Structural Eng. / Earthquake Eng. Vol. 4. No. 2, 299s-309s. October 1987
Japan Society of Civil Engineers (Proc. of JSCE No. 386/I-8)

ELLIPTIC INTEGRAL SOLUTIONS FOR EXTENSIONAL ELASTICA WITH CONSTANT INITIAL CURVATURE

By Yoshiaki GOTO, Tokiharu YAMASHITA**, and Sei MATSUURA****

It is well known that closed-form solutions for elastica problems can be expressed by elliptic integrals. However, the problems solved so far are mostly based on inextensional beam theory where the elongation of member axis is ignored.

Herein, general solutions are derived for the extensional theories such as finite displacements with finite strains and those with small strains. In these derivations, special efforts are made to reduce the elliptic integrals to normal forms in order to obtain highly accurate solutions.

Keywords : elastica, finite displacement, elliptic integral

1. INTRODUCTION

The governing differential equations for the finite displacement plane beam theory become highly nonlinear, and, hence, it is very difficult to solve these equations analytically. Therefore, in order to simplify the solution procedures, the method with the separation of rigid body displacements[1),2)], combined with finite element techniques, is most widely used in general practice.

However, the closed-form solutions are still important to the practical point that the accuracy of approximate solutions can be precisely evaluated by these solutions, to say nothing of the mathematical importance. There have been presented sevaral methods directly to solve the highly nonlinear differential equations for plane beam theory. Nevertheless all the methods except that with elliptic integrals[3)~8)] cannot yield closed-form solutions, thus resulting in the difficulty to obtain accurate solution near buckling points. Indeed, it is well known that closed-form solutions are obtained for elastica problems, utilizing elliptic integrals, but the solutions presented so far are restricted to those of inextensional beam theory where the elongation of member axis is ignored. Furthermore, with a few exceptions[5)], the solutions derived are mostly for specific structures, that is, straight cantilevers[3),4),6),7)].

Herein, closed-from solutions with general expressions are derived for the members with constant initial curvature, based on the extensional theories such as finite displacements with finite strains and those with small strains. In the derivation of the solutions, special efforts are made to reduce elliptic integrals to normal forms. This is because the highly accurate methods in the calculation of elliptic integrals have been developed primarily for normal forms. Using the closed-form solutions thus derived, initially curved

* Member of JSCE, Dr. Eng., Assoc. Prof. of Civil Eng., Nagoya Institute of Technology (Showaku, Nagoya)
** Member of JSCE, M. Eng., Civil Eng. Dept., Hitachi Zosen Corporation (Konohanaku, Osaka)
*** Member of JSCE, Dr. Eng., Prof. of Civil Eng., Nagoya Institute of Technology (Showaku, Nagoya)

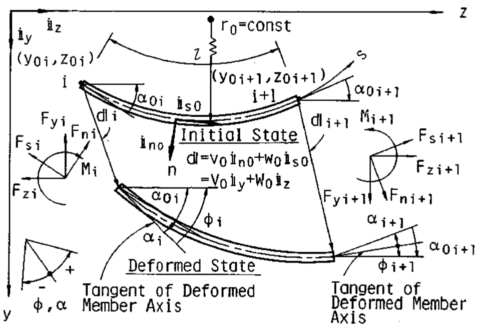

Fig. 1 Geometry of the Initial and the Deformed Beam Element.

Table 1 Governing Equations with Lagrangian Expression.

Equilibrium Equaions	Boundary Conditions	
	Mechanical	Geometrical
$F_n' - \frac{1}{r_0} F_s = 0$ $F_s' + \frac{1}{r_0} F_n = 0$	$F_n = \overline{F_n}$ $F_s = \overline{F_s}$ $M = \overline{M}$	$v_0 = \overline{v_0}$ $w_0 = \overline{w_0}$ $\alpha = \overline{\alpha}$
Theories	F_n , F_s	Stress Resultants vs. Displacements
a) Finite Displacements with Finite Strains	$F_n = N \sin\alpha + \frac{M'}{\sqrt{g_0}} \cos\alpha$ $F_s = N \cos\alpha - \frac{M'}{\sqrt{g_0}} \sin\alpha$	$N = EA(\sqrt{g_0} - 1)$ $M = -EI\alpha'$
b) Finite Displacements with Small Strains	$F_n = N \sin\alpha + M' \cos\alpha$ $F_s = N \cos\alpha - M' \sin\alpha$	$N = EA(\sqrt{g_0} - 1)$ $M = -EI\alpha'$
c) Inextensional Finite Displacements	$F_n = N \sin\alpha + M' \cos\alpha$ $F_s = N \cos\alpha - M' \sin\alpha$	$\sqrt{g_0} = 1$ $M = -EI\alpha'$

Remarks: The following notations are used throughout Tables.
E=Young's Modulus, $A=\int_A r_0/\hat{r} \cdot dA$, $I=\int_A n^2 r_0/\hat{r} \cdot dA$, A=Cross Sectional Area, (r_0=Radius of Initial Curvature of Centroidal Axis, $\hat{r}=r_0+n$) N=Axial Stress Resultant, $g_0=(v_0'-w_0/r_0)^2+(1+v_0/r_0+w_0')^2$, $'=d/ds$

cantilevers are analyzed in order to demonstrate the validity of the solutions as well as to examine the quantitative difference among the theories that are customarily used in the analysis of plane frames.

2. GOVERNING DIFFERENTIAL EQUATIONS

The governing differential equations used for this study are derived, introducing the customary assumptions of Bernoulli-Euler hypothesis and no change of cross sectional shapes. Following the general practice for curved members, the equations are expressed by the components defined in terms of the orthogonal curvilinear coordinates (n, s) shown in Fig. 1, where coordinate s is taken along the centroidal axis of a curved member before deformation. The governing equations, thus obtained, are summarized in Table 1[1], in which it should be noted that the rotational angle α can be expressed as

$$\sin\alpha=(v_0'-w_0/r_0)/\sqrt{g_0}, \quad \cos\alpha=(1+w_0'+v_0/r_0)/\sqrt{g_0} \quad \cdots\cdots (1\cdot a, b)$$

As seen from Table 1, distributed forces are not considered in the equilibrium equations, since the governing differential equations with distributed forces cannot be integrated analytically. In the table force-components vs. sectional force relations as well as the constitutive relations are classified into three levels of nonlinearity. The first is the theory of a) finite displacement with finite strains, which has no restrictions on the magnitude of strains except the beam assumptions. The other theories are obtained by introducing restrictions on the magnitude of strains. The second is called the theory of b) finite displacements with small strains, where the axial strains are assumed negligibly small compared with unity. The third is the theory of c) inextensional finite displacements, which is most simplified, introducing the assumption of inextensional member axis. The customary analysis using elliptic integrals are mostly based on the third theory.

3. INTEGRATION OF GOVERNING EQUATIONS

Although it is a general practice for the governing equations of curved members to be expressed in terms of the orthogonal curvilinear coordinate system (n, s), the differential equations with this expression can not be integrated in closed form. Hence, the components of physical quantities have to be transformed into a right handed rectangular Cartesian coordinate system (y, z), as shown in Fig. 1.

In the first place, the equilibrium equations are transformed into those expressed by the y- and

z-components. Following the rule of orthogonal transformation, the equilibrium equations are expanded into the directions of the y- and z-axes as

$$\left.\begin{array}{l}(F'_s+F_n/r_0)\sin\alpha_0+(F'_n-F_s/r_0)\cos\alpha_0=0\\ -(F'_n-F_s/r_0)\sin\alpha_0+(F'_s+F_n/r_0)\cos\alpha_0=0\end{array}\right\} \quad \cdots\cdots (2\cdot\text{a, b})$$

where α_0 is the angle between z-coordinate and the tangent of the member axis before deformation.

If the positive curvature of r_0 be defined as shown in Fig. 1, the following relation holds between α_0 and r_0.

$$\alpha'_0=-1/r_0 \quad \cdots\cdots (3)$$

in which α'_0 is constant from the assumption of constant initial curvature.

Substituting eq. (3) into eq. (2), eq. (2) yields

$$(F_s\sin\alpha_0)'+(F_n\cos\alpha_0)'=0, \quad -(F_n\sin\alpha_0)'+(F_s\cos\alpha_0)'=0 \quad \cdots\cdots (4\cdot\text{a, b})$$

Integrating eq. (4) with respect to s and introducing the boundary conditions at node i lead to

$$F_s\sin\alpha_0+F_n\cos\alpha_0=F_{yi}, \quad -F_n\sin\alpha_0+F_s\cos\alpha_0=F_{zi} \quad \cdots\cdots (5\cdot\text{a, b})$$

where (F_{yi}, F_{zi}) is the y-and z-components of the nodal force at i-th end.

In the following integration procedure, the expressions of the equations differ according to the theories. Here, for simplicity, the procedure is shown only for the theory of a) finite displacements with finite strains. The integration procedures for the other theories, however, are almost the same and these procedures can be easily understood by this example.

Using the (F_n, F_s) vs. (N, M')-relations given in Table 1, eq. (5) can be solved for N and M' as

$$M'=\sqrt{g_0}\,\{F_{yi}\cos(\alpha+\alpha_0)-F_{zi}\sin(\alpha+\alpha_0)\}, \quad N=F_{yi}\sin(\alpha+\alpha_0)+F_{zi}\cos(\alpha+\alpha_0) \quad \cdots\cdots (6\cdot\text{a, b})$$

Sectional forces N and M along with g_0 can further be eliminated from eq. (6), helped by the constitutive relations in Table 1. Thus, the governing differential equation is finally reduced to

$$-EI\alpha''=[1+\{F_{yi}\sin(\alpha+\alpha_0)+F_{zi}\cos(\alpha+\alpha_0)\}/EA]\{F_{yi}\cos(\alpha+\alpha_0)-F_{zi}\sin(\alpha+\alpha_0)\} \quad \cdots\cdots (7)$$

Considering $\alpha''_0=0$ from the assumption of constant initial curvature and multiplying both sides of the above equation by $\alpha'+\alpha'_0$, eq. (7) can be integrated as follows

$$\begin{aligned}(\alpha'+\alpha'_0)^2=&(M_i/EI+1/r_0)^2-2(F_{yi}/EI)\{\sin(\alpha+\alpha_0)-\sin(\alpha_i+\alpha_{0i})\}\\ &-2(F_{zi}/EI)\{\cos(\alpha+\alpha_0)-\cos(\alpha_i+\alpha_{0i})\}-(1/E^2AI)[F_{yi}F_{zi}\{\sin 2(\alpha+\alpha_0)\\ &-\sin 2(\alpha_i+\alpha_{0i})\}+(F^2_{zi}-F^2_{yi})\{\cos 2(\alpha+\alpha_0)-\cos 2(\alpha_i+\alpha_{0i})\}/2] \quad \cdots\cdots (8)\end{aligned}$$

where the integral constant is determined introducing the boundary conditions at node i. Here, for simplicity, a new variable ϕ is introduced to express $\alpha+\alpha_0$. As indicated in Fig. 1, ϕ is interpreted as an angle between the z axis and the tangent of the deformed member axis. Therefore, ϕ' can be solved from eq. (8) as

$$\phi'=f \quad \cdots\cdots (9\cdot\text{a})$$

$$\begin{aligned}f=&-\text{sign}(M+EI/r_0)[(M_i/EI+1/r_0)^2-2(F_{yi}/EI)(\sin\phi-\sin\phi_i)-2(F_{zi}/EI)(\cos\phi-\cos\phi_i)\\ &-(1/E^2AI)\{F_{yi}F_{zi}(\sin 2\phi-\sin 2\phi_i)+(F^2_{zi}-F^2_{yi})(\cos 2\phi-\cos 2\phi_i)/2\}]^{1/2} \quad \cdots\cdots (9\cdot\text{b})\end{aligned}$$

where sign $(\cdot)$ is defined to take the value of ± 1 according to the $\pm$ of $(\cdot)$.

Equation (9·a) can further be integrated in the form

$$s_{i+1}-s_i=l=\int_{\phi_i}^{\phi_{i+1}}(1/f)d\phi \quad \cdots\cdots (10)$$

This is the integral equation to calculate rotational angles.

In order to obtain displacement, displacement components (v_0, w_0) have to be transformed into the components (V_0, W_0) in the directions of y- and z-coordinates. From simple geometrical consideration, v_0 and w_0 are related to V_0 and W_0 as

$$v_0=V_0\cos\alpha_0-\sin\alpha_0, \quad w_0=W_0\cos\alpha_0+V_0\sin\alpha_0 \quad \cdots\cdots (11\cdot\text{a, b})$$

Substituting eq. (11) into eq. (1), eq (1) can be solved for V'_0 and W'_0 as

$$V'_0=\sqrt{g_0}\sin(\alpha+\alpha_0)-\sin\alpha_0, \quad W'_0=\sqrt{g_0}\cos(\alpha+\alpha_0)-\cos\alpha_0 \quad \cdots\cdots (12\cdot\text{a, b})$$

Noting that

Table 2 Integral Solutions.

$l=\int_{\phi_i}^{\phi_{i+1}} \frac{1}{f}\, d\phi$ $A_{i+1}=A_i-\int_{\phi_i}^{\phi_{i+1}} \frac{K_m}{f}(B_i \cdot s-C_i \cdot c)\, d\phi$	$\frac{W_{0_{i+1}}+z_{0_{i+1}}}{l}=\frac{W_{0_i}+z_{0_i}}{l}+\int_{\phi_i}^{\phi_{i+1}} \frac{K_w\, c}{f}\, d\phi$ $B_{i+1}=B_i$	$\frac{V_{0_{i+1}}+y_{0_{i+1}}}{l}=\frac{V_{0_i}+y_{0_i}}{l}+\int_{\phi_i}^{\phi_{i+1}} \frac{K_v\, s}{f}\, d\phi$ $C_{i+1}=C_i$	
Theories	a) Finite Displacements with Finite Strains	b) Finite Displacements with Small Strains	c) Inextensional Finite Displacements
f	$-\mathrm{sign}(M+EI/r_0)\{(A_i+l/r_0)^2-2B_i(c-c_i)$ $-2C_i(s-s_i)-\frac{B_iC_i}{\lambda^2}(s2-s2_i)$ $-\frac{B_i^2-C_i^2}{2\lambda^2}(c2-c2_i)\}^{1/2}$	$-\mathrm{sign}(M+EI/r_0)\{(A_i+l/r_0)^2-2B_i(c-c_i)$ $-2C_i(s-s_i)\}^{1/2}$	$-\mathrm{sign}(M+EI/r_0)\{(A_i+l/r_0)^2$ $-2B_i(c-c_i)-2C_i(s-s_i)\}^{1/2}$
K_w K_v K_m	$K_w=K_v=K_m=\sqrt{g_0}$	$K_w=K_v=\sqrt{g_0}$ $K_m=1.0$	$K_w=K_v=K_m=1.0$

Remarks: $A_i=\frac{M_i l}{EI}$, $B_i=\frac{F_{z_i} l^2}{EI}$, $C_i=\frac{F_{y_i} l^2}{EI}$, $\lambda=l/\sqrt{I/A}$,

$\phi=\alpha+\alpha_0$, $\phi_i=\alpha_i+\alpha_{0i}$, $\phi_{i+1}=\alpha_{i+1}+\alpha_{0i+1}$, $c=\cos\phi$, $s=\sin\phi$, $c_i=\cos\phi_i$, $s_i=\sin\phi_i$, $c2=\cos 2\phi$, $s2=\sin 2\phi$, $c2_i=\cos 2\phi_i$, $s2_i=\sin 2\phi_i$

l=Original Length of Curved Beam Element, sign$(\cdot)=\pm 1$ according to the $\pm$ of $(\cdot)$, $\sqrt{g_0}=1+(B_i \cdot c+C_i \cdot s)/\lambda^2$

$$dz_0/ds=\cos\alpha_0, \quad dy_0/ds=\sin\alpha_0 \qquad (13\cdot a,\ b)$$

eq. (12) yields

$$(V_0+y_0)'=\sqrt{g_0}\sin\phi, \quad (W_0+z_0)'=\sqrt{g_0}\cos\phi \qquad (14\cdot a,\ b)$$

Substituting eq. (9) into eq. (14), eq. (14) can be integrated as follows.

$$\left.\begin{aligned} V_{0i+1}+y_{0i+1}&=V_{0i}+y_{0i}+\int_{\phi_i}^{\phi_{i+1}}\{\sqrt{g_0}\sin\phi/f\}\,d\phi \\ W_{0i+1}+z_{0i+1}&=W_{0i}+z_{0i}+\int_{\phi_i}^{\phi_{i+1}}\{\sqrt{g_0}\cos\phi/f\}\,d\phi \end{aligned}\right\} \qquad (15\cdot a,\ b)$$

where

$$\sqrt{g_0}=(1/EA)(F_{zi}\cos\phi+F_{yi}\sin\phi)+1 \qquad (16)$$

From eq. (5), y- and z-components of nodal force are constant with respect to s, and the nodal force at node $i+1$ is given by

$$F_{yi+1}=F_{yi}, \quad F_{zi+1}=F_{zi} \qquad (17\cdot a,\ b)$$

Next, the integral equation for moment is derived. Here employed is the same technique as was used in the derivation of eq. (15). Thus, eq. (6·a) can be integrated as

$$M_{i+1}=M_i+\int_{\phi_i}^{\phi_{i+1}}\sqrt{g_0}\,(F_{yi}\cos\phi-F_{zi}\sin\phi)/f\cdot d\phi \qquad (18)$$

Equations (10), (15), (17), and (18) are the closed-form solutions in integral expression for the theory of a) finite displacements with finite strains. The nondimensionalized integral equations, thus obtained for the theories shown in Table 1, are summarized in Table 2.

From Table 2, it can be easily noticed that the integral equations expressed in terms of coordinates (y, z) are the same as those for straight members subject to additional bending moment of EI/r_0. Therefore, the beams with initial curvature can be analyzed making use of the integral equations for straight members, if only the initial curvature is taken into account as an additional moment.

3. REDUCTION OF ELLIPTIC INTEGRALS TO NORMAL FORMS

(1) Components of integrals

In Table 2, sign $(M+EI/r_0)$ can be let out of the integral sign by dividing the integration interval into subintervals such that the sign of $M+EI/r_0$ becomes either positive or negative throughout the respective subintervals. Hence, the transformation method does not lose its generality, even if sign $(M+1/r_0)$ is assumed 1 in Table 2. Accordingly, the integral equations shown in Table 2 are composed of the following independent integral components.

$$I_1=\int_{\phi_i}^{\phi_{i+1}} 1/f_1\cdot d\phi,\quad I_2=\int_{\phi_i}^{\phi_{i+1}} \sin\phi/f_1\cdot d\phi,\quad I_3=\int_{\phi_i}^{\phi_{i+1}} \cos\phi/f_1\cdot d\phi,$$
$$I_4=\int_{\phi_i}^{\phi_{i+1}} \sin\phi\cos\phi/f_1\cdot d\phi,\quad I_5=\int_{\phi_i}^{\phi_{i+1}} \sin^2\phi/f_1\cdot d\phi \qquad \cdots\cdots (19\,\text{a}\sim\text{e})$$

$$f_1=|f|=(a_0+a_1\cos\phi+a_2\sin\phi+a_3\sin\phi\cos\phi+a_4\cos^2\phi+a_5\sin^2\phi)^{1/2} \cdots\cdots (20)$$

where I_4 and I_5 are not included in the theory of c) inextensional finite displacements and the coefficients a_0 and $a_3\sim a_5$ differ according to the theories

Herein, the above integral components $I_1\sim I_5$ are transformed to be expressed by the three kinds of Legendre-Jacobi's normal forms[9),10)] described in Table 3. In this transformation, it is possible to simplify the procedures, if an appropriate method is employed depending on the form of f_1. According to the form of f_1, the theories in Table 1 can be classified into two groups. One group is of the theory of a) finite displacements with finite strains and the other is of the theories of b) finite displacements with small strains and c) inextensional finite displacemnts. Therefore, the transformation procedures are shown for these two groups respectively.

Table 3 Legendre-Jacobi's Normal Forms of Elliptic Integrals.

The Normal Elliptic Integral of the First Kind
$F=\int_{\hat{z}_i}^{\hat{z}_{i+1}} \frac{d\hat{z}}{\sqrt{(1-\hat{z}^2)(1-k^2\hat{z}^2)}}$
The Normal Elliptic Integral of the Second Kind
$E=\int_{\hat{z}_i}^{\hat{z}_{i+1}} \sqrt{\frac{1-k^2\hat{z}^2}{1-\hat{z}^2}}\, d\hat{z}$
The Normal Elliptic Integral of the Third Kind
$\Pi=\int_{\hat{z}_i}^{\hat{z}_{i+1}} \frac{d\hat{z}}{(1-\alpha\hat{z}^2)\sqrt{(1-\hat{z}^2)(1-k^2\hat{z}^2)}}$

(2) Finite displacements with finite strains

In this case, as far as authors examined, no simple methods of transformation were found out and, hence, a general method[9)], explained briefly as follows, is employed for this problem.

In the first place, introducing a new independent variable $\tilde{x}$ by the substitution

$$\tilde{x}=\tan\phi/2 \cdots\cdots (21)$$

the trigonometric integrands of eq. (19) are transformed into algebraic ones as

$$I_1=\int_{\tilde{x}_i}^{\tilde{x}_{i+1}} 2/f_2\cdot d\tilde{x} \qquad I_j=\int_{\tilde{x}_i}^{\tilde{x}_{i+1}} R_{j-1}/f_2\cdot d\tilde{x}\,(j=2\sim5) \cdots\cdots (22\cdot\text{a, b})$$

where

$$R_1=4\tilde{x}/(1+\tilde{x}^2),\ R_2=2(1-\tilde{x}^2)/(1+\tilde{x}^2),\ R_3=4\tilde{x}(1-\tilde{x}^2)/(1+\tilde{x}^2)^2,$$
$$R_4=8\tilde{x}^2/(1+\tilde{x}^2)^2,\ f_2=(b_0\tilde{x}^4+b_1\tilde{x}^3+b_2\tilde{x}^2+b_3\tilde{x}+b_4)^{1/2} \cdots\cdots (23\cdot\text{a}\sim\text{e})$$

The coefficients of eq. (23·e) are given by

$$b_0=a_0-a_1+a_4,\ b_1=2(a_2-a_3),\ b_2=2(a_0-a_4+2a_5),$$
$$b_3=2(a_2+a_3),\ b_4=a_0+a_1+a_4 \cdots\cdots (24\cdot\text{a}\sim\text{e})$$

Henceforth, the algebraic integrands of eq. (22) are reduced to normal forms, following the customary method shown in Ref 9).

Making use of the four roots $\gamma_i\,(i=1\sim4)$[a)] of the equation

$$(f_2)^2=b_0\tilde{x}^4+b_1\tilde{x}^3+b_2\tilde{x}^2+b_4=0 \cdots\cdots (25)$$

$\tilde{x}$ in eq. (21) is further transformed to a new variable $\tilde{y}$ defined by the following substitution

$$\text{I}:\ \tilde{x}=\tilde{y}+m,\ \text{when}\ \gamma_1+\gamma_2=\gamma_3+\gamma_4,\quad \text{II}:\ \tilde{x}=(m+n\tilde{y})/(1+\tilde{y}),\ \text{when}\ \gamma_1+\gamma_2\neq\gamma_3+\gamma_4 \cdots\cdots (26\cdot\text{a, b})$$

in which m for I and (m, n) for II are respectively given by

$$\text{I}:\ m=(\gamma_1+\gamma_2)/2,$$
$$\text{II}:\ \left.\begin{matrix} m \\ n \end{matrix}\right\}=[-(\gamma_1\gamma_2-\gamma_3\gamma_4)\pm\{(\gamma_1-\gamma_3)(\gamma_2-\gamma_3)(\gamma_1-\gamma_4)(\gamma_2-\gamma_4)\}^{1/2}]/(-\gamma_1-\gamma_2+\gamma_3+\gamma_4) \cdots\cdots (27\cdot\text{a, b})$$

It should be noted here that the relation $mn=1$ holds specifically for the present case.

a) If equation (25) has imaginary roots, γ_1 and γ_2 are defined as a pair of complex roots. When the equation has only real roots, γ_1 and γ_2 are arbitrary two adjacent roots arranged in the decreasing order of magnitude. In the specific case when the equation has multiple roots, all the integrals of eq. (22) are reduced to elementary integrals.

With the substitution given by eq. (26), integrals of eq. (22) are reduced to

$$I_1=e_1\int_{\tilde{y}_i}^{\tilde{y}_{i+1}} 1/f_3\cdot d\tilde{y},\quad I_j=e_j\int_{\tilde{y}_i}^{\tilde{y}_{i+1}} R_{2j+1}/f_3\cdot d\tilde{y}+e_j\int_{\tilde{y}_i}^{\tilde{y}_{i+1}} R_{2(j+1)}\,\tilde{y}/f_3\cdot d\tilde{y}\ (j=2\sim5) \quad \cdots\cdots (28\cdot a,\ b)$$

$$f_3=\{(c_1+c_2\tilde{y}^2)(c_3+c_4\tilde{y}^2)\}^{1/2} \quad \cdots\cdots (29)$$

where $R_5\sim R_{12}$ are the rational functions in terms of $\tilde{y}^2$, as given in Appendix A, while $e_1\sim e_5$ and $c_1\sim c_4$ are constants shown in Appendix B.

It is easily understood that the second integral of eq. (28・b) can be expressed by elementary functions, and, hence, eq. (28・a) along with the first integral of eq. (28・b) are of primary concern in the transformation hereinafter.

The first integrand of eq. (28・b) is further resolved into partial fractions. The results of the resolution are different according to whether eq. (26・a) or eq. (26・b) is used in the substitution. In case when eq. (26・a) is used, the expansion is given by the form

$$\int_{\tilde{y}_i}^{\tilde{y}_{i+1}} R_k/f_3\cdot d\tilde{y}:(k=5,\ 7)\rightarrow\left\{\begin{array}{l}\int_{\tilde{y}_i}^{\tilde{y}_{i+1}}\delta_1/f_3\cdot d\tilde{y}+\int_{\tilde{y}_i}^{\tilde{y}_{i+1}}\delta_2/\{(\tilde{y}^2-\beta_1)f_3\}d\tilde{y}\\ +\int_{\tilde{y}_i}^{\tilde{y}_{i+1}}\delta_3/\{(\tilde{y}^2-\beta_2)f_3\}d\tilde{y}\end{array}\right\} \quad \cdots\cdots (30\cdot a)$$

$$\int_{\tilde{y}_i}^{\tilde{y}_{i+1}} R_k/f_3\cdot d\tilde{y}:(k=9,\ 11)\rightarrow\left\{\begin{array}{l}\int_{\tilde{y}_i}^{\tilde{y}_{i+1}}\delta_1/f_3\cdot d\tilde{y}+\int_{\tilde{y}_i}^{\tilde{y}_{i+1}}\delta_2/\{(\tilde{y}^2-\beta_1)f_3\}d\tilde{y}\\ +\int_{\tilde{y}_i}^{\tilde{y}_{i+1}}\delta_3/\{(\tilde{y}^2-\beta_2)f_3\}d\tilde{y}+\int_{\tilde{y}_i}^{\tilde{y}_{i+1}}\delta_4/\{(\tilde{y}^2-\beta_1)^2f_3\}d\tilde{y}\\ +\int_{\tilde{y}_i}^{\tilde{y}_{i+1}}\delta_5/\{(\tilde{y}^2-\beta_2)^2f_3\}d\tilde{y}\end{array}\right\} \quad \cdots\cdots (30\cdot b)$$

in which β_1, β_2 and $\delta_1\sim\delta_5$ are contants that differ according to the rational functions of R_j ($j=5,\ 7,\ 9,\ 11$).

On the other hand, if eq. (26・b) is applied, the expansion is simplified due to the relation of $mn=-1$ as

$$\int_{\tilde{y}_i}^{\tilde{y}_{i+1}} R_k/f_3\cdot d\tilde{y}:(k=5,\ 7)\rightarrow\int_{\tilde{y}_i}^{\tilde{y}_{i+1}}\delta_1/f_3\cdot d\tilde{y}+\int_{\tilde{y}_i}^{\tilde{y}_{i+1}}\delta_2/\{(\tilde{y}^2-\beta_1)f_3\}d\tilde{y} \quad \cdots\cdots (31\cdot a)$$

Table 4 Transformation and Expression with Normal Forms.

f_3	$\lvert\tilde{y}\rvert$ / Transformation	Eq. (32)	Expression with Normal Forms	k	$\tilde{\alpha}$
1) $\{(r^2-\tilde{y}^2)(q^2-\tilde{y}^2)\}^{1/2}$	$0\leq\lvert\tilde{y}\rvert\leq q<r$	a	$F/(rt)$	$\frac{q}{r}$	$\frac{q^2}{\beta}$
	$\tilde{y}=q\tilde{z}$	b	$-\Pi/(rt\beta)$		
		c	$\Pi'/(rt\beta^2)$		
2) $\{(\tilde{y}^2-r^2)(\tilde{y}^2-q^2)\}^{1/2}$	$q<r\leq\lvert\tilde{y}\rvert$	a	$-F/(rt)$	$\frac{q}{r}$	$\frac{\beta}{r^2}$
	$\tilde{y}=r/\tilde{z}$	b	$(F-\Pi)/(rt\beta)$		
		c	$(-F+2\Pi-\Pi')/(rt\beta^2)$		
3) $\{(\tilde{r}^2-y^2)(\tilde{y}^2-q^2)\}^{1/2}$	$q\leq\lvert\tilde{y}\rvert\leq r$	a	$-F/(rt)$	$\frac{(r^2-q^2)^{1/2}}{r}$	$\frac{r^2-q^2}{r^2-\beta}$
	$\tilde{y}^2=r^2-(r^2-q^2)\tilde{z}^2$	b	$-\Pi/\{rt(r^2-\beta)\}$		
		c	$-\Pi'/\{rt(r^2-\beta)^2\}$		
4) $\{(r^2+\tilde{y}^2)(q^2-\tilde{y}^2)\}^{1/2}$	$0\leq\lvert\tilde{y}\rvert\leq q$	a	$-F/\{(r^2+q^2)^{1/2}t$	$\frac{q}{(r^2+q^2)^{1/2}}$	$\frac{q}{q^2-\beta}$
	$\tilde{y}^2=q^2(1-\tilde{z}^2)$	b	$-\Pi/\{(r^2+q^2)^{1/2}t(q^2-\beta)\}$		
		c	$-\Pi'/\{(r^2+q^2)^{1/2}t(q^2-\beta)^2\}$		
5) $\{(r^2+\tilde{y}^2)(\tilde{y}^2-q^2)\}^{1/2}$	$q\leq\lvert\tilde{y}\rvert$	a	$F/\{(r^2+q^2)^{1/2}t\}$	$\frac{r}{(r^2+q^2)^{1/2}}$	$\frac{\beta}{\beta-q^2}$
	$\tilde{y}^2=q^2/(1-\tilde{z}^2)$	b	$\{-F+q^2\Pi/(q^2-\beta)\}/\{(r^2+q^2)^{1/2}t\beta\}$		
		c	$\{F-2q^2\Pi/(q^2-\beta)+q^4\Pi'/(q^2-\beta)^2\}/\{(r^2+q^2)^{1/2}t\beta^2\}$		
6) $\{(\tilde{y}^2+r^2)(\tilde{y}^2+q^2)\}^{1/2}$	$q<r$	a	$F/(rt)$	$\frac{(r^2-q^2)^{1/2}}{r}$	$\frac{q^2+6}{\beta}$
	$\tilde{y}^2=q^2\tilde{z}^2/(1-\tilde{z}^2)$	b	$(-F-q^2\Pi/\beta)/\{rt(q^2+\beta)\}$		
		c	$(F+2q^2\Pi/\beta+q^4\Pi'/\beta^2)/\{rt(q^2+\beta)^2\}$		

Remarks: $r=(\lvert c_1/c_2\rvert)^{1/2}$, $q=(\lvert c_3/c_4\rvert)^{1/2}$, $t=(\lvert c_2\cdot c_4\rvert)^{1/2}$, $\beta=\beta_i$

$$\Pi'=\left(\left[\frac{\tilde{\alpha}^2\tilde{z}\{(1-\tilde{z}^2)(1-k^2\tilde{z}^2)\}^{1/2}}{(1-\tilde{\alpha}\tilde{z}^2)}\right]_{\tilde{z}_i}^{\tilde{z}_{i+1}}+(\tilde{\alpha}-k^2)F-\tilde{\alpha}E+\{3k^2-2(1+k^2)\tilde{\alpha}+\tilde{\alpha}^2\}\Pi\right)/\{2(\tilde{\alpha}-k^2)(\tilde{\alpha}-1)\},$$

E, F, and Π are the normal forms of elliptic integrals described in Table 3.

$$\int_{\tilde{y}_i}^{\tilde{y}_{i+1}} R_k/f_3\cdot d\tilde{y} : (k=9,\ 11)\rightarrow\left\{\begin{array}{l}\int_{\tilde{y}_i}^{\tilde{y}_{i+1}}\delta_1/f_3\cdot d\tilde{y}+\int_{\tilde{y}_i}^{\tilde{y}_{i+1}}\delta_2/\{(\tilde{y}^2-\beta_1)f_3\}d\tilde{y}\\ +\int_{\tilde{y}_i}^{\tilde{y}_{i+1}}\delta_3/\{(\tilde{y}^2-\beta_1)^2f_3\}d\tilde{y}\end{array}\right\} \cdots\cdots (31\cdot b)$$

where β_1, β_2 and $\delta_1\sim\delta_3$ are all real constants. From eqs. (30) and (31), it can be seen that the integrals $I_1\sim I_5$ of eq. (28) are expressed by the following three kinds of integrals in addition to elementary integrals.

$$\int_{\tilde{y}_i}^{\tilde{y}_{i+1}} 1/f_3\cdot d\tilde{y},\quad \int_{\tilde{y}_i}^{\tilde{y}_{i+1}} 1/\{(\tilde{y}^2-\beta_i)f_3\}d\tilde{y},\quad \int_{\tilde{y}_i}^{\tilde{y}_{i+1}} 1/\{(\tilde{y}^2-\beta_i)^2f_3\}\,d\tilde{y}\quad (i=1,\ 2) \cdots\cdots (32\cdot a\sim c)$$

Finally, the above three integals are transformed into normal forms. The method of this transformation, however, differs according to the limits of integrals as well as the values of $c_1\sim c_4$ in eq. (29). Therefore, the methods and the results of transformation are classified in Table 4.

As is noticed from Table 4, the integrals $I_1\sim I_5$ include the same components of normal forms regardless of the method of transformation and the components of these integrals can be symbolically expressed as follows, using the notations for normal forms indicated in Table 3.

$$I_1\rightarrow F,\ I_2\rightarrow F+\Pi+G,\ I_3\rightarrow F+\Pi+G,\ I_4\rightarrow F+E+\Pi+G,\ I_5\rightarrow F+E+\Pi+G \cdots\cdots (33\cdot a\sim e)$$

where G denotes an elementary integral.

(3) Finite displacements with small strains and inextensional finite displacements

The method of transformation used in the previous section is versatile and this method can also be applied for the theories of b) finite displacements with small strains and c) inextensional finite displacements. However, the above method is very much complicated because it requires a lot of transformations, which also differ according to the limits of integrals as well as the values of coefficients $b_0\sim b_4$ in eq. (25).

In the present case, the function f in Table 2 is more simplified than that of a) finite displacements with finite strains. Accordingly, it is possible to use a simpler method of transformation, compared with that shown in the previous section.

As is easily from Table 2, the trigonometric functions in function f can be combined as

$$f_4=|f|=(P_i+Q_i\cos\psi)^{1/2} \cdots\cdots (34)$$

in which

$$\left.\begin{array}{l}\psi=\phi-\theta_i,\ P_i=(A_i+l/r_0)^2-Q_i\cos\psi_i,\ Q_i=2(B_i^2+C_i^2)^{1/2},\\ \cos\theta_i=-2B_i/Q_i,\ \sin\theta_i=-2C_i/Q_i\end{array}\right\} \cdots\cdots (35\cdot a\sim e)$$

With eq. (35·a), eqs. (19·a~e) are transformed to

$$\left.\begin{array}{l}I_1=I_1^*,\ I_2=\cos\theta_iI_2^*+\sin\theta_iI_3^*,\ I_3=\cos\theta_iI_3^*-\sin\theta_iI_2^*,\\ I_4=\cos 2\theta_iI_4^*-\sin 2\theta_iI_5^*+(\sin 2\theta_i/2)I_1^*,\\ I_5=\cos 2\theta_iI_5^*+\sin 2\theta_iI_4^*+\sin^2\theta_iI_1^*\end{array}\right\} \cdots\cdots (36\cdot a\sim e)$$

where $I_1^*\sim I_5^*$ are independent integral components given by

$$\left.\begin{array}{l}I_1^*=\int_{\psi_i}^{\psi_{i+1}}1/f_4\cdot d\psi,\ I_2^*=\int_{\psi_i}^{\psi_{i+1}}\sin\psi/f_4\cdot d\psi,\ I_3^*=\int_{\psi_i}^{\psi_{i+1}}\cos\psi/f_4\cdot d\psi,\\ I_4^*=\int_{\psi_i}^{\psi_{i+1}}\sin\psi\cos\psi/f_4\cdot d\psi,\ I_5^*=\int_{\psi_i}^{\psi_{i+1}}\sin^2\psi/f_4\cdot d\psi\end{array}\right\} \cdots\cdots (37\cdot a\sim e)$$

These integral components are to be transformed into normal forms. Considering that the beams always deform such that the value within the parentheses of eq. (34) becomes positive, a new independent variable $\tilde{y}$ is introduced by the substitution

$$\tilde{y}^2=P_i+Q_i\cos\psi \cdots\cdots (38)$$

With this substitution, eqs. (37·a~e) are reduced to

$$I_1^*=-2\int_{\tilde{y}_i}^{\tilde{y}_{i+1}}\mathrm{sign}(\sin\psi)/f_5\cdot d\tilde{y},\ I_2^*=-(2/Q_i)(\tilde{y}_{i+1}-\tilde{y}_i),$$

$$I'_3=-(2/Q_i)\int_{\tilde{y}_i}^{\tilde{y}_{i+1}}\mathrm{sign}(\sin\phi)(\tilde{y}^2-P_i)/f_5\cdot d\tilde{y},\quad I'_4=-(2/Q_i^2)\{(\tilde{y}_{i+1}^3-\tilde{y}_i^3)/3-P_i(\tilde{y}_{i+1}-\tilde{y}_i)\},$$
$$I'_5=-(2/Q_i^2)\int_{\tilde{y}_i}^{\tilde{y}_{i+1}}\mathrm{sign}(\sin\phi)f_5\cdot d\tilde{y} \qquad \cdots\cdots (39\cdot a\sim e)$$

in which

$$f_5=[\{\tilde{y}^2-(P_i-Q_i)\}\{(P_i+Q_i)-\tilde{y}^2\}]^{1/2} \cdots\cdots (40)$$

Since I'_2 and I'_4 are elementary functions, I'_1, I'_3, and I'_5 are the integrals further to be reduced to normal forms. In eq. (39) sign ($\sin\phi$) can be removed for the same reason as was explained in section (1). Equations (39) and (40) are of the form similar to that of eqs. (28) and (29) and the method applied for eqs. (28) and (29) can also be utilized in the present transformation.

Considering from eq. (35) the relation

$$P_i+Q_i>P_i-Q_i,\quad P_i+Q_i>0 \cdots\cdots (41)$$

only the following two cases are possible among those indicated in Table 4, thus resulting in a simple transformation.

If $P_i>Q_i$, the method 3) in Table 3 is applied by letting

$$r=\sqrt{P_i+Q_i},\quad q=\sqrt{P_i-Q_i},\quad t=1 \cdots\cdots (42\cdot a\sim c)$$

With the above method, integrals I'_1, I'_3, and I'_5 are respectively reduced to

$$I'_1=2F/\sqrt{P_i+Q_i},\quad I'_3=2\{(P_i+Q_i)E-P_iF\}/(Q_i\sqrt{P_i+Q_i})$$
$$I'_5=4\sqrt{P_i+Q_i}\{(Q_i-P_i)F+P_iE-Q_i[\tilde{z}f'_5]_{\tilde{z}_i}^{\tilde{z}_{i+1}}\}/3Q_i^2 \qquad \cdots\cdots (43\cdot a\sim c)$$

in which

$$f'_5=\sqrt{(1-\tilde{z}^2)(1-k^2\tilde{z}^2)},\quad k=\sqrt{2Q_i/(P_i+Q_i)} \cdots\cdots (44\cdot a, b)$$

In case when $P_i<Q_i$, letting

$$r=\sqrt{Q_i-P_i},\quad q=\sqrt{P_i+Q_i},\quad t=1 \cdots\cdots (45\cdot a\sim c)$$

the expression with normal forms are given by

$$I'_1=2F/\sqrt{2Q_i},\quad I'_3=2(2E-F)/\sqrt{2Q_i} \cdots\cdots (46\cdot a\sim c)$$
$$I'_5=4\{(Q_i-P_i)F+2P_iE-(P_i+Q_i)[\tilde{z}f'_5]_{\tilde{z}_i}^{\tilde{z}_{i+1}}\}/(3Q_i\sqrt{2Q_i})$$

where

$$f'_5=\sqrt{(1-\tilde{z}^2)(1-k^2\tilde{z}^2)},\quad k=\{(P_i+Q_i)/2Q_i\}^{1/2} \cdots\cdots (47\cdot a, b)$$

If $P_i=Q_i$, all the integrals are reduced to elementry integrals.

Similar to the theory of a) finite displacements with finite strains, the integrals $I_1\sim I_5$, thus obtained, include the same integral components of normal forms regardless of the method of transformation. Therefore, the components of these integrals can be symbolically expressed as follows in the same way as was used in eq. (33)

$$I_1\rightarrow F,\quad I_2\rightarrow F+E+G,\quad I_3\rightarrow F+E+G,$$
$$I_4\rightarrow F+E+G,\quad I_5\rightarrow F+E+G \cdots\cdots (48\cdot a\sim e)$$

Different from eq. (33), it should be noticed that eqs. (48·b ~ e) do not include the normal elliptic integral of the third kind, which is replaced here by that of the second kind.

4. NUMERICAL EXAMPLES

An initially curved cantilever with concentrated load at free end is analyzed as an example to demonstrate the validity of the derived solutions as well as to examine the quantitative difference among the theories.

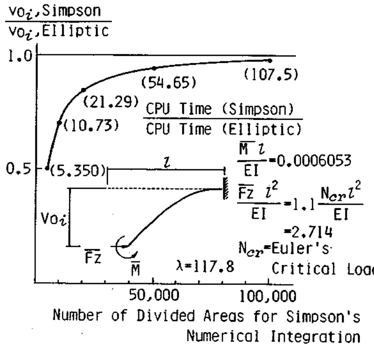

Fig. 2 Accuracy of Numerical Integration for Finite Displacements with Finite Strains.

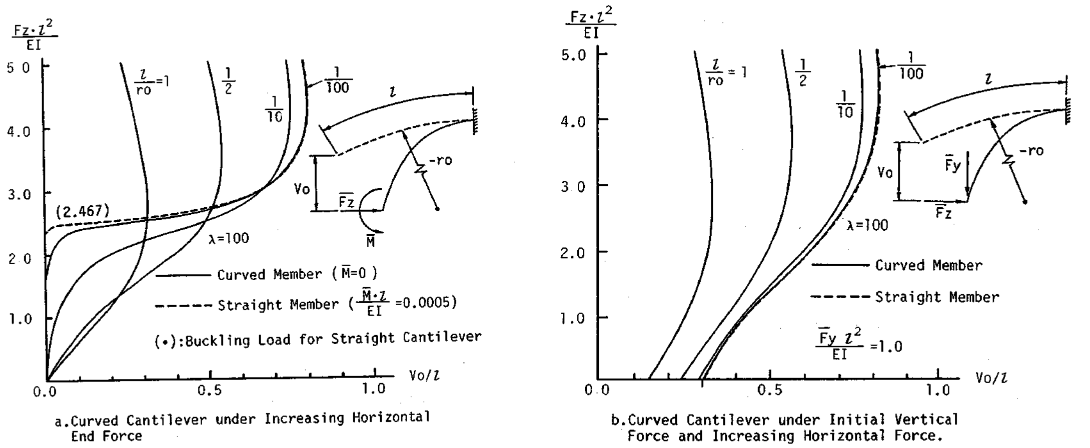

Fig. 3 Effect of Initial Curvature.

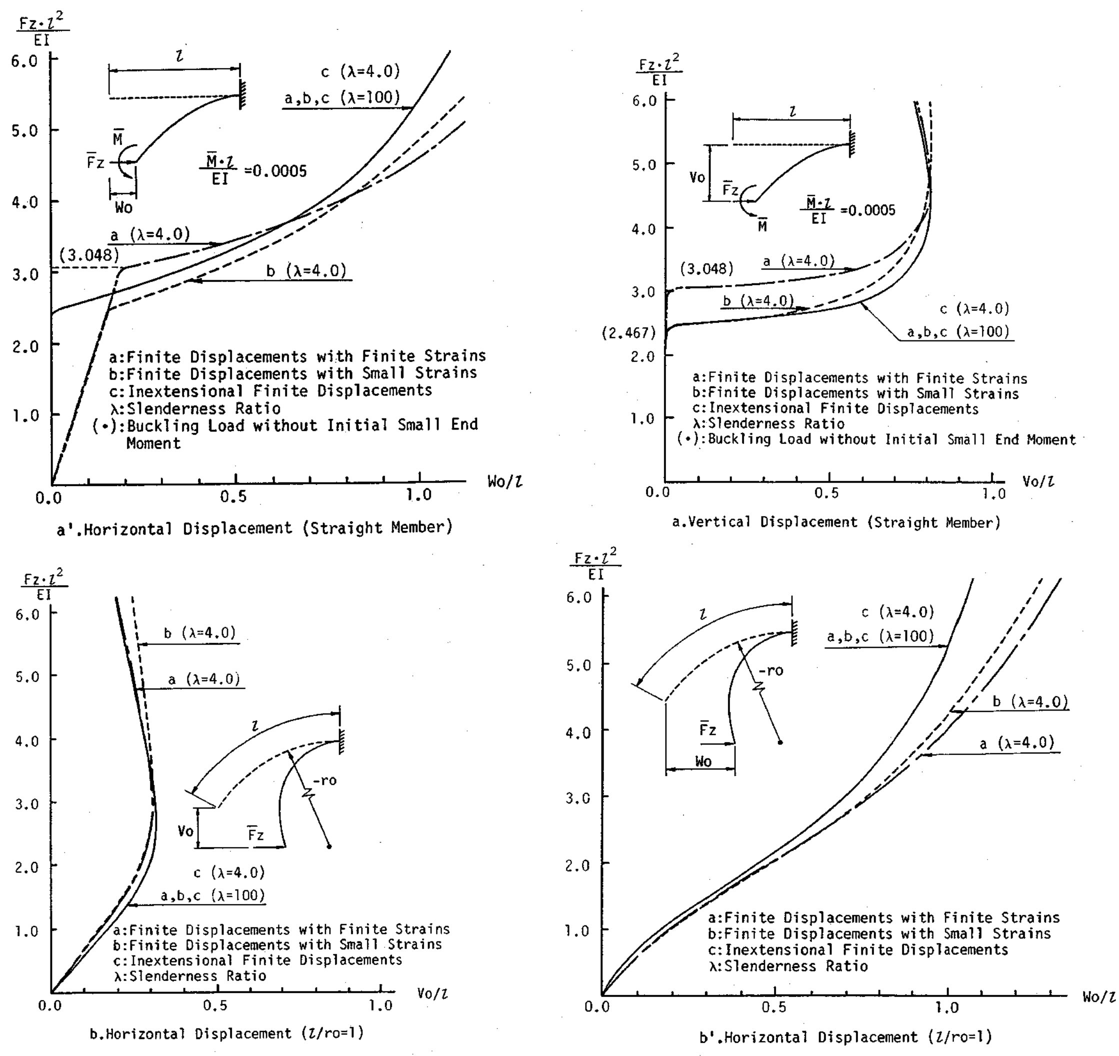

Fig. 4 Quantitative Difference among the Theories.

The boundary conditions of this structure are expressed by

$$M_i=-\overline{M},\ F_{yi}=-\overline{F}_y,\ F_{zi}=-\overline{F}_z,\ V_{0i+1}=0,\ W_{0i+1}=0,\ \phi_{i+1}=0 \qquad (49\cdot a\sim f)$$

Herein, the solution procedure for this problem is explained, using the equations for the theory of a) finite displacements with finite strains.

The angle ϕ_i is firstly calculated from the integral equation of eq. (10), substituting the boundary conditions given by eq. (49·a~c, f). Since the integral equation connot be solved explicitly for ϕ_i, bisection method is employed as an iterative method to obtain ϕ_i. Then, substituting ϕ_i, thus obtained, along with the boundary conditions given by eq. (49), the other physical quatities, i.e. V_{0i}, W_{0i}, F_{yi+1}, F_{zi+1}, and M_{i+1} can be directly calculated from eqs. (15), (17), and (18) without iteration. The solution procedures for the other theories are exactly the same.

In the first place, the present integration using normal forms is compared with the customary numerical integration in terms of their accuracy and efficiency. As a typical numerical integration, the integration with Simpson's 1/3 rule is chosen for comparison. The results of this comparison are summarized in Fig. 2. It can be understood from Fig. 2 that the accuracy and the efficiency of the integration are drastically improved by the use of normal forms.

Next, in order to show the effect of initial curvature, several cantilevers with different curvatures are analyzed under two loading conditions, as explained in Fig. 3. This analysis is based on the theory of a) finite displacements with finite strains and the cantilevers analyzed here have the same slenderness ratio λ of 100 defined by

$$\lambda=l/\sqrt{I/A} \qquad (50)$$

Load-displacement relations obtained from the above analysis are plotted in Fig. 3.

Further examined are the quantitative difference among the theories. In order to reveal the difference more clearly, cantilevers with the slenderness ratio of 4 are added to the numerical examples, though such stocky members are not practical. Load-displacement relations calculated for these examples are shown in Fig. 4, classified according to whether the cantilevers are straight or initially curved.

In case when $\lambda=100$, as is seen from the respective figures, the load-displacement relations obtained using three different theories, are represented by one curve. Hence, it can be said in this case that the difference among the theories has little effect on the results of analysis.

On the other hand, if $\lambda=4$, an obvious difference among the theories is observed in the calculated results and this is evident especially for the straight cantilever. The above difference is characterized by the point that the approximate theories, such as b) finite displacements with small strains and c) inextensional finite displacements, are apt to overestimate the displacements around the buckling point.

It should be noted for the theory of c) inextensional finite displacements that the load-displacement relations are represented by one curve in a exact sence, regardless of the value of λ. This is because the slenderness ratio λ is not included in the nondimensionalized integral equations in Table 2.

5. CONCLUSIONS

Closed-form solutions with integral expressions are derived for extensional elastica. The theories used here consider the extensional deformation of member axis, different from the inextensional theory customarily used in the analysis of elastica. The solutions, thus derived, include elliptic integrals. These integrals are further reduced to Legendre-Jacobi's normal forms in order to utilize the accurate method available for the calculation of elliptic integrals. As a result, it is known that the solution for the theory of a) finite displacements with finite strains includes all the three normal forms, while the other theories such as a) finite dispacements with small strains and c) inextensional finite dispacements are composed of the normal forms of the first and second kind. With the integrals reduced to normal forms, the accuracy of the solutions is drastically improved, compared with those obtained by ordinary numerical integrations.

APPENDIX A $R_5 \sim R_{12}$

1) Substitution given by Eq. (26·a)

$R_5=\{-m\tilde{y}^2+m(1+m^2)\}/g,\ \ R_6=\{\tilde{y}^2+(1-m^2)\}/g,\ \ R_7=(-\tilde{y}^4+2m^2\tilde{y}^2+1-m^4)/g,\ \ R_8=-4m/g,$

$R_9=\{m\tilde{y}^6-m(5+3m^2)\tilde{y}^4-m(5-6m^2-3m^4)\tilde{y}^2+m(1+m^2)^2(1-m^2)\}/g^2,$

$R_{10}=\{-\tilde{y}^6-(1-3m^2)\tilde{y}^4+(1+6m^2-3m^4)\tilde{y}^2+(1+m^2)(1-6m^2+m^4)\}/g^2,$

$R_{11}=\{\tilde{y}^6+(2-m^2)\tilde{y}^4+(1-4m^2-m^4)\tilde{y}^2+m^2(1+m^2)^2\}/g^2,$

$R_{12}=\{-2m\tilde{y}^4+4m^3\tilde{y}^2+2m(1-m^2)(1+m^2)\}/g^2$

$g=\tilde{y}^4+2(1-m^2)\tilde{y}^2+(1+m^2)^2$

2) Substitution given by Eq. (26·b)

$R_5=(n\tilde{y}^2+m)/g,\ \ R_6=(m+n)/g,\ \ R_7=\{(1-n^2)\tilde{y}^2+1-m^2\}/g,\ \ R_8=4/g,$

$R_9=\{n(1-n^2)\tilde{y}^4+6(m+n)\tilde{y}^2+m(1-m^2)\}/g^2,\ \ R_{10}=\{(m-n^3+6n)\tilde{y}^2+6m+n-m^3\}/g^2,$

$R_{11}=\{n^2\tilde{y}^4+(m^2+n^2-4)\tilde{y}^2+m^2\}/g^2,\ \ R_{12}=\{2(n^2-1)\tilde{y}^2+2(m^2-1)\}/g^2,$

$g=(1+n^2)\tilde{y}^2+1+m^2$

APPENDIX B $e_1 \sim e_5,\ c_1 \sim c_4$

1) Substitution given by Eq. (26·a)

$e_1=2,\ \ e_2=4,\ \ e_3=2,\ \ e_4=4,\ \ e_5=8,\ \ c_1=b_0(\gamma_1\gamma_2-m^2),\ \ c_2=b_0,\ \ c_3=\gamma_3\gamma_4-m^2,\ \ c_4=1$

2) Substitution given by Eq. (26·b)

$e_1=2(n-m),\ \ e_2=4(n-m),\ \ e_3=2(n-m),\ \ e_4=4(n-m),\ \ e_5=8(n-m)$

$c_1=b_0\{m^2-(\gamma_1+\gamma_2)m+\gamma_1\gamma_2\},\ \ c_2=b_0\{n^2-(\gamma_1+\gamma_2)n+\gamma_1\gamma_2\},\ \ c_3=m^2-(\gamma_3+\gamma_4)m+\gamma_3\gamma_4,$

$c_4=n^2-(\gamma_3+\gamma_4)n+\gamma_3\gamma_4$

REFERENCES

1) Goto, Y., Hasegawa, A. and Nishino, F. : Accuracy and Convergence of the Separation of Rigid Body Displacements for Plane Curved Frames, Proc. of JSCE, Div. of Structural Eng./Earthquake Eng., Vol. 1, pp. 29s~39s, Apr., 1984.
2) Goto, Y., Hasegawa, A., Nishino, F. and Matsuura, S. : Accuracy and Convergence of the Separation of Rigid Body Displacements for Space Frames, Proc. of JSCE, Div. of Structural Eng./Earthquake Eng., Vol. 2, No. 2, pp. 85s~95s, Apr., 1985.
3) Timoshenko, S. P. and Gere, J. M. : Theory of Elastic Stability, 2 nd Ed., McGraw-Hill, pp. 76~82, 1961.
4) Sliter, G. E. and Boresi, A. P. : Elastica Supported at Midpoint by a Spring, ASCE, Vol. 90, No. EM 2, pp. 1~15, Apr., 1964.
5) Lee, S. L., Manuel, F. S. and Rossow, E. C. : Large Deflections and Stability of Elastic Frames, ASCE, Vol. 94, No. EM 2, pp. 521~547, Apri., 1968.
6) Britvec, S. J. : The Stability of Elastic Systems, Pergamon Press, pp. 137~195, 1973.
7) Lau, J. H. : Large Deflections of Beams with Combined Loads, ASCE, Vol. 108, No. EM 1, pp. 180~185, Feb. 1982.
8) Seide, P. : Large Deflections of a Simply Supported Beam Subjected to Moment at One End, Journal of Applied Mechanics, ASME, Vol. 51, pp. 519~525, Sept. 1984.
9) Ando, S. : Introduction to Elliptic Integrals and Elliptic Functions, Nisshin Book Co., 1970 (in Japanese).
10) Byrd, P. F. and Friedman, M. D. : Handbook of Elliptic Integrals for Engineers and Phyicists, Springer Verlag, pp. 223~239, 1954.

(Received December 5 1986)