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SYSTEM MODAL IDENTIFICATION USING FREE VIBRATION DATA

By N. C. MICKLEBOROUGH and Yong Lin PI***

System modal identification for structures is a fundamental problem in structural dynamics. The identified natural frequencies, damping ratios and mode shapes can be used to better understand the behaviour of structures, modify structures, design control system for structures, and can also be used in direct analysis of structures. A time domain method for identification of modal parameters for a test structure using free vibration responses is presented in this paper. The method in based on the multidimensional autoregressive model of a vibrating structure. The method is efficient with computing time and has the advantage of being able to identify very close natural frequencies and highly coupled modes. A numerical example and a test case are used to illustrate the usefulness and results of the method.

Keywords : identification, modal parameter, vibrating structure, time domain

1. INTRODUCTION

The identification of modal parameters characterizing the important vibration modes of a structure is a basic problem in structural dynamics. Time domain modal parameter identification methods include Least Squares Complex Exponential method[2], Poly Reference Time Domain method[3], Ibrahim Time Domain (ITD) method[4),5] and mathematical model method[6]. Among these methods, the ITD method has been studied most intensively. The basic concept of ITD method is introduced as follows.

For a structure vibrating freely, the response at any time t can be expressed as

$$x=\sum_{j=1}^{2n} \Psi_j e^{\lambda_j t} \quad (1)$$

If there are n structural modes to be determined, the free responses will be used to construct the $2n \times N$ structure response matrices X and $\hat{X}$ such that

$$X=[x_i(t_j)] \quad (2)$$

and

$$\hat{X}=[x_i(t_j+\delta t_1)] \quad (3)$$

N is the number of time data points which should be equal to or greater than $2n$.

From equation (1), the X and $\hat{X}$ can be written as

$$X=\Psi\Lambda \quad (4)$$

and

$$\hat{X}=\Psi P\Lambda \quad (5)$$

where

* School of Civil Engineering, University of New South Wales, Australia
** School of Civil Engineering, University of New South Wales, Australia

$$\Lambda=\begin{bmatrix}\ddots & & \\ & e^{\lambda_i t_j} & \\ & & \ddots\end{bmatrix} \qquad (6)$$

and

$$\boldsymbol{P}=\begin{bmatrix}\ddots & & \\ & e^{\lambda_i \delta t_1} & \\ & & \ddots\end{bmatrix} \qquad (7)$$

are $2n\times N$ and $2n\times 2n$ matrices.

By eliminating Λ from equations (4) and (5), an eigenvalue problem is obtained.

$$\boldsymbol{H\Psi}=\boldsymbol{\Psi P}. \qquad (8)$$

The matrix $\boldsymbol{H}$ must satisfy the equation

$$\boldsymbol{HX}=\hat{\boldsymbol{X}} \qquad (9)$$

Estimate of matrix $\boldsymbol{H}$ can be obtained by least square method using equation (9). The eigenvectors of the matrix $\boldsymbol{H}$ are the eigenvectors of the structure and eigenvalues of the matrix $\boldsymbol{H}$ are related to the eigenvalues of the structure by the equation

$$p_i=e^{\lambda_i \delta t_1} \qquad (10)$$

In this paper an autoregressive (AR) model for a free vibrating structure is derived. At the same time, it is proved that the matrix $\boldsymbol{H}$ has the form

$$\boldsymbol{H}=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{I}_r & \cdots & \boldsymbol{0}\\ \vdots & \vdots & \cdots & \vdots\\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{I}_r\\ \boldsymbol{H}_s & \boldsymbol{H}_{s-1} & \cdots & \boldsymbol{H}_1\end{bmatrix} \qquad (11)$$

Therefore, only the lower parts of matrix $\boldsymbol{H}$ are required to be calculated. The derived AR model is used to estimate the lower submatrix $\boldsymbol{H}_1$, $\boldsymbol{H}_2\cdots$ and $\boldsymbol{H}_s$, from which the modal parameters of the original vibrating structure are identified.

2. BACKGROUND

Free vibration of a linear structural dynamic system of n degrees of freedom is described by the matrix differential equation of motion

$$\boldsymbol{M}\ddot{\xi}(t)+\boldsymbol{C}\dot{\xi}(t)+\boldsymbol{K}\xi(t)=0 \qquad (12)$$

where $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ are $n\times n$ mass, damping and stiffness matrices, respectively, $\xi(t)$, $\dot{\xi}(t)$ and $\ddot{\xi}(t)$ are $n\times 1$ vectors of response displacement, velocity and acceleration, respectively.

The eigenvalue problem associated with equation (12) is given by

$$(\lambda_i^2\boldsymbol{M}+\lambda_i\boldsymbol{C}+\boldsymbol{K})\boldsymbol{\Psi}_i=0,\quad i=1, 2,\cdots, n \qquad (13)$$

where λ_i and $\boldsymbol{\Psi}_i$ are the eigenvalues and eigenvectors. The eigenvalues and vectors can be real or complex, and when complex they occur in pairs of conjugates. The eigenvalues can be expressed as

$$\lambda_i=-\eta_i\omega_i\pm j\omega_i\sqrt{1-\eta_i^2} \qquad (14)$$

where $j^2=-1$ and η_i and ω_i are the damping ratio and the frequency of damped oscillation associated with the i-th mode, respectively.

The aim of the analysis is to identify the eigenvalues and eigenvectors of equation (13) using response measurements such as displacements, velocities or accelerations in time domain.

Equation of motion (12) is not a convenient form for modal identification analysis and a $2n\times 1$ state vector

$$\boldsymbol{x}^T(t)=[\xi^T(t)\,|\,\dot{\xi}^T(t)] \qquad (15)$$

is introduced to the form of state equation

$$\dot{\boldsymbol{x}}(t)=\boldsymbol{Ax}(t) \qquad (16)$$

where

$$A=\begin{bmatrix} 0 & I_n \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \quad \cdots\cdots (17)$$

is $2n\times2n$ matrix and is called system matrix.

If the exponential function $\exp(At)$ of the matrix At is defined by the Taylor series

$$\exp(At)=\sum_{i=0}^{\infty}\frac{(At)^i}{i!} \quad \cdots\cdots (18)$$

the solution of equation (16) is obtained as

$$x(t)=\Phi(t)x(0) \quad \cdots\cdots (19)$$

$x(0)$ is the initial condition of the dynamic structural system and $\Phi(t)$ is the $2n\times2n$ state transition matrix of the linear free vibrating structure with

$$\Phi(t)=\exp(At) \quad \cdots\cdots (20)$$

In the practical test, however, the sampling can be performed only at discrete time instants t_k, t_{k+1}, $\cdots$. Once the sampling intervals Δt are constant, $t_{k+1}-t_k=\Delta t=T$, and t_0 is taken to be zero with $x(t_k)=x(kT)$ and $x(t_k)$ represented as $x(k)$.

It follows from equation (19) that

$$x(k+i)=\Phi^i x(k) \quad \cdots\cdots (21)$$

with $\Phi=\Phi(T)=\text{const.}$

Practically, not all the state variables can be measured. Thus the measurement equation describing the relationship between measurement responses and state variables is introduced. It can be assumed that only the first r displacements of the structure are measured and that the measurement $q(k)$ is subjected to measurement noise $\delta(k)$ such that

$$q(k)=Cx(k)+\delta(k) \quad \cdots\cdots (22)$$

then $q(k)=r\times1$ measured displacement (column) vector at time $t=(kT)$; and $\delta(k)=r\times1$ measurement noise, independent of $q(k)$, with zero mean and finite covariance matrix V_δ,
and

$$C=[I_r \quad 0] \quad \cdots\cdots (23)$$

is $r\times2n$ matrix with I being an $r\times r$ unit matrix and 0 being an $r\times(2n-r)$ null matrix.

A further assumption made is that the vibrating structure is completely observable. The observability condition is

$$L=\begin{bmatrix} C \\ C\Phi \\ C\Phi^2 \\ \cdot\cdot \\ C\Phi^{(s-1)} \end{bmatrix} \quad \cdots\cdots (24)$$

is nonsingular with an integer $s=2n/r$.

The proposed method is derived assuming the measurement is noise free and the system is observable with an index of two.

3. AUTOREGRESSIVE MODEL OF FREE VIBRATING STRUCTURE

An autoregressive (AR) model of a linear free vibrating structure is derived as follows. Considering equations (21) and (22) yields that

$$q(k+i)=C\Phi^i x(k) \quad \cdots\cdots (25)$$

where $q(k+i)$ is an $n\times1$ vector and C is an $n\times2n$ matrix when the observation index of the system is assumed to be two.

Hence,

$$\begin{bmatrix} q(k+1) \\ q(k+2) \end{bmatrix} = \begin{bmatrix} C \\ C\Phi \end{bmatrix} \Phi x(k) \quad \cdots\cdots (26)$$

and

$$\begin{bmatrix} q(k+2) \\ q(k+3) \end{bmatrix} = \begin{bmatrix} C \\ C\Phi \end{bmatrix} \Phi^2 x(k) \quad \cdots\cdots (27)$$

Simultaneously solving the two equations (26) and (27) to eliminate the $x(k)$ yields

$$\begin{bmatrix} q(k+2) \\ q(k+3) \end{bmatrix} = H \begin{bmatrix} q(k+1) \\ q(k+2) \end{bmatrix} \quad \cdots\cdots (28)$$

where

$$H = \begin{bmatrix} C \\ C\Phi \end{bmatrix} \Phi \begin{bmatrix} C \\ C\Phi \end{bmatrix}^{-1} \quad \cdots\cdots (29)$$

Equation (29) can be proved to have the form (refer to APPENDIX A)

$$H = \begin{bmatrix} 0 & I_n \\ H_2 & H_1 \end{bmatrix} \quad \cdots\cdots (30\cdot a)$$

When the observation index is s,

$$H = \begin{bmatrix} 0 & I_r & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & I_r \\ H_s & H_{s-1} & \cdots & H_1 \end{bmatrix} \quad \cdots\cdots (30\cdot b)$$

Hence, equation (28) becomes

$$\begin{bmatrix} q(k+2) \\ q(k+3) \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ H_2 & H_1 \end{bmatrix} \begin{bmatrix} q(k+1) \\ q(k+2) \end{bmatrix} \quad \cdots\cdots (31)$$

where H_1 and H_2 are $n \times n$ matrices.

By expanding equation (31) and considering the last row of the equation, the autoregressive model of the linear free vibrating structure in equation (12) is obtained such as

$$q(k+3) = H_2 q(k+1) + H_1 q(k+2) + w(k+3), \quad (k=0, 1, \cdots, N) \quad \cdots\cdots (32\cdot a)$$

where $w(k+3)$ is an $n \times 1$ error vector, representing the response measuring noise.

Similarly, when the observation index of the system is s, the corresponding autoregressive model will be

$$q(k+s+1) = H_s q(k+1) + H_{s-1} q(k+2) + \cdots + H_1 q(k+s) + w(k+s+1)$$
$$\text{for} \quad (k=0, 1, \cdots, N) \quad \cdots\cdots (32\cdot b)$$

4. MODAL IDENTIFICATION

The relationships between eigenvalues and eigenvectors of the system matrix A and the state transition matrix Φ are derived and they are related to the eigenvalues and eigenvectors of matrix H and those in equation (13). Once these relationships are established, the eigenvalues and eigenvectors of the linear free vibrating structure can be identified by solving eigenvalue problem of the matrix H.

Let Z and Λ be eigenvector matrix and eigenvalue matrix of the system matrix A, that is,

$$AZ = Z\Lambda \quad \cdots\cdots (33)$$

where

$$\Lambda = \begin{bmatrix} \ddots & & \\ & \lambda_i & \\ & & \ddots \end{bmatrix} \quad \text{and} \quad Z = [z_1 \, z_2 \cdots z_{2n}] \quad \cdots\cdots (34)$$

with λ_i and z_i being the i th eigenvalue and eigenvectors of A.

It can be shown that the eigenvalue matrix Λ and eigenvector matrix Z are related to the eigenvalues λ_i

and eigenvectors Ψ_i of the dynamic structure described by equation (13) as

$$Z=\begin{bmatrix} \Psi \\ \Psi\Lambda \end{bmatrix} \quad (35)$$

Ψ is a matrix with eigenvectors Ψ_i being its column vectors.
A transformation of variables is introduced as

$$x(t)=Zy(t) \quad (36)$$

and is substituted into equation (16) to obtain

$$Z\dot{y}(t)=AZy(t) \quad (37)$$

and

$$\dot{y}(t)=Z^{-1}AZy(t) \quad (38)$$

Similar to the solution of equation (16), the solution of equation (38) can be written as

$$y(t)=\exp(Z^{-1}AZt)y(0) \quad (39)$$

From equation (33)

$$Z^{-1}AZ=\Lambda \quad (40)$$

Hence

$$y(t)=\exp(\Lambda t)y(0) \quad (41)$$

Expanding exp $[\Lambda t]$ in the exponential series,

$$\exp(\Lambda t)=I+\begin{bmatrix} \ddots & & \\ & \lambda_i t & \\ & & \ddots \end{bmatrix}+\begin{bmatrix} \ddots & & \\ & \frac{(\lambda_i t)^2}{2!} & \\ & & \ddots \end{bmatrix}+\cdots=\begin{bmatrix} \ddots & & \\ & e^{\lambda_i t} & \\ & & \ddots \end{bmatrix} \quad (42)$$

Considering transformation (36) yields

$$x(t)=Z\exp(\Lambda t)Z^{-1}x(0) \quad (43)$$

Comparison of equation (43) and (19), and considering equation (35), the relationships of the state transition matrix Φ to the eigenvalue matrix Λ and eigenvector matrix Z of the system matrix and so to those described by equation (13) are established

$$\Phi(t)=Z\exp(\Lambda t)Z^{-1}=\begin{bmatrix} \Psi \\ \Psi\Lambda \end{bmatrix}\exp(\Lambda t)\begin{bmatrix} \Psi \\ \Psi\Lambda \end{bmatrix}^{-1} \quad (44)$$

On the other hand, from equation (29)

$$H\begin{bmatrix} C \\ C\Phi \end{bmatrix}=\begin{bmatrix} C \\ C\Phi \end{bmatrix}\Phi \quad (45)$$

Substituting equation (44) into the equation (45) and considering Z^{-1} being nonsingular yield an eigenvalue problem

$$H\begin{bmatrix} \Psi \\ \Psi\exp(\Lambda t) \end{bmatrix}=\begin{bmatrix} \Psi \\ \Psi\exp(\Lambda t) \end{bmatrix}\exp(\Lambda t) \quad (46)$$

Equation (46) gives the relationship of matrix H to the eigenvalue matrix and eigenvector matrix of the system matrix A. The upper halves of the eigenvectors of the matrix H are the eigenvectors of the original equation (13). The eigenvalues of the matrix H are $e^{\lambda_i T}$, from which the eigenvalues or natural frequencies and damping factors of the dynamic structure described by equation (13) can be calculated. To this end the eigenvalues $e^{\lambda_i T}$ are written as

$$\alpha_i=e^{(-\eta_i\omega_i\pm i\omega_i\sqrt{1-\eta^2_i})T} \quad (47)$$

Thus

$$-\eta_i\omega_i=\frac{1}{2T}\ln(\alpha_i\alpha_i^*) \quad (48)$$

and

$$\omega_i\sqrt{1-\eta_i^2}=\begin{bmatrix} \frac{1}{T}\tan^{-1}\left|\frac{\alpha_i-\alpha_i^*}{\alpha_i+\alpha_i^*}\right| \\ \frac{1}{T}\left(\pi-\tan^{-1}\left|\frac{\alpha_i-\alpha_i^*}{\alpha_i+\alpha_i^*}\right|\right) \quad \text{if } \alpha_i+\alpha_i^*<0 \end{bmatrix} \quad (49)$$

from which the natural frequencies and damping factors can be obtained as follows.

$$\omega_i=\sqrt{(-\eta_i\omega_i)^2+(\omega_i\sqrt{1-\eta_i^2})^2} \quad (50)$$

and

$$\eta_i=\frac{|-\eta_i\omega_i|}{\omega_i} \quad (51)$$

Key problem is to find matrix $\boldsymbol{H}$. Once it is found, the modal identification of a linear dynamic system can be solved as previously mentioned.

5. ESTIMATE OF MATRIX $\boldsymbol{H}$

Estimate of matrix $\boldsymbol{H}$ using response measurements can be done in two ways from the auto-regressive model developing processing. The first algorithm uses the auto-regressive model (32) which is written in the following form for the convenience of numerical implementation,

$$\boldsymbol{q}(k+2)=[\boldsymbol{H}_2\ \boldsymbol{H}_1]\begin{bmatrix}\boldsymbol{q}(k)\\ \boldsymbol{q}(k+1)\end{bmatrix}+\boldsymbol{w}(k+2) \quad (52\cdot a)$$

for $k=1, 2, \cdots, N$, where N indicates the number of time data points and usually equal to or greater than $2n$. In order to use identical regression vectors for the different output components, a $(2n\times n)\times 1$ dimensional column vector θ and a $(2n\times n)\times n$ matrix $\boldsymbol{\Xi}$ to transform Eqn. (52·a) into

$$\boldsymbol{q}(k+2)=\boldsymbol{\Xi}^T(k)\boldsymbol{\theta}+\boldsymbol{w}(k+2) \quad (52\cdot b)$$

where

$$\boldsymbol{\Xi}(k)=\begin{bmatrix}\boldsymbol{q}(k)\\ \boldsymbol{q}(k+1)\end{bmatrix}\otimes\boldsymbol{I}_n \text{ and } \boldsymbol{\theta}=\text{column}[\boldsymbol{H}_2\ \boldsymbol{H}_1]$$

where the operator $\otimes$ indicates the Kronecker product of matrices and the operator column indicates the column operation on a matrix. (refer to APPENDIX B).

Collecting all the equations (52) for the time instants $k=1, 2, \cdots, N$,

$$\hat{\boldsymbol{q}}=\hat{\boldsymbol{Q}}\boldsymbol{\theta}+\hat{\boldsymbol{w}} \quad (53)$$

where $\hat{\boldsymbol{q}}$ and $\hat{\boldsymbol{w}}$ are of dimension $(N\times n)\times 1$, $\hat{\boldsymbol{Q}}$ is of dimension $(N\times n)\times(2n\times n)$ and $\boldsymbol{\theta}$ is a $(2n\times n)\times 1$ vector

$\hat{\boldsymbol{q}}=[\boldsymbol{q}^T(3)\boldsymbol{q}^T(4)\cdots\boldsymbol{q}^T(N+2)]^T$, $\theta=\text{column}[\boldsymbol{H}_2\ \boldsymbol{H}_1]$,

$\hat{\boldsymbol{w}}=[\boldsymbol{w}^T(3)\boldsymbol{w}^T(4)\cdots\boldsymbol{w}^T(N+2)]^T$ and $\hat{\boldsymbol{Q}}=[\boldsymbol{\Xi}(1)\,\boldsymbol{\Xi}(2)\cdots\boldsymbol{\Xi}(N)]^T$

When the observation index of the system is s, the same equation as Eqn. (53) can be formed and

$\boldsymbol{q}=[\boldsymbol{q}^T(s+1)\boldsymbol{q}^T(s+2)\cdots\boldsymbol{q}^T(s+N)]^T$, $\theta=\text{column}[\boldsymbol{H}_s\ \boldsymbol{H}_{s-1}\cdots\boldsymbol{H}_1]$,

$\hat{\boldsymbol{w}}=[\boldsymbol{w}^T(s+1)\boldsymbol{w}^T(s+2)\cdots\boldsymbol{w}^T(s+N)]^T$ and $\hat{\boldsymbol{Q}}=[\boldsymbol{\Xi}(1)\,\boldsymbol{\Xi}(2)\cdots\boldsymbol{\Xi}(N)]^T$

with

$$\boldsymbol{\Xi}(k)=\begin{bmatrix}\boldsymbol{q}(k)\\ \boldsymbol{q}(k+1)\\ \vdots\\ \boldsymbol{q}(k+s-1)\end{bmatrix}\otimes\boldsymbol{I}_r \qquad r\times s=2n$$

The ordinary least square estimate method is applied to equation (53), the estimate of θ in the least square sense is

$$\bar{\boldsymbol{\theta}}_{LS}=(\hat{\boldsymbol{Q}}^T\boldsymbol{Q})^{-1}\hat{\boldsymbol{Q}}^T\hat{\boldsymbol{q}} \quad (54)$$

The second algorithm can be developed from equation (31) similarly. To this end equation (31) can be rewritten as

$$\hat{q}(k)=\Xi^T(k)\theta+\hat{w}(k) \cdots\cdots (55)$$

where

$$\hat{q}(k)=\begin{bmatrix} q(k+1) \\ q(k+2) \end{bmatrix},\ \Xi(k)=\begin{bmatrix} q(k) \\ q(k+1) \end{bmatrix}\otimes I_n,\ \hat{w}(k)=\begin{bmatrix} w(k+1) \\ w(k+2) \end{bmatrix}$$

and $\hat{q}(k)$ is a $2n\times1$ vector, Ξ is a $(2n\times n)\times n$ matrix, θ is a $(2n\times2n)\times1$ vector and $w(k)$ is a $2n$ vector representing the measurement noise.

Equation (55) can be further simplified into the form

$$\hat{q}=\hat{Q}\theta+\hat{w} \cdots\cdots (56)$$

for N measurements and $k=1, 2, \cdots, N$, where

$\hat{q}=[\hat{q}^T(k)\,\hat{q}^T(k+1)\cdots\hat{q}^T(k+N)]^T$, $\hat{Q}=[\Xi(k)\,\Xi(k+1)\cdots\Xi(k+N)]^T$,

$\hat{w}=[\hat{w}^T(k)\hat{w}^T(k+1)\cdots\hat{w}^T(k+N)]^T$ and $\theta=$column H

with N indicating the number of time data points, which usually is equal to or greater than $2n$.

When the observation index of the system is s, Eqn. (56) can also be used and

$$\hat{q}(k)=\begin{bmatrix} q(k+1) \\ q(k+2) \\ \vdots \\ q(k+s) \end{bmatrix},\ \Xi(k)=\begin{bmatrix} q(k) \\ q(k+1) \\ \vdots \\ q(k+s-1) \end{bmatrix}\otimes I_n,\ \hat{w}(k)=\begin{bmatrix} w(k+1) \\ w(k+2) \\ \vdots \\ w(k+s) \end{bmatrix}$$

If least square parameter estimation method is used to determine the unknown θ,

$$\overline{\theta}_{LS}=(\hat{Q}^T\hat{Q})^{-1}\hat{Q}^T\hat{q} \cdots\cdots (57)$$

The second algorithm has the similar formulation as the ITD method.

When the system is observable with an index s higher than two, the eigenvalue problem in equation (46) is readily extended to

$$H\begin{bmatrix} CZ \\ CZ\exp(\Lambda t) \\ \cdot\cdot \\ CZ\exp((s-1)\Lambda t) \end{bmatrix}=\begin{bmatrix} CZ \\ CZ\exp(\Lambda t) \\ \cdot\cdot \\ CZ\exp((s-1)\Lambda t) \end{bmatrix}\exp(\Lambda t) \cdots\cdots (58)$$

From relationships (35) and (58) the top sub-matrix CZ of the eigenvector matrix has the columns which are the vectors with the first r elements of the eigenvectors Ψ_i of the original vibrating system described by equation (13). When the number of measurement stations is the same as the order of the dynamic structure, equation (58) will reduce back to equation (46).

The flow scheme for the modal identification can then be summarized as follows.

(1) Collect the measured response data to form the matrices Q and $\hat{Q}$,

(2) Evaluate the estimate of H_2 and H_1 according to the equation (54) or the estimate of H according to equation (57),

(3) Form the matrix H according to the equation (30),

(4) Calculate the eigenvalues and eigenvectors of the matrix H,

(5) Calculate the natural frequencies and damping ratios of the original structure.

6. APPLICATION

(1) Determination of the order of the test structure

Neither the parameters nor the degrees of the structure are known before a test. The first step for processing the identification of a vibrating structure is therefore to determine the order of the structure, i. e., the degrees of freedom of the structure.

Form measurement matrix $\overline{Q}$

$$\overline{Q}=\begin{bmatrix} q(1) & q(2) & \cdots & q(N) \\ q(2) & q(3) & \cdots & q(N+1) \\ \vdots & \vdots & \cdots & \vdots \\ q(s-1) & q(s) & \cdots & q(N+s-2) \end{bmatrix} \qquad (59)$$

The following three methods can be used with this measurement matrix to determine the degrees of freedom excited in a test. The first method is to successively calculate the determinant $|\boldsymbol{D}_k|$ of the coveriance matrix $\boldsymbol{D}=\overline{\boldsymbol{Q}}\overline{\boldsymbol{Q}}^T$ by starting with assuming k degrees of freedom being equal to 1, 2, $\cdots$ and ratio $r_1=|\boldsymbol{D}_2|/|\boldsymbol{D}_1|$, $r_k=|\boldsymbol{D}_{k+1}|/|\boldsymbol{D}_k|$, until the ratio suddenly drops. Assume $k=m$ at that time, it means that $|\boldsymbol{D}_{k+1}|$ is very small and the matrix $\boldsymbol{D}_{k+1}$ becomes singular. Hence degrees of freedom of the structure excited will be m.

The second method is to perform the singular-value decomposition of the matrix $\boldsymbol{D}$ such as

$$\boldsymbol{D}=\boldsymbol{U}\boldsymbol{S}\boldsymbol{U}^T \qquad (60)$$

where $\boldsymbol{S}$ is a diagonal matrix with the singular values s_i in monotonic decreasing order; $\boldsymbol{U}$ is an orthogonal matrix.

For a tolerance ε, if

$$s_m/s_1 \geqq \varepsilon > s_{m+1}/s_1 \qquad (61)$$

then the rank of the matrix $\boldsymbol{D}$ will be m. Tolerance should be somewhat larger than the relative precision of the numbers of the calculation which depends on the computer used for the identification process.

The third method is to perform an orthogonal-triangular decomposition. By permuting the columns of $\boldsymbol{D}$, the diagonal elements of $\boldsymbol{R}$ can be rearranged to be monotone decreasing, i.e., there is a permutation matrix $\boldsymbol{P}$ such that

$$\boldsymbol{D}\boldsymbol{P}=\boldsymbol{U}\boldsymbol{R} \qquad (62)$$

where $\boldsymbol{U}$ is an orthogonal matrix; $\boldsymbol{R}$ is an upper triangular matrix with diagonal elements $r_{11} \geqq r_{22} \geqq \cdots \geqq r_{pp} \geqq 0$.

If $\boldsymbol{D}$ is of rank of m, a sharp break is expected after r_{mm},

$$\text{i.e.} \quad r_{mm} \gg r_{ii} \quad \text{when } i>m \qquad (63)$$

While the first two methods can detect well-defined rank of the matrix $\boldsymbol{D}$, the third method may be conservative for determining the rank of the matrix $\boldsymbol{D}$.

(2) The number of measurement stations

The number of measurement stations depends on the purpose of a test, the degrees of freedom to be excited and accuracy requirement. From equation (46), if only the frequencies and damping ratios need to be identified, a single measurement station will be enough for the identification. It can also be seen from equation (46) that in order to identify the eigenvectors of a test structure, at least two measurement stations will be required. One measurement station should be fixed in a series of tests as a reference measuring point. Other measurement stations move each time in the series of tests.

(3) Numerical computing

The matrix $\hat{\boldsymbol{Q}}^T\hat{\boldsymbol{Q}}$ in Eqns. (54) and (57) may be ill-conditioned, in particular when its dimension is high. There exist methods to find $\overline{\theta}_{LS}$ that much better numerically behaved. The so called "square-root filtering algorithms" are recommended to use in the least squares estimation. The details may be referred for example to Lawson and Hanson (1974).

From Eqns. (54) and (57), it can be seen that only half number of the parameters in Eqn. (54) compared with that of Eqn. (57) need to be estimated. This is one of the computation advantages of the proposed method. In addition to this, the matrix for the eigenvalue problem in Eqn. (30) is a Hessenberg matrix, the eigenvalue problem of which is much easier to be solved than that of ITD method and computation time can also be saved in the solution of the eigenvalue problem of matrix $\boldsymbol{H}$.

(4) Sampling rate

Through simple manipunations of trigonometric formulas, the sampling rate f_s should satisfy the following relationship

$$f_s > \left(\frac{2}{k+1}\right) f_{\max} \cdots\cdots (64)$$

where $f_{\max}$ is the maximum frequency can be identified by the sampling rate f_s and k is determined according to the interested range of frequencies. For the range of 0 to $f_{\max}$, $k=0$. For the range of $f_{\min} >$ 0 to $f_{\max}$, $k=0, 1, \cdots, p$ and $p < \frac{f_{\min}}{f_{\max} - f_{\min}}$. A frequency $f_N = f_s/2$ is called Nyquist frequency. The information about frequencies higher than the Nyquist frequency will be lost.

Hence when $f_{\max}$ is excessively high, the whole range of $(0-f_{\max})$ requires extremely high sampling rate which may present practical difficulties. In this case, there are two methods recommended to solve the problem. The first method is to divide the whole range of frequencies into some sub-range band. This approach allows smaller sampling rates to be used for each frequency band. Each frequency band must be studied using response information which contains only frequency components in the range of interest (through filtering or other means). The second solution is to use high recording speed tape recorder to record the signals and then play them back at a lower speed during the digitization process. The ratio of two speeds is used later as a correction factor to obtain the actual structural frequencies.

7. NUMERICAL EXAMPLES

Two examples are presented in this section to illustrate the results of the application of the proposed method.

The first example is a simulated experiment. In the simulated experiment, a mathematical model of a structure is given and free dynamic responses of the structure in the time domain are calculated under some initial conditions. Eigenvectors, natural frequencies and damping ratios are calculated from the original mathematical model. From the free responses by the proposed method, the eigenvectors, natural frequencies and damping ratios are identified. The identified modal parameters are then compared to those calculated to illustrate the effectiveness of the proposed method.

The example is a marine riser with five clamps as shown in Fig. 1. Thirty two normal modes were calculated by finite element method. The first eight modes were used to generate a set of free decay response data. In order to illustrate the capability of identification of closely spaced frequencies, the frequencies calculated by finite element method are not used. Instead, the frequencies are arbitrarily assigned as 12, 12.5, 40, 48, 56, 76, 100 and 114 Hz with damping factor 0.02 for all modes. Since the calculated modes are normal instead of complex, the following formula is used to compute the free decay response data.

$$\boldsymbol{q}(k) = \sum_{i=1}^{8} (\boldsymbol{\phi}_i e^{(-\eta_i \omega_i t_k)} \cos(\omega_i \sqrt{1-\eta_i^2}\ t_k))$$

In practical observation, the noise is inevitable, a set of random white noise of 25% of the noise to signal ratio was added to simulate an actual situation. The free decay responses without noise and with noise are shown in Figs. 2 and 3.

Twenty four simulated measurement stations were arranged along the riser equally spaced. Sampling frequency was taken as 160 Hz and 480 samples were recorded, which corresponds to a recording of 3 seconds. The corresponding Nyquist frequency is 160/2=80 Hz and the information about frequencies higher than 80 Hz is thus lost by sampling. When analyzing the free decay response data it will be assumed that the modal parameters which generate the data are not known. The measurement data matrix $\overline{\boldsymbol{Q}}$ was formed according to Eqn. (59). The rank of the matrix $\boldsymbol{D}$ was found to be six and six modes were identified. The identified modal parameters are given in Table 1. So called "Mode Shape Correlation Constant" (MSCC)[5),11)] was used to assess the accuracy of identified mode shapes. The following equation is used to compute the MSCC

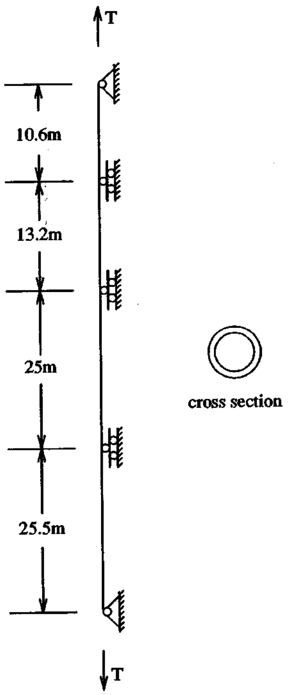

Fig. 1 Marine Riser.

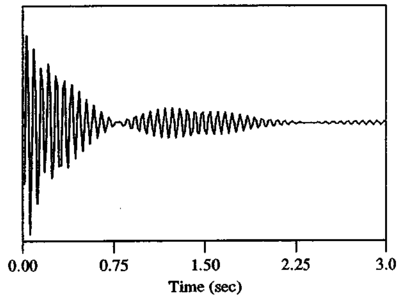

Fig. 2 Free Decay Responses Without Noise.

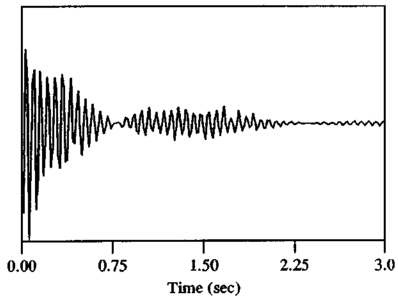

Fig. 3 Free Decay Responses With Noise.

$$MSCC=\frac{|\phi_a^* \times \phi_b|^2}{[\phi_a^* \times \phi_a] \times [\phi_b^* \times \phi_b]} \times 100$$

where

ϕ_a is the assumed input mode ; ϕ_b is the identified mode ;

*means the conjugate transpose of a vector ; | | indicates the amplitude of a mode.

The accuracy of identified frequencies and damping factors were qualified by direct comparison. It is observed that the natural frequencies, damping ratios and eigenvectors were identified with reasonable accuracy even though the frequencies are very closely spaced.

The second example is steel cantilever which was tested in the laboratory of structural engineering at the UNSW. The geometry of the cantilever is shown in Fig. 4.

The free decay responses were generated after sudden termination of excitation of the cantilever by a pseudo random noise signal. The exciter was located at 10 mm from the free end of the cantilever. To limit the range of interested frequencies, the accelerometer outputs were filtered to eliminate frequency components higher than 800 Hz. The sampling rate was 1 600 Hz. The recording time was 1.5 sec and 2 400 samples were recorded. Four accelerometers were arranged at 10 mm, 90 mm, 180 mm and 270 mm from the free end of the cantilever. To minimize the bias in the data, the excitation was set to a constant value for a reasonable long time before it was terminated.

Table 1 Identification Results for the Beam for Six Modes 25% Noise and 24 Degrees of Freedom.

MODE No.	FREQUENCY (Hz)	DAMPING RATIO (Hz)	MSCC WITH INPUT MODE No 1	2	3	4	5	6
1	12.06	1.87	100	0	0	0	0	0
2	12.53	2.12	3	100	0	0	0	0
3	39.93	2.11	0	0	100	0	0	0
4	48.09	1.82	0	0	0	100	0	0
5	55.87	2.06	0	2	0	0	100	0
6	76.17	2.08	0	0	3	0	0	100

Table 2 Identified Frequencies and Damping Ratios for the Cantilever.

Mode	Experimental Results				Theoretical Results	
	Proposed method		Circle fit method		Analytical	Finite element
	Frequency (Hz)	Damping ratio	Frequency (Hz)	Damping ratio	Frequency (Hz)	Frequency (Hz)
1	11.03	0.081	11.27	0.084	12.55	12.44
2	71.63	0.016	71.51	0.017	79.22	77.85
3	200.25	0.015	200.32	0.008	220.06	218.10
4	395.75	0.004	398.07	0.004	431.31	427.95
5	652.25	0.002	657.25	0.003	712.99	709.67

The tested results using the first proposed algorithm are listed in Tables 2 and 3. For comparison, test results by frequency domain method-circle fit using pseudo random noise excitation are also listed in these tables. Theoretical modal parameters of the cantilever were calculated without consideration of damping. Analytical and numerical results by finite element method are included in these tables.

It is noted there is good consistency between the results for modal parameters obtained by two experimental methods.

8. CONCLUSIONS

This paper presents a time domain method for modal identification of a linear vibrating structure using the autoregressive model. The method can reduce the computation time compared with ITD method without loss of the advantage of capability of identifying closely spaced modes or highly coupled modes.

A simulated example and a test case illustrated the effectiveness of the method.

Table 3 Identified Modes for the Cantilever.

	Complex mode		Normal Mode
	Proposed method	Circle fit method	Finite element method
Mode 1	1.000	1.000	1.000
	0.732+i0.008	0.709+i0.005	0.862
	0.584+i0.221	0.539+i0.187	0.725
	0.462-i0.063	0.410-i0.028	0.590
Mode 2	1.000	1.000	1.000
	0.482+i0.010	0.450+i0.007	0.515
	0.091+i0.021	0.082+i0.017	0.055
	-0.564+i0.010	-0.539+i0.006	-0.336
Mode 3	1.000	1.000	1.000
	0.094+i0.002	-0.089+i0.015	0.001
	-0.676-i0.023	-0.637-i0.015	-0.429
	-0.778+i0.009	-0.748+i0.007	-0.684
Mode 4	1.000	1.000	1.000
	-0.332-i0.021	-0.309-i0.018	-0.095
	-0.868-i0.031	-0.837-i0.026	-0.684
	-0.151-i0.002	-0.147-i0.001	-0.398
Mode 5	1.000	1.000	1.000
	-0.667-i0.032	-0.647-i0.027	-0.353
	-0.591+i0.005	-0.575+i0.004	-0.623
	0.763+i0.015	0.741+i0.011	0.271

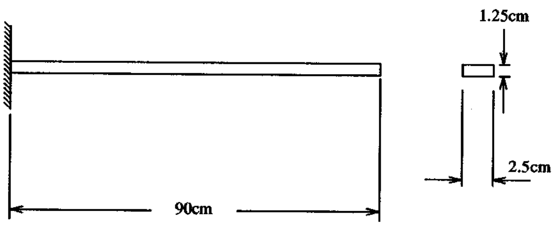

Fig. 4 Dimension of the cantilever beam.

APPENDIX A

Equation (29) may be rewritten as

$$H=\begin{bmatrix} C \\ C\Phi \end{bmatrix} I \begin{bmatrix} C\Phi^{-1} \\ C \end{bmatrix}^{-1} \qquad \text{(A·1)}$$

Assume

$$\begin{bmatrix} C\Phi^{-1} \\ C \end{bmatrix}^{-1}=[E \ \ F] \qquad \text{(A·2)}$$

then

$$\begin{bmatrix} C\Phi^{-1} \\ C \end{bmatrix}[E \ \ F]=\begin{bmatrix} C\Phi^{-1}E & C\Phi^{-1}F \\ CE & CF \end{bmatrix}=\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Hence

$$C\Phi^{-1}E=I_n \ \ C\Phi^{-1}F=0 \ \ CF=I_n \ \ CE=0 \qquad \text{(A·3)}$$

Substituting equation (A·2) into equation (A·1) and considering equation (A·3) obtain

$$H=\begin{bmatrix} C \\ C\Phi \end{bmatrix}[E \;\; F]=\begin{bmatrix} CE & CF \\ C\Phi E & C\Phi F \end{bmatrix}=\begin{bmatrix} 0 & I_n \\ H_2 & H_1 \end{bmatrix} \qquad (A\cdot 4)$$

with

$[H_2 \;\; H_1]=[C\Phi E \;\; C\Phi F]$

APPENDIX B

The Kronecker product of an $m\times n$ matrix $A=[a_{ij}]$ and a $p\times r$ matrix $B=[b_{ij}]$ is defined as

$$A\otimes B=\begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \qquad (B\cdot 1)$$

This is an $mp\times nr$ matrix.

The operator "column" is defined as the operation to form a column vector out of a matrix by stacking its columns on top of each other

$$\text{column}\, B=\begin{bmatrix} B^1 \\ B^2 \\ \vdots \\ B^r \end{bmatrix} \qquad (B\cdot 2)$$

where B^j is the j th column of B.

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