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PERTURBATION TECHNIQUE TO APPROXIMATE THE EFFECT OF DAMPING NONPROPORTIONALITY IN MODAL DYNAMIC ANALYSIS

By Benito M. PACHECO and Yozo FUJINO***

A general second-order perturbation technique is applied in approximating the complex eigenvectors and eigenvalues of MDOF system with moderately nonproportional viscous damping. Only the nonproportionality, not the overall level of damping itself, is assumed to be either moderate or weak. Perturbation coefficients explicitly relate the complex eigenvectors and eigenvalues to the nonproportional damping matrix of the system and the natural frequencies and mode shapes of the counterpart undamped system. Using the perturbed complex "modes", the dynamic response is expressed in a form analogous to modal superposition for proportionally damped system, but with additional terms explicitly representing nonproportionality effect. Numerical examples are given to illustrate the accuracy of the technique. It is pointed out that once the mode shapes and natural frequencies of the counterpart undamped system are known, in the present technique there is no need for numerically solving another eigenvalue problem, which would be bigger, when the damping is to be considered. This computational advantage is even more significant when designing or else identifying the system damping; either task requires reanalysis everytime that the damping matrix is changed. As a second advantage of the method, additional physical insights into the mathematical analysis are obtained. For example, the mode shapes of the counterpart undamped system are seen to couple to form the complex eigenvectors.

Keywords : perturbation, nonproportional damping, modal analysis, eigenproblem

1. INTRODUCTION

Often it is necessary to take into account not only the overall level but also the distribution pattern of damping within the structure. In many structures, damping capacity is distributed spacewise in manner not proportional to either mass or stiffness, nor satisfying the generalized criterion of proportionality that has been established by Caughey and O' Kelly[1]. The damping matrix in the linear multi-degree-of-freedom (MDOF) mathematical model corresponding to such a structure is nonproportional as to make the eigenvectors complex and different from the mode shapes of the counterpart undamped system. The nonproportionality is often either weak or moderate. Examples of nonproportionally damped system include : structure with tuned mass damper (TMD) ; structure with some high-damping elements or members ; structure-equipment system ; and soil-structure system.

In this paper, a brief review of the 30-year old generalized modal method of dynamic analysis for nonproportionally damped system is first given, along with reasons for its apparent unpopularity among engineers. The discussion is followed by a derivation of second-order perturbations to obtain the complex eigenvectors and eigenvalues, in such a way that the only eigenproblem that has to be directly solved is that of the counterpart undamped system. This is to lessen the computation effort, as well as to enhance the physical interpretation of complex eigenvector by relating the latter to the mode shapes of the counterpart undamped system.

The perturbed complex "modes" are next substituted into the generalized modal superposition equation for dynamic response. The resulting equation, involving only real quantities, is analogous to classical

* Member of JSCE, Dr. Eng., Asst. Prof., Dept. of Civil Eng., Univ. of Tokyo, Hongo 7-3-1, Bunkyo, Tokyo
** Member of JSCE, Ph. D., Asso. Prof., Eng. Research Inst., Univ. of Tokyo, Yayoi 2-11-16, Bunkyo, Tokyo

modal superposition for proportionally damped system, but with additional terms due to effects of moderate nonproportionality in damping.

All the equations necessary for implementing the method are listed. Numerical examples are given to demonstrate the computational accuracy and physical interpretation of the perturbations. More examples and further discussions are given in a companion paper[12].

2. REVIEW OF GENERALIZED MODAL METHOD

Representing the mass by matrix $\boldsymbol{M}$, the stiffness by $\boldsymbol{K}$, and the damping by $\boldsymbol{C}$, all of size $n\times n$ for system with n degrees of freedom, the equation of motion subject to external forces represented by vector $\boldsymbol{f}$, may be set up as in Eq. (1) below. It is assumed that the coordinates $\boldsymbol{x}$ have been so selected that the matrices $\boldsymbol{M}$ and $\boldsymbol{K}$ are positive definite. To be considered are cases where $\boldsymbol{C}$ is positive definite and the overall damping level may be high but still subcritical, the latter as may be checked, for instance, by the criterion of Inman and Andry[2] . Inequality (2) is Caughey and O' Kelly's[1] criterion for damping nonproportionality. For free vibration (Eq. (3)) in "mode shape" $\boldsymbol{y}_j$ and "frequency" λ_j (Eq. (4)), a quadratic eigenvalue problem (Eq. (5)) is obtained.

$$\boldsymbol{M}\ddot{\boldsymbol{x}}+\boldsymbol{C}\dot{\boldsymbol{x}}+\boldsymbol{K}\boldsymbol{x}=\boldsymbol{f} \qquad \boldsymbol{C}\boldsymbol{M}^{-1}\boldsymbol{K}\neq\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{C} \qquad (1),(2)$$

$$\boldsymbol{M}\ddot{\boldsymbol{x}}+\boldsymbol{C}\dot{\boldsymbol{x}}+\boldsymbol{K}\boldsymbol{x}=\boldsymbol{0} \qquad \boldsymbol{x}_j=\boldsymbol{y}_j\exp(\lambda_j t) \qquad (3),(4)$$

$$(\lambda_j^2\boldsymbol{M}+\lambda_j\boldsymbol{C}+\boldsymbol{K})\boldsymbol{y}_j=\boldsymbol{0},\ j=1,2,\cdots,r,\cdots,n,\ n+1,\cdots,\ n+r,\cdots,\ 2n \qquad (5)$$

Eq. (5) has $2n$ solutions, i. e. pairs of eigenvector $\boldsymbol{y}_j$ and eigenvalue λ_j. In a system as specified above, each eigenvalue is complex with negative real part; an ordering is assumed here such that the $(n+r)$-th eigenvalue is conjugate of the r-th. The eigenvectors also are complex. These solutions of the above eigenvalue problem not only show the natural dynamic properties of the system but can also uncouple the system of second-order differential equations of Eq. (1) into first-order differential equations. As pointed out by Foss[3], the dynamic response $\boldsymbol{x}(t)$ including initial displacement $\boldsymbol{x}_0$ and initial velocity $\dot{\boldsymbol{x}}_0$ may be obtained by a generalized modal superposition:

$$\boldsymbol{x}(t)=2\,\mathrm{Re}\sum_{j=1}^{n}P_j(t)\,\boldsymbol{y}_j=\sum_{j=1}^{n}(2\,\mathrm{Re}\,P_j)(\mathrm{Re}\,\boldsymbol{y}_j)-(2\,\mathrm{Im}\,P_j)(\mathrm{Im}\,\boldsymbol{y}_j) \qquad (6)$$

where Re and Im stand for " real part of" and "imaginary part of", respectively. The scalar function P_j, which might be called modal complex coordinate or modal participation function, is:

$$P_j=\boldsymbol{y}_j^T\exp(\lambda_j t)\left[\int_0^t \boldsymbol{f}(\tau)\exp(-\lambda_j\tau)\,d\tau+\lambda_j\boldsymbol{M}\boldsymbol{x}_0+\boldsymbol{C}\boldsymbol{x}_0+\boldsymbol{M}\dot{\boldsymbol{x}}_0\right]/r_j \qquad (7)$$

$$r_j=\boldsymbol{y}_j^T(2\lambda_j\boldsymbol{M}+\boldsymbol{C})\,\boldsymbol{y}_j \qquad (8)$$

The generalized modal method of Eq. (6), although mathematically well established, did not find early extensive application in structural engineering practice. The eigenvectors are complex, instead of real; hence the otherwise convenient and widely used modal method of analysis became burdened with some computational and conceptual difficulties. Veletsos and Ventura[4] observed that: (a) the generalized method is inherently more involved than the classical; (b) when used in conjunction with the response spectrum approach in seismic analysis, it has had to rely on approximations of questionable accuracy; and (c) perhaps most important, the physical meanings of the elements of the solution for this method have not been identifed as well as those for the classical modal method.

Several papers have since appeared in the literature that assume the eigenproperties to be known and concentrate the efforts on efficiently calculating the equivalent of P_j of Eq. (6). That is not to deny, however, that the computational effort required in solving Eq. (5) itself, can be much more than the requirement of the eigenvalue problem of an undamped system. While each of $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ is of size $n\times n$, numerical algorithms to solve Eq. (5) actually solve the eigenproblem of a $2n\times 2n$ matrix. Techniques of reducing both storage and computing time are indeed much welcome[5]. This is particularly true when either designing or identifying the system damping; either task requires reanalysis everytime that the damping matrix is changed. Some perturbation techniques have been proposed for lightly damped systems[6)~8)] that may avoid increasing the eigenproblem size from $n \times n$ to $2n \times 2n$. Chung and Lee[9)], applying the technique of Meirovitch and Ryland[8)], proposed to use a counterpart proportionally damped system as the unperturbed system in obtaining the eigenproperties. Cronin[10)], likewise regarding the nonproportionality as a perturbation, perturbed the harmonic response to first order and explored the possibility of approximating C by a "best" proportional damping matrix that is frequency dependent and load distribution dependent.

The present authors recently proposed[11)] a general second-order perturbation technique assuming that the nonproportionality is moderate, and derived explicit real-form formulas for the perturbations on frequencies, modal damping ratios, and nonproportionally damped "modes". The approach is equivalent in order, but different in formulation from Chung and Lee's[9)]. Details of the method are presented below.

3. PERTURBATIONS OF EIGENVALUES AND EIGENVECTORS

(1) Basic assumptions of the present perturbation method

The real eigenvalues ω_{0j} and real eigenvectors $\boldsymbol{y}_{0j}$ of the counterpart undamped system described by Eqs. (9)~(11) below are assumed to be known, with the eigenvectors normalized as in Eq. (12). Eq. (13) is the other orthogonality property. Note that $i=\sqrt{-1}$ and δ_{jk} is Kronecker delta.

$$\boldsymbol{M}\ddot{\boldsymbol{x}}+\boldsymbol{K}\boldsymbol{x}=\boldsymbol{0} \qquad \boldsymbol{x}_j=\boldsymbol{y}_{0j}\exp(i\,\omega_{0j}t) \quad \cdots (9),(10)$$

$$(-\omega_{0j}^2\boldsymbol{M}+\boldsymbol{K})\boldsymbol{y}_{0j}=\boldsymbol{0},\ j=1,2\cdots,\ r,\cdots n \quad \cdots (11)$$

$$\boldsymbol{y}_{0k}^T\boldsymbol{M}\boldsymbol{y}_{0j}=\delta_{jk} \qquad \boldsymbol{y}_{0k}^T\boldsymbol{K}\boldsymbol{y}_{0j}=\omega_{0j}^2\delta_{jk} \quad \cdots (12),(13)$$

In popular terminology, ω_{0j} is natural frequency and $\boldsymbol{y}_{0j}$ is mode shape. Transforming the damping matrix C using the modal matrix Y_0 as in Eq. (14) below, and separating the diagonal and offdiagonal elements, it is possible to formally identify a counterpart proportional damping matrix C_p and a damping nonproportionality matrix C_n uniquely.

$$\boldsymbol{Y}_0^T\boldsymbol{C}\boldsymbol{Y}_0=\mathrm{diag}[2\,\omega_{0j}\xi_{0j}]+\mathrm{offdiag}\,\tilde{\boldsymbol{C}} \quad \cdots (14)$$

$$\boldsymbol{C}_p=\boldsymbol{Y}_0\boldsymbol{M}\,\mathrm{diag}[2\,\omega_{0j}\xi_{0j}]\,\boldsymbol{M}\boldsymbol{Y}_0^T \quad \cdots (15)$$

$$\boldsymbol{C}_n=\boldsymbol{Y}_0\boldsymbol{M}\,\mathrm{offdiag}\,\tilde{\boldsymbol{C}}\boldsymbol{M}\boldsymbol{Y}_0^T \quad \cdots (16)$$

By the assumption of moderate nonproportionality, the norm of C_n is one order smaller than the corresponding norm of C_p. The quadratic eigenproblem of Eq. (5) may now be rewritten as:

$$(\lambda_j^2\boldsymbol{M}+\lambda_j(\boldsymbol{C}_p+\boldsymbol{C}_n)+\boldsymbol{K})\boldsymbol{y}_j=\boldsymbol{0},\ j=1,2,\cdots,r,\cdots,n,\ n+1,\cdots,n+r,\cdots,2n \quad \cdots (17)$$

where C_n may be viewed as a perturbation due to damping nonproportionality.

(2) Unperturbed system

From Eq. (17), by neglecting C_n, the unperturbed (or zero-order perturbed) eigenproblem of Eq. (18) below may be obtained, with solutions known from Eqs. (19)~(20).

$$(\lambda_j^2\boldsymbol{M}+\lambda_j\boldsymbol{C}_p+\boldsymbol{K})\boldsymbol{y}_j=\boldsymbol{0} \quad \cdots (18)$$

$$\lambda_j=\lambda_{0j}=-\omega_{0j}\xi_{0j}+i\omega_{0j}\sqrt{1-\varepsilon_{0j}^2},\ j=1,2,\cdots,r,\cdots n \quad \cdots (19)$$

$$\xi_{0j}=\boldsymbol{y}_{0j}^T\boldsymbol{C}_p\boldsymbol{y}_{0j}/2\,\omega_{0j} \quad \cdots (20)$$

$$\boldsymbol{y}_j=\boldsymbol{y}_{0j} \quad \cdots (21)$$

where C_p in Eq. (20) is replaced by C in actual calculation of (unperturbed) ξ_{0j}. Note that the eigenvectors (Eq. (21)) are real and identical to the undamped modes, while the eigenvalues (Eq. (19)) are complex. λ_{j+n} is conjugate of λ_j; $\boldsymbol{y}_{j+n}$ and $\boldsymbol{y}_j$ are identical.

(3) Perturbations up to second order

The complex-valued perturbations are assumed to have the following form.

$$\lambda_j=\lambda_{0j}+\lambda_{1j}+\lambda_{2j} \quad \cdots (22)$$

$$\boldsymbol{y}_j=\boldsymbol{y}_{0j}+\boldsymbol{y}_{1j}+\boldsymbol{y}_{2j} \qquad (23)$$

$$\boldsymbol{y}_{1j}=\sum_{k=1}^{n} a_{jk}(1-\delta_{jk})\,\boldsymbol{y}_{0k} \qquad \boldsymbol{y}_{2j}=\sum_{k=1}^{n} b_{jk}(1-\delta_{jk})\,\boldsymbol{y}_{0k} \qquad (24),\ (25)$$

where the first of two subscripts in Eqs. (22), (23) indicates the order of perturbation. In Eqs. (24), (25) for $\boldsymbol{y}_{1j}$ and $\boldsymbol{y}_{2j}$, it is not necessary to include $k=j$, i. e. $\boldsymbol{y}_{0j}$. The vector set $\boldsymbol{y}_{01}, \boldsymbol{y}_{02}, \cdots, \boldsymbol{y}_{0n}$ constitutes a compete vector space, in terms of which the expansion of $\boldsymbol{y}_j$ can be written ; however $\boldsymbol{y}_{0j}$ is already included in the expansion (Eq. (23)) as the first term.

The perturbation coefficients λ_{1j}, λ_{2j}, a_{jk} and b_{jk} are obtainable by : substitution of Eqs. (22), (23) into Eq. (17) ; grouping of terms of the same order of magnitude to yield three separate matrix equations, namely zero-order (Eq. (26)), first order (Eq. (27)) and second-order (Eq. (28)) ; and application of ortho-normalization properties of Eqs. (12), (13) and expansions Eqs. (24), (25).

$$(\lambda_{0j}^2\boldsymbol{M}+\lambda_{0j}\boldsymbol{C}_p+\boldsymbol{K})\,\boldsymbol{y}_{0j}=\boldsymbol{0} \qquad (26)$$

$$(\lambda_{0j}^2\boldsymbol{M}+\lambda_{0j}\boldsymbol{C}_p+\boldsymbol{K})\,\boldsymbol{y}_{1j}=-(2\,\lambda_{0j}\lambda_{1j}\boldsymbol{M}+\lambda_{1j}\boldsymbol{C}_p+\lambda_{0j}\boldsymbol{C}_n)\,\boldsymbol{y}_{0j} \qquad (27)$$

$$(\lambda_{0j}^2\boldsymbol{M}+\lambda_{0j}\boldsymbol{C}_p+\boldsymbol{K})\,\boldsymbol{y}_{2j}=-(2\,\lambda_{0j}\lambda_{1j}\boldsymbol{M}+\lambda_{1j}\boldsymbol{C}_p+\lambda_{0j}\boldsymbol{C}_n)\,\boldsymbol{y}_{1j}-((2\,\lambda_{0j}\lambda_{2j}+\lambda_{1j}^2)\,\boldsymbol{M}+\lambda_{2j}\boldsymbol{C}_p+\lambda_{1j}\boldsymbol{C}_n)\,\boldsymbol{y}_{0j} \qquad (28)$$

Observe that Eq. (26) is identical to Eq. (18). As for Eq. (27), after some tedious but straightforward matrix algebra, it can be reduced to formulas for λ_{1j} and a_{jk} ; likewise Eq. (28) yields formulas for λ_{2j} and b_{jk}. Denoting the elements of $\tilde{C}$ as $\tilde{c}_{jk}$, the complex perturbations may be expressed as follows.

$$\tilde{c}_{jk}=\boldsymbol{y}_{0k}^T\boldsymbol{C}\boldsymbol{y}_{0j} \qquad (29)$$

$$\lambda_{1j}=0 \qquad (30)$$

$$\lambda_{2j}=-\lambda_{0j}\sum_{k=1}^{n} a_{jk}(1-\delta_{jk})\,\tilde{c}_{jk}/2\,(\lambda_{0j}+\omega_{0j}\xi_{0j}) \qquad (31)$$

$$a_{jk}=\lambda_{0j}\tilde{c}_{jk}/(\lambda_{0k}-\lambda_{0j})\,(\lambda_{0k}+2\,\omega_{0k}\xi_{0k}+\lambda_{0j}) \qquad (32)$$

$$b_{jk}=\lambda_{0j}\sum_{l=1}^{n} a_{jl}(1-\delta_{jl})\,\tilde{c}_{kl}/(\lambda_{0k}-\lambda_{0j})\,(\lambda_{0k}+2\,\omega_{0k}\xi_{0k}+\lambda_{0j}) \qquad (33)$$

With Eqs. (19) ~ (25) and (29) ~ (33), the complex eigenvectors and eigenvalues of Eq. (5) are now expressed in terms of the real eigenvectors, or mode shapes, and real eigenvalues, or natural frequencies, of Eq. (11). This can mean a big reduction in the numerical calculations. The denominators of Eqs. (32) and (33) indicate that eigenvector perturbation is particularly large when both $\omega_{0j}\approx\omega_{0k}$ and $\xi_{0j}\approx\xi_{0k}$.

(4) Perturbations in real form

The complex eigenvalues and eigenvectors may be rewritten explicitly in terms of their respective real and imaginary parts. The forms in Eqs. (34) ~ (37) below are so chosen that the perturbations may take on some physical significance. For example, Eq. (34) is analogous to Eq. (19). By this analogy, ω_j is pseudo natural frequency, and ξ_j is pseudo damping ratio of "mode" j.

$$\lambda_j=-\omega_j\xi_j+i\omega_j\sqrt{1-\xi_j^2} \qquad (34)$$

$$\omega_j=\omega_{0j}\sqrt{1+\alpha_j} \qquad \xi_j=\xi_{0j}\sqrt{1+\beta_j} \qquad (35),\ (36)$$

$$\boldsymbol{y}_j=\boldsymbol{y}_{0j}+\sum_{k=1}^{n}\zeta_{jk}\boldsymbol{y}_{0k}+i\sum_{k=1}^{n}\eta_{jk}\boldsymbol{y}_{0k} \qquad (37)$$

$$Re\,\boldsymbol{y}_j=\boldsymbol{y}_{0j}+\sum_{k=1}^{n}\zeta_{jk}\boldsymbol{y}_{0k} \qquad Im\,\boldsymbol{y}_j=\sum_{k=1}^{n}\eta_{jk}\boldsymbol{y}_{0k} \qquad (37\cdot a,\ b)$$

Then α_j and β_j may be regarded as nonproportionality-induced perturbations of natural frequency and modal damping, respectively. As for the eigenvector, Eq. (37) shows that an eigenvector being complex is equivalent to damping-induced "coupling" of natural modes. As the perturbations ζ_{jk} and η_{jk} are generally not the same for all pairs of j and k, the relative values of these perturbations indicate which natural modes of the counterpart undamped system are significantly coupled due to damping nonproportionality. The formulas for α_j, β_j, ζ_{jk} and η_{jk} are summarized below. For compactness of expressions, Eqs. (38) ~

(42) are introduced as definitions.

$$\sigma_{0j}=\omega_{0j}\xi_{0j} \qquad \phi_{0j}=\omega_{0j}\sqrt{1-\xi_{0j}^2} \qquad (38),(39)$$

$$R_{jk}=[\{(\sigma_{0k}-\sigma_{0j})^2+(\phi_{0k}^2-\phi_{0j}^2)\}\sigma_{0j}-\{2(\sigma_{0k}-\sigma_{0j})\phi_{0j}\}\phi_{0j}]/D_{jk} \qquad (40)$$

$$I_{jk}=-[\{(\sigma_{0k}-\sigma_{0j})^2+(\phi_{0k}^2-\phi_{0j}^2)\}\phi_{0j}+\{2(\sigma_{0k}-\sigma_{0j})\phi_{0j}\}\sigma_{0j}]/D_{jk} \qquad (41)$$

$$D_{jk}=\{(\sigma_{0k}-\sigma_{0j})^2+(\phi_{0k}^2-\phi_{0j}^2)\}^2+\{2(\sigma_{0k}-\sigma_{0j})\phi_{0j}\}^2 \qquad (42)$$

$$\gamma_j=\sum_{k=1}^{n}(R_{jk}-I_{jk}\sigma_{0j}/\phi_{0j})\,\tilde{c}_{jk}^2/2\,\sigma_{0j} \qquad (43)$$

$$\kappa_j=-\sum_{k=1}^{n}(R_{jk}\sigma_{0j}/\phi_{0j}+I_{jk})\,\tilde{c}_{jk}^2/2\,\phi_{0j} \qquad (44)$$

$$\alpha_j=\xi_{0j}^2[(1+\gamma_j)^2-(1+\kappa_j)^2]+(2\,\kappa_j+\kappa_j^2) \qquad (45)$$

$$\beta_j=(\gamma_j^2+2\,\gamma_j-\alpha_j)/(1+\alpha_j) \qquad (46)$$

$$\zeta_{jk}=R_{jk}\tilde{c}_{jk}+\sum_{l=1}^{n}(R_{jk}R_{jl}-I_{jk}I_{jl})\,\tilde{c}_{jl}\tilde{c}_{kl} \qquad (47)$$

$$\eta_{jk}=I_{jk}\tilde{c}_{jk}+\sum_{l=1}^{n}(R_{jk}I_{jl}+I_{jk}R_{jl})\,\tilde{c}_{jl}\tilde{c}_{kl} \qquad (48)$$

The equations may look cumbersome ; but they are in fact explicit formulas ready for computer coding. These may be easily added to standard subroutines that are originally intended for the eigenvalue problem of Eq. (11) subject to eigenvector normalization of Eq. (12). Note that Eq. (12) is a very common choice of normalization in computer implementation of classical modal analysis.

4. DYNAMIC RESPONSE ANALYSIS BY SUPERPOSITION OF PERTURBED MODES

When the perturbed nonproportionally damped "modes" are substituted into Eqs. (6)~(8), to obtain the displacement response $\boldsymbol{x}$ to a general load $\boldsymbol{f}$ including effects of initial conditions $\boldsymbol{x}_0$ and $\dot{\boldsymbol{x}}_0$, the following form emerges. Note that all perturbations in the following are real (not complex).

$$\boldsymbol{x}=\boldsymbol{x}_1+\boldsymbol{x}_2+\boldsymbol{x}_3 \qquad (49)$$

$$\boldsymbol{x}_1=\sum_{j=1}^{n}\boldsymbol{y}_{0j}^T\boldsymbol{p}_j\boldsymbol{y}_{0j} \qquad (50)$$

$$\boldsymbol{x}_2=\sum_{j=1}^{n}\sum_{k=1}^{n}[\{\zeta_{jk}\boldsymbol{y}_{0k}^T\boldsymbol{p}_j+\eta_{jk}\boldsymbol{y}_{0k}^T\boldsymbol{q}_j\}\boldsymbol{y}_{0j}+\{\zeta_{jk}\boldsymbol{y}_{0j}^T\boldsymbol{p}_j+\eta_{jk}\boldsymbol{y}_{0j}^T\boldsymbol{q}_j\}\boldsymbol{y}_{0k}] \qquad (51)$$

$$\boldsymbol{x}_3=\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}[\{(\zeta_{jk}\zeta_{jl}-\eta_{jk}\eta_{jl})\boldsymbol{y}_{0k}^T\boldsymbol{p}_j\}\boldsymbol{y}_{0l}+\{(\eta_{jk}\zeta_{jl}+\zeta_{jk}\eta_{jl})\boldsymbol{y}_{0k}^T\boldsymbol{q}_j\}\boldsymbol{y}_{0l}] \qquad (52)$$

where the eigenvector perturbations ζ and η vanish, as do $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$, when damping is proportional. The term $\boldsymbol{x}_3$ is expected to be negligible compared to the sum $\boldsymbol{x}_1+\boldsymbol{x}_2$; hence for practical purposes the second term $\boldsymbol{x}_2$ may be regarded as the main effect of damping nonproportionality on the displacement response. Analogous equations, and similar comment as the foregoing, can be derived for velocity or acceleration response.

The disturbance-related vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ are defined below (Eqs. (58), (59)). Also for compactness of formulas, definitions are given by Eqs. (53), (54) that are analogous to Eqs. (38), (39). The convolution integrals of the force vector $\boldsymbol{f}$ have been written in terms of the impulse response function h_j (Eq. (55)) and its derivative with respect to time t.

$$\sigma_j=\omega_j\xi_j \qquad \phi_j=\omega_j\sqrt{1-\xi_j^2} \qquad (53),(54)$$

$$h_j(t)=\exp(-\sigma_j t)\sin(\phi_j t)/\phi_j \qquad (55)$$

$$\chi_j=-2\,\sigma_{0j}\gamma_j-2\,\sigma_{0j}(1+\gamma_j)\sum_{k=1}^{n}(\zeta_{jk}^2-\eta_{jk}^2)-4\,\phi_{0j}(1+\kappa_j)\sum_{k=1}^{n}\zeta_{jk}\eta_{jk}$$
$$+2\sum_{k=1}^{n}\sigma_{0k}(\zeta_{jk}^2-\eta_{jk}^2)+2\sum_{k=1}^{n}\zeta_{jk}\tilde{c}_{jk}+\sum_{k=1}^{n}\sum_{l=1}^{n}(\zeta_{jk}\zeta_{jl}-\eta_{jk}\eta_{jl})\,\tilde{c}_{kl} \qquad (56)$$

$$\psi_j=2\,\phi_{0j}\kappa_j+2\,\phi_{0j}(1+\kappa_j)\sum_{k=1}^{n}(\zeta_{jk}^2-\eta_{jk}^2)-4\,\sigma_{0j}(1+\gamma_j)\sum_{k=1}^{n}\zeta_{jk}\eta_{jk}$$

$$+4\sum_{k=1}^{n}\sigma_{0k}\zeta_{jk}\eta_{jk}+2\sum_{k=1}^{n}\eta_{jk}\tilde{c}_{jk}+\sum_{k=1}^{n}\sum_{l=1}^{n}(\zeta_{jk}\eta_{jl}+\eta_{jk}\zeta_{jl})\,\tilde{c}_{kl} \quad (57)$$

$$\begin{aligned}\boldsymbol{p}_j(t)=[&\{2(2\phi_{0j}+\psi_j)\phi_j+2\chi_j\sigma_j\}\int_0^t \boldsymbol{f}(\tau)\,h_j(t-\tau)\,d\tau+2\chi_j\int_0^t \boldsymbol{f}(\tau)\,\dot{h}_j(t-\tau)\,d\tau\\ &+\{(2(2\phi_{0j}+\psi_j)\phi_j+2\chi_j\sigma_j)(\boldsymbol{C}\boldsymbol{x}_0+\boldsymbol{M}\dot{\boldsymbol{x}}_0)-2\chi_j\omega_j^2(\boldsymbol{M}\boldsymbol{x}_0)\}\,h_j\\ &+\{(2(2\phi_{0j}+\psi_j)\phi_j-2\chi_j\sigma_j)(\boldsymbol{M}\boldsymbol{x}_0)+2\chi_j(\boldsymbol{C}\boldsymbol{x}_0+\boldsymbol{M}\dot{\boldsymbol{x}}_0)\}\,\dot{h}_j]\\ &/[4\phi_{0j}^2+4\phi_{0j}\psi_j+\chi_j^2+\psi_j^2]\end{aligned} \quad (58)$$

$$\begin{aligned}\boldsymbol{q}_j(t)=[&\{2(2\phi_{0j}+\psi_j)\sigma_j-2\chi_j\phi_j\}\int_0^t \boldsymbol{f}(\tau)\,h_j(t-\tau)\,d\tau+2(2\phi_{0j}+\psi_j)\int_0^t \boldsymbol{f}(\tau)\,\dot{h}_j(t-\tau)\,d\tau\\ &+\{(2(2\phi_{0j}+\psi_j)\sigma_j-2\chi_j\phi_j)(\boldsymbol{C}\boldsymbol{x}_0+\boldsymbol{M}\dot{\boldsymbol{x}}_0)-2(2\phi_{0j}+\psi_j)w_j^2(\boldsymbol{M}\boldsymbol{x}_0)\}\,h_j\\ &+\{(-2(2\phi_{0j}+\psi_j)\sigma_j-2\chi_j\phi_j)(\boldsymbol{M}\boldsymbol{x}_0)+2(2\phi_{0j}+\psi_j)(\boldsymbol{C}\boldsymbol{x}_0+\boldsymbol{M}\dot{\boldsymbol{x}}_0)\}\,\dot{h}_j]\\ &/[4\phi_{0j}^2+4\phi_{0j}\psi_j+\chi_j^2+\psi_j^2]\end{aligned} \quad (59)$$

χ_j and ψ_j are perturbations in Eqs. (58), (59); but unlike α_j, β_j, ζ_{jk} and η_{jk}, they have the dimension of ω instead of being dimensionless. An efficient computer algorithm may be written to numerically evaluate the two convolution integrals in Eqs. (58), (59) simultaneously.

5. NUMERICAL EXAMPLES

Systems with nine degrees of freedom are given here as examples. The structure is a cantilever with eight lumped masses and a tuned mass damper (TMD) attached at the free end, as illustrated in Fig. 1. To emphasize the immediate source of damping nonproportionality, the original 8-DOF structure itself (without the TMD) is chosen to be proportionally damped. Two example values of TMD damping ratio, ξ_{tmd}, are considered. Accordingly, the two 9-DOF examples have different levels of nonproportionality.

The mass, damping, and stiffness matrices of the total system are of the following form:

$$\boldsymbol{M}=\mathrm{diag}[m_1\ m_2\ m_3\ \cdots\cdots\ m_9]$$

$$\boldsymbol{C}=\begin{bmatrix} c_1+c_2 & -c_2 & 0 & \cdots\cdots & 0\\ & c_2+c_3 & -c_3 & \cdots\cdots & 0\\ \text{sym.} & & \cdots\cdots & \cdot\cdot\vdots\cdot\cdot & \vdots\\ & & & c_8+c_9 & -c_9\\ & & & & c_9 \end{bmatrix}$$

$$\boldsymbol{K}=\begin{bmatrix} k_1+k_2 & -k_2 & 0 & \cdots\cdots & 0\\ & k_2+k_3 & -k_3 & \cdots\cdots & 0\\ \text{sym.} & & \cdots\cdots & \cdots\cdots & 0\\ & & & k_8+k_9 & -k_9\\ & & & & k_9 \end{bmatrix}$$

The TMD mass, m_9, is considered to be one percent of the generalized mass of the fundamental mode of the structure. The TMD frequency $(\sqrt{k_9/m_9})$ is tuned to the first natural frequency of the original 8-DOF system. Denote the mass matrix of the original eight-degree-of-freedom cantilever as $\boldsymbol{M}_S$; its fundamental frequency as ω_{1S}; and the corresponding first mode shape as $\boldsymbol{y}_{1S}$, which is renormalized such that the modal displacement of the eighth degree of freedom, or cantilever free-end, is unity. Then the TMD properties are calculated as follows.

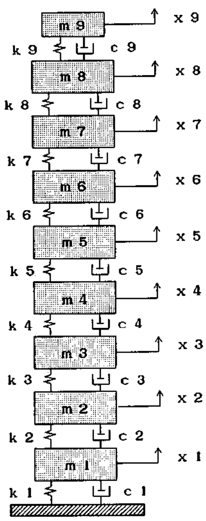

Fig. 1 9-DOF cantilever with TMD at free end.

Table 1 Maximum perturbations for the two examples.

	ξ_{TMD} = 5.0%	ξ_{TMD} = 7.5%
max α_j	0.0057	0.0151
max β_j	0.0072	0.0212
max ζ_{jk}	0.0089	0.0197
max η_{jk}	0.1971	0.2872

$m_9=0.01\ \boldsymbol{y}_{1s}^T\boldsymbol{M}_s\boldsymbol{y}_{1s}$ $k_9=m_9\omega_{1s}^2$ $c_9=2\ m_9\omega_{1s}\xi_{tmd}$

As numerical examples, the following structural properties in consistent units are used.

$m_1=m_2=\cdots=m_8=1$

$c_1=c_2=\cdots=c_8=4$

$k_1=k_2=\cdots=k_8=340$

$m_9=0.04286$ $k_9=0.4955$ $c_9=0.01457\ (\xi_{tmd}=5\ \%)$ or $0.02186\ (\xi_{tmd}=7.5\ \%)$

There being nine degrees of freedom, the real perturbation coefficients consist of nine pairs of α and β, and nine sets of eight pairs of ζ and η. The maximum values of these perturbations are listed in Table 1 for the two examples. It is to be noted that in the case of the more highly damped TMD ($\xi_{tmd}=7.5\ \%$), the nonproportionality of overall damping is higher, as may be understood from the larger perturbations. However, the cause of stronger nonproportionality is not the higher value of ξ_{tmd} alone; the stonger nonproportionality is due to the bigger discrepancy between TMD damping and structural damping.

As for comparing the perturbations on natural frequency, damping ratio and mode, it is observed that the dominant effect is on the eigenvector; the perturbation η, which converts real mode to complex eigenvector, is most significant.

For verifying the computational accuracy, pseudo frequency (ω), pseudo damping ratio (ξ) and complex "mode" ($\boldsymbol{y}$) are calculated by the method suggested by Foss[3], and values so obtained are treated as exact. For comparison, each complex vector, say vector j, is renormalized such that the complex element with largest modulus, say element m, obtains a modulus of unity ($L_{jm}=1$) and zero phase angle or argument ($\theta_{jm}=0$). The modulus and phase of element k of the renormalized j-th vector are denoted as L_{jk} and θ_{jk}. Then for each real scalar quantity : ω_j, ξ_j, L_{jk} and θ_{jk},

$$\varepsilon=\text{percentage error}=\text{abs}\left(\frac{\text{exact}-\text{perturbed}}{\text{perturbed}}\right)\times 100\ \%$$

Tables 2, 3 list for each mode the errors in ω_j and ξ_j; only the maximum errors among L_{jk} and θ_{jk} are shown. As comparison of the two tables shows, the errors are bigger when the perturbations are larger (Table 3 corresponding to $\xi_{tmd}=7.5\ \%$) ; this is consistent with the basic assumptions of the method. Comparing the various "modes", in Table 2 for instance, it is observed that the errors (and the perturbations) are larger for modes 1 and 2. These are the modes that have nearly equal natural frequencies ($\omega_{01}=\omega_{02}$), as to be expected since the TMD has been tuned to the first mode of the original 8-DOF cantilever ($\omega_{1s}=3.4$). Other examples that are partly reported in the companion paper[12] likewise show that nonproportionality tends to be particularly significant when natural frequencies are close. Numerical examples have shown that the perturbation coefficients, particularly η, are themselves a partial check of the accuracy of the method. Examples herein and in the companion paper[12] indicate that the pecentage error in phase, θ, does not exceed 10 % if the magnitude of perturbation η does not exceed 0.3 ; the errors of the other perturbations are then much smaller. Should the perturbations prove to be too large, the technique shall have served as indicator of necessity of a direct solution of the quadratic eigenproblem.

Table 2 Percentage errors when $\xi_{tmd}=5.0\ \%$.

Mode	ε_ω	ε_ξ	max ε_L	max ε_θ
1	0.011 %	0.049	0.223	3.633
2	0.011	0.018	0.241	3.627
3	0.000	0.000	0.001	0.037
4	0.000	0.000	0.001	0.029
5	0.000	0.000	0.001	0.009
6	0.000	0.000	0.001	0.008
7	0.000	0.000	0.001	0.004
8	0.000	0.000	0.001	0.004
9	0.000	0.000	0.000	0.004

Table 3 Percentage errors when $\xi_{tmd}=7.5\ \%$.

Mode	ε_ω	ε_ξ	max ε_L	max ε_θ
1	0.068 %	0.356	1.110	7.629
2	0.063	0.288	1.220	7.593
3	0.000	0.000	0.006	0.072
4	0.000	0.000	0.005	0.068
5	0.000	0.000	0.003	0.044
6	0.000	0.000	0.003	0.018
7	0.000	0.000	0.003	0.009
8	0.000	0.000	0.003	0.017
9	0.000	0.000	0.003	0.030

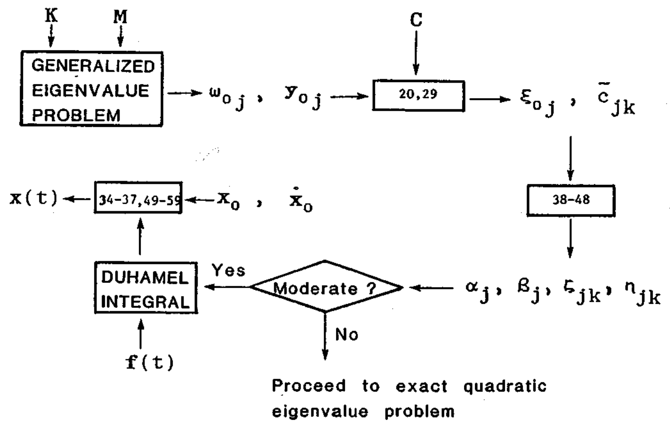

Fig. 2 Flowchart of the perturbation technique. Numbers in boxes refer to formulas in the paper. Generalized eigenvalue problem refers to equations (11) and (12).

Concerning computational efficiency, the present technique reduced to about one-fifth the time required to calculate the eigenvalues and eigenvectors of the 9-DOF examples. The reduction is expected to be more significant for larger systems. It may also depend on the system configuration.

6. SUMMARY AND CONCLUDING REMARKS

The method presented here can be summarized as in Fig. 2. All the formulas that are required in calculating the modal perturbations have been listed. By avoiding a direct numerical solution of the quadratic eigenvalue problem associated with nonproportionally damped system, the present technique reduces the calculations. The accuracy check is indirectly accomplished by verifying that none of the perturbations are too large; and tentative criterion has been indicated by numerical examples. Physical interpretation can be given to the real valued perturbations. Having expressed the complex "modes" in terms of natural modes, the perturbation technique may also pave the way for other rational extensions of classical modal combination rules for response spectrum method of seismic analysis.

The twin advantages of the present technique are even more significant when either designing or identifying the system damping. In either task, reanalysis is required everytime that the damping matrix is changed. The present technique requires only the eigenproblem of counterpart undamped system to be solved directly, and only once.

APPENDIX - LIST OF SYMBOLS

Matrices

$\boldsymbol{C}$=damping matrix

$\tilde{\boldsymbol{C}}$=offdiagonal matrix from transformation of $\boldsymbol{C}$ by $\boldsymbol{Y}_0$ (Eq. (14))

$\boldsymbol{C}_n$=nonproportional part of $\boldsymbol{C}$ (Eq. (16))

$\boldsymbol{C}_p$=proportional part of $\boldsymbol{C}$ (Eq. (15))

$\boldsymbol{K}$=stiffness matrix

$\boldsymbol{M}$=mass or inertia matrix

$\boldsymbol{Y}_0$=modal matrix where column j is natural mode $\boldsymbol{y}_{0j}$

Vectors

$\boldsymbol{f}(t)$=external force

$\boldsymbol{p}(t)$=main disturbance vector (Eqs. (58) and (50)~(52))

$\boldsymbol{q}(t)$=auxiliary disturbance vector (Eqs. (59) and (51), (52))

x_0=initial displacement

$\dot{x}_0$=initial velocity

$x(t)$=displacement

$x_1(t)$=primary part of $x(t)$ including only eigenvalue perturbations (Eqs. (49) and (50))

$x_2(t)$=secondary part of $x(t)$ due to eigenvector perturbations (Eqs. (49) and (51))

$x_3(t)$=tertiary part of $x(t)$ due to eigenvector perturbations (Eqs. (49) and (52))

$\dot{x}(t)$=velocity

$\ddot{x}(t)$=acceleration

y_j=complex j-th eigenvector (Eqs. (4) and (5))

y_{0j}=j-th mode, mode shape, or real eigenvector (Eqs. (10) and (11))

y_{1j}=complex first-order perturbation on j-th mode (Eqs. (23) and (24))

y_{2j}=complex second-order perturbation on j-th mode (Eqs. (23) and (25))

Common scalars

i=unit imaginary number

t=time

ε=percentage error

Scalars pertaining to mode j

$h_j(t)$=impulse response function (Eq. (55))

$P_j(t)$=complex coordinate or participation function (Eqs. (6) and (7))

$r_j(t)$=complex normalization constant (Eq. (8))

L_{jk}=modulus of k-th element of renormalized j-th complex eigenvector

θ_{jk}=phase angle of k-th element of renormalized j-th complex eigenvector

α_j=perturbation on natural frequency (Eqs. (35) and (45))

β_j=perturbation on damping ratio (Eqs. (36) and (46))

γ_j=perturbation paired with $\varkappa_j$ (Eq. (43))

$\varkappa_j$=perturbation paired with γ_j (Eq. (44))

λ_j=complex eigenvalue (Eqs. (4) and (5))

λ_{0j}=complex unperturbed eigenvalue (Eqs. (19) and (22))

λ_{1j}=complex first-order perturbation on eigenvalue (Eqs. (22) and (30))

λ_{2j}=complex second-order perturbation on eigenvalue (Eqs. (22) and (31))

ξ_{0j}=damping ratio when proportionally damped (Eq. (20))

ξ_j=pseudo damping ratio (Eqs. (34) and (36))

σ_{0j}=absolute value of real part of λ_{0j} (Eqs. (19) and (38))

σ_j=absolute value of real part of λ_j (Eqs. (34) and (53))

ϕ_{0j}=imaginary part of λ_{0j} (Eqs. (19) and (39))

ϕ_j=imaginary part of λ_j (Eqs. (34) and (54))

χ_j=perturbation paired with ψ_j (Eq. (56))

ψ_j=perturbation paired with χ_j (Eq. (57))

ω_{0j}=natural frequency (Eqs. (10) and (11))

ω_j=pseudo natural frequency (Eqs. (34) and (35))

Scalars relating modes j and k

D_{jk}=(Eq. (42))

I_{jk}=(Eq. (41))

R_{jk}=(Eq. (40))

a_{jk}=complex coefficient of first-order perturbation on j-th mode (Eqs. (24) and (32))

b_{jk}=complex coefficient of second-order perturbation on j-th mode (Eqs. (25) and (33))

$\tilde{c}_{jk}$=element of $\tilde{C}$ (Eq. (29))

δ_{jk}=Kronecker delta

ζ_{jk}=perturbation coefficient on real part of j-th mode (Eqs. (37) and (47))

η_{jk}=perturbation coefficient on imaginary part of j mode (Eqs. (37) and (48))

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