulting equations, and integrating with respect to x and z, leads to

$$\ddot{Q}_{mn}(t)+\omega_{mn}^2 Q_{mn}(t)=F_{mn}(t)/M_{mn}, \quad (18)$$

where

$$F_{mn}(t)=\frac{1}{\rho}\int_0^H\int_0^L (f_x U_m \cos k_n x+f_z W_m \sin k_n x)\, dxdz, \quad (19)$$

$$M_{mn}=\int_0^H\int_0^L (U_m^2\cos^2 k_n x+W_m^2\sin^2 k_n x)\, dxdz$$

$$=\frac{1}{2}LH\int_0^1 \{U_m^2(\zeta)+W_m^2(\zeta)\}\, d\zeta, \quad (20)$$

in which $\zeta=z/H$. $F_{mn}(t)$ and M_{mn} are called the generalized load and the generalized mass, respectively.

When the force is an impulse f_x applied at (x_0, z_0) in the x-direction, Eqs. (9), (10) and (19) give,

$$F_{mn}(t)=\frac{1}{\rho}U_m(z_0) f_x \cos k_n x_0\, \delta(t). \quad (21)$$

Similarly, for an impulse f_z applied in the z-direction, Eq. (19) gives

$$F_{mn}(t)=\frac{1}{\rho}W_m(z_0) f_z \sin k_n x_0\, \delta(t). \quad (22)$$

Thus, for a unit impulse due to both f_x and f_z, the generalized load is expressed by,

$$F_{mn}(t)=\frac{1}{\rho}\{U_m(z_0)f_x\cos k_nx_0+W_m(z_0)f_z\sin k_nx_0\}\,\delta(t). \qquad (23)$$

In Eq. (23), it should be noticed that the terms in the brackets represent a scaler product between a displacement vector and an applied force vector. Hence, a maximum modal response can be expected when the impulse is applied in the direction parallel to the modal displacement vector at the point, and a modal response is not excited if the impulse is applied in the direction normal to the modal displacement vector.

When a vertical unit impulse is applied at a surface point (x_0, H), the normal coordinate $Q_{mn}(t)$ in Eq. (18) becomes,

$$Q_{mn}(t)=\frac{W_m(H)\sin k_nx_0}{\rho M_{mn}\omega_{mn}}\sin\omega_{mn}t$$
$$=\frac{2W_m(H)\sin k_nx_0}{\rho LH\,\omega_{mn}\int_0^1\{U_m^2(\zeta)+W_m^2(\zeta)\}d\zeta}\sin\omega_{mn}t. \qquad (24)$$

Denoting the relative amplitudes in the x- and z-directions at the surface by A_x and A_z, respectively, Eqs. (9), (10) and (24) leads to,

$$A_x=\frac{|2U_m(H)W_m(H)|}{\omega_{mn}\int_0^1\{U_m^2(\zeta)+W_m^2(\zeta)\}d\zeta}, \qquad (25)$$

$$A_z=\frac{2W_m^2(H)}{\omega_{mn}\int_0^1\{U_m^2(\zeta)+W_m^2(\zeta)\}d\zeta}. \qquad (26)$$

Likewise, when a horizontal unit impulse is applied at the surface point, $Q_{mn}(t)$ in Eq. (18) results in,

$$Q_{mn}(t)=\frac{2U_m(H)\cos k_nx_0}{\rho LH\,\omega_{mn}\int_0^1\{U_m^2(\zeta)+W_m^2(\zeta)\}d\zeta}\sin\omega_{mn}t, \qquad (27)$$

and

$$A_x=\frac{2U_m^2(H)}{\omega_{mn}\int_0^1\{U_m^2(\zeta)+W_m^2(\zeta)\}d\zeta}, \qquad (28)$$

$$A_z=\frac{|2U_m(H)W_m(H)|}{\omega_{mn}\int_0^1\{U_m^2(\zeta)+W_m^2(\zeta)\}d\zeta}. \qquad (29)$$

Note that A_x in Eq. (25) is identical to A_z in Eq. (29), which simply states the Maxwell's law of reciprocal deflections.

(2) Modal amplitudes

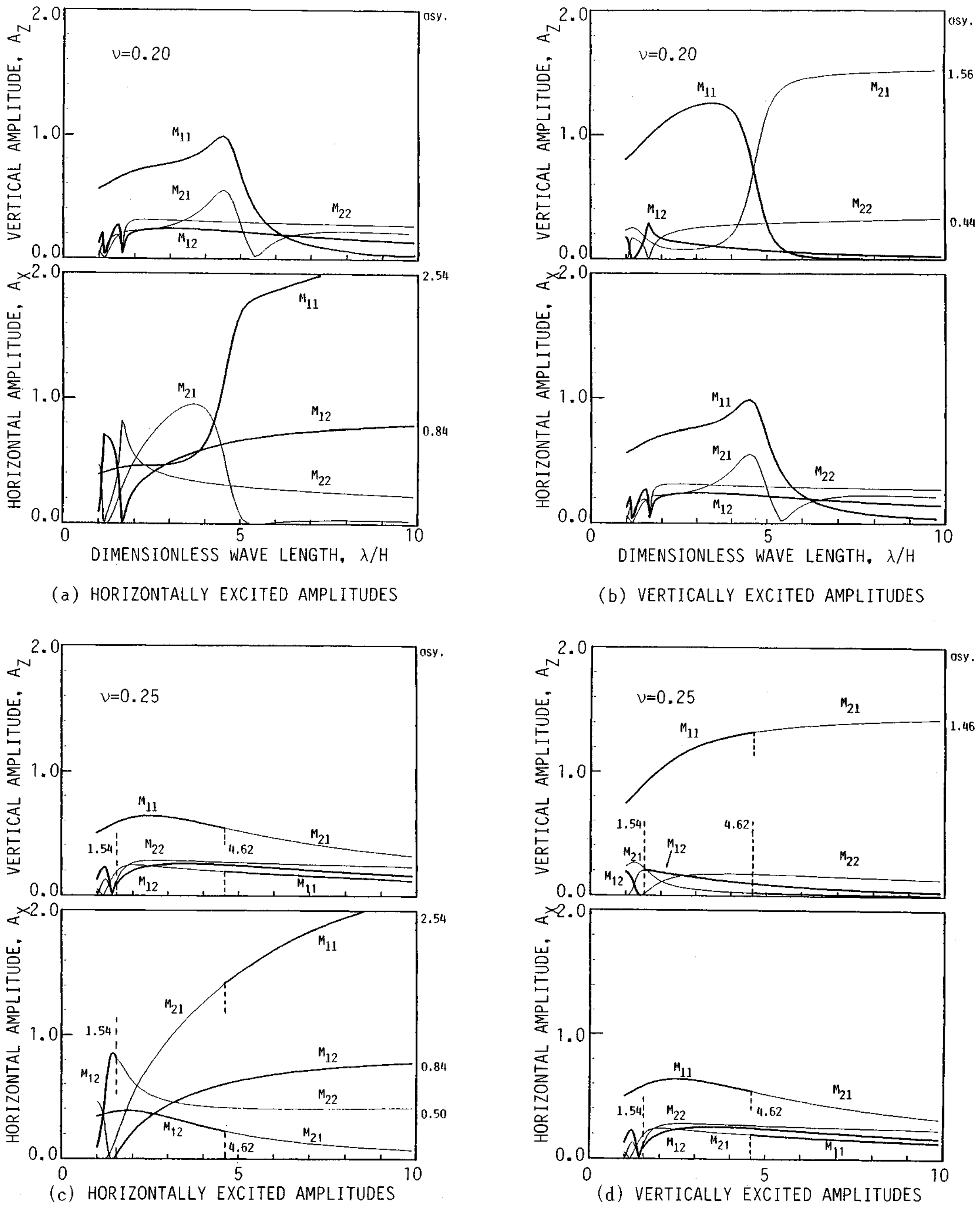
A pair of vertical and horizontal amplitudes A_x and A_z calculated from Eqs. (25) and (26), or from (28) and (29) are plotted against the dimensionless wave length in **Fig. 5**. At rightmost position of each figure in **Fig. 5**, asymptotic values of the modal amplitude at infinite wave length are shown in the case where they are different from zero. In **Fig. 5**, when Poisson's ratio is 0.25, the mode M_{11} appears to interchange its amplitude characteristics with the mode M_{21} at the wave length $\lambda=4.62\,H$, and the mode M_{12} does similarly with the mode M_{22} at $\lambda=1.54\,H$. These are similar to those mentioned previously with respect to the amplitude ratio.

Regarding the dispersive Rayleigh wave in the layer, several findings can be drawn from **Fig. 5**, which include :

1) The mode M_{11} or M_{21} makes larger contribution to displacement amplitude than higher modes.

2) When a horizontal impulse is applied, the component of the larger wave length of the mode M_{11} makes the larger contribution to the amplitude regardless of Poisson's ratio.

3) When a vertical impulse is applied, the largest contribution to the amplitude is attained by either the mode M_{21} or the mode M_{11}, which depends on Poisson's ratio.



(a) HORIZONTALLY EXCITED AMPLITUDES (b) VERTICALLY EXCITED AMPLITUDES

(c) HORIZONTALLY EXCITED AMPLITUDES (d) VERTICALLY EXCITED AMPLITUDES

Fig. 5 Amplitudes of Response to Horizontal or Vertical Excitation (Continued).

4) It is evident from the comparison of figures in Fig. 5 that the largest contribution to the amplitude is not necessarily given by the frequency component which shows the minimum group velocity in a dispersion curve.

4. CONCLUSION

Vibration of a finite elastic layer underlain by a rigid basement has been formulated by the mode-superposition procedure. The formulated results and their associated illustrations have revealed

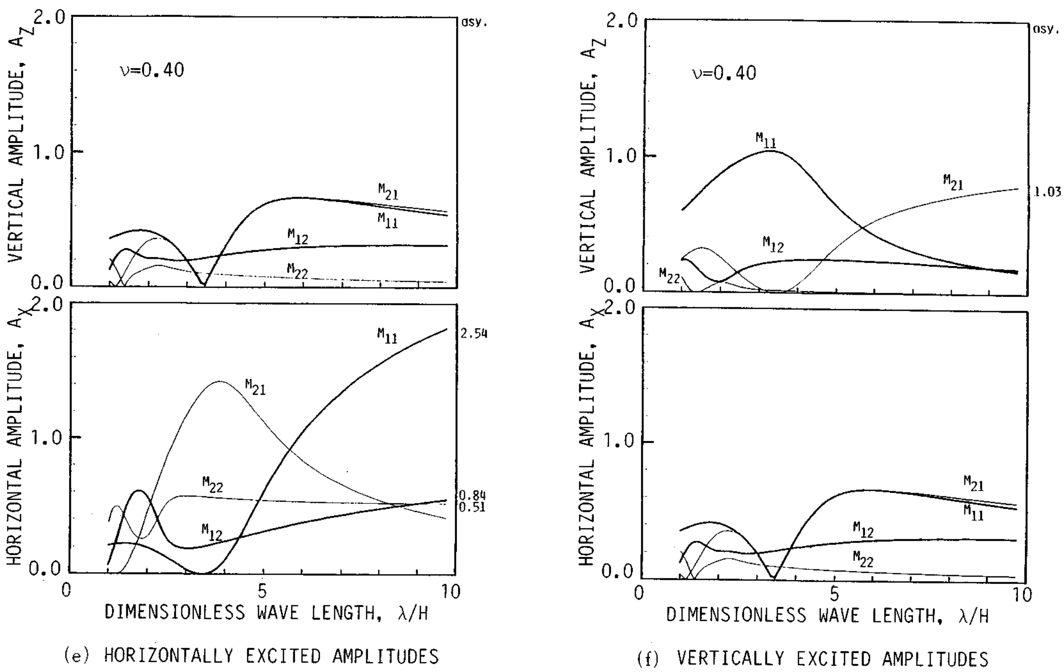


Fig. 5 Amplitudes of Response due to Horizontal or Vertical Excitation.

many characteristics of the dispersive Rayleigh wave. The characteristics revealed by the vibration analysis include;

(1) From a natural mode of vibration of the layer, it is possible to specify the direction of particle orbit of the associated Rayleigh wave; that is, either prograde or retrograde.

(2) The modal amplitude ratio at a free surface shows remarkable change with the change in Poisson's ratio ν as well as in wave length λ. For instance, the ratio of the mode M_{21} becomes infinite at $\lambda \rightarrow \infty$ when $\nu > 0.40$, while it vanishes at $\lambda \rightarrow \infty$ when $\nu < 0.30$.

(3) In a special case where Poisson's ratio is 0.25, a pair of modes, say the modes M_{11} and M_{21}, exchange their nature smoothly at a certain wave length.

(4) Regardless of Poisson's ratio, group velocity of the modes M_{11} and M_{12} is always positive, while there is a range of period or wave length where the modes M_{21} and M_{22} possess negative group velocity.

(5) Relative contribution of an impulse to the generation of Rayleigh wave is predictable by means of a scaler product between the impulse and a modal displacement.

(6) Even if Rayleigh wave is generated by an impulse which has a white-noise frequency spectrum, the largest contribution to its amplitude is not necessarily given by the frequency component which gives the minimum group velocity.

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