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GROUP THEORETIC DESCRIPTION OF BIFURCATION BEHAVIOR OF AXISYMMETRIC REGULAR-POLYGONAL TRUSS DOMES

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Bifurcation buckling behavior of polygonal-shaped, truss dome structures is studied with the aid of a group theoretic method. This study extends the previous research on simple regular-polygonal truss domes to those with greater number of degree of freedoms and with much more realistic configuration. Unlike the previous research, radial and rotational displacements of the domes are included here in describing their bifurcation modes, in addition to vertical displacements. The method has permitted one to obtain potential bifurcation modes of the domes and hierarchies of bifurcation paths under axisymmetric loadings without performing bifuraction tracing analyses. The applicability of group theoretic method to regular-polygonal domes has been fully assessed through this study.
Keywords : symmetry, bifurcation behavior, polygon, truss domes.

1. INTRODUCTION

Dome structures display highly complex bifurcation buckling behavior[1),2)]. Extensive research conducted by several engineers[3)~7)] has enabled us to trace such complex behavior, often at the expense of awkward computations. Nonetheless the complexity somewhat slowed the progress of theoretical description of the behavior. At present, a theory of elastic stability established by Thompson[8)] and others early in the 1970 s remains to be the most widely accepted way to describe bifurcation behavior. Their theory, formulated in terms of derivatives of total potential energy function, is rather more theoretically complete than it is practical in conventional analyses since one cannot expect estimation of the derivatives as a part of such analyses.

Applied mathematicians, by contrast, have developed in last decades a group theoretic method to describe bifurcation behavior[9)~12)], focusing on geometric symmetry. They succeeded in describing various physical phenomena in a methodical manner utilizing compact groups, which include dihedral, rotation groups, etc., as a mathematical tool to represent symmetry.

Extending this method, the authors[13)~17)] described bifurcation behavior of reticulated, regular-polygonal truss domes shown in Fig. 1. Just as the method described well various physical phenomena, it explained well the domes' bifurcation behavior. The domes under axisymmetric loadings were proved to be covariant with dihedral groups, where bifurcation behavior of systems covariant with a group has been proven to be described mathematically by the group[9),10)] (Refs. 18 and 19 may be appropriate for textbooks for dihedral groups). Furthermore, the groups permitted one to arrive at potential bifurcation modes of the domes and their bifurcation mode hierarchies, without carrying out tedious bifurcation tracing

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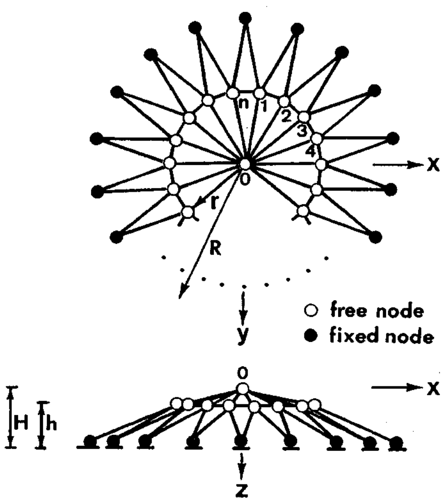


Fig. 1 Simple Regular-Polygonal Dome.

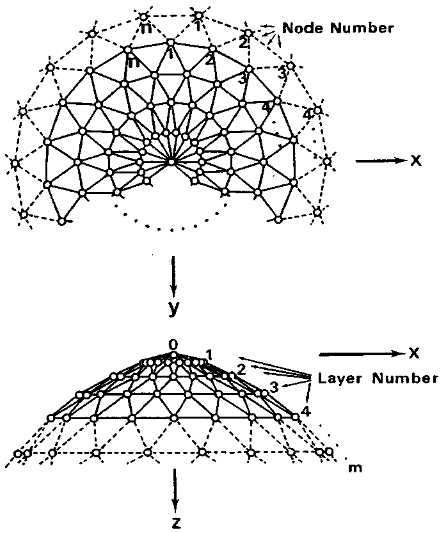


Fig. 2 General Regular-Polygonal Dome with Layers of Free Nodes.

analyses. Most of the qualitative bifurcation behavioral characteristics can be obtained through mere observation of geometric configurations of domes. By contrast, bifurcation tracing analysis techniques were indispensable in arriving at quantitative aspects of the behavior, such as buckling loads, nodal displacement values, and so on. The group theoretic method and the analysis techniques focus on different aspects of the behavior.

While displaying a great promise at the least for those truss domes, the group theoretic method somewhat lacked in generality since its application was limited to simple regular-polygonal domes and was formulated by considering only vertical displacements of the domes. A lot to be studied remained regarding the applicability of the method to more general cases, such as the description of bifurcation behavior of realistic dome structures with greater number of degree of freedoms with reference to their vertical, radial, and rotational displacements.

At the first step to extend its applicability, the method is employed in this paper to describe bifurcation behavior of general polygonal truss domes made up of layers of regular-polygonal-shaped free nodes (see Fig. 2) with reference to all the displacements. As can be seen, these domes hold much more complex and realistic geometric configurations and greater number of degree of freedoms compared with the simple polygonal domes (see Fig. 1). Those domes, accordingly, are expected to be of great assistance in deriving general rules regarding bifurcation behavior of dome structures. We prove the domes under axisymmetric loadings are covariant with dihedral groups, thus validating the usability of dihedral groups in describing the domes' bifurcation behavior. In addition, we obtain bifurcation modes of the general polygonal domes to find that they possess the same types of possible bifurcation modes irrespective of the number of layers of the domes. The data regarding bifurcation modes of simple polygonal domes advanced in Ref. 14 are refined herein to be applicable to more general cases. A suggestion regarding future studies concludes this paper.

2. VERIFICATION OF D_n-COVARIANCY OF POLYGONAL TRUSS DOMES

The authors study bifurcation behavior of the polygonal truss dome structures with m-layers of n-gonal-shaped free nodes (see Fig. 2) subjected to axisymmetric vertical loadings throughout this paper. The members of the trusses have identical sectional and material properties so that the trusses have axisymmetric stiffness distribution. Prior to applying a dihedral group D_n to the description of the behavior, let us prove D_n-covariancy of the domes. This verification is identical with that performed in Ref. 13 regarding the simple polygonal domes (see Fig. 1), except for the presence of multiple layers of regular-polygonal-shaped free nodes.

Using the equilibrium equations for the finite-displacement problem of elastic truss members advanced in Ref. 7, one can write equilibrium equations for those truss domes as :

$$H_{i(k)}(f, x) = K_{i(k)j(l)}(x) \cdot x_{j(l)} - F_{i(k)}(x) - f \cdot f_{i(k)} = 0 \qquad i, j=0, 1, 2, \cdots\cdots, n; \quad k, l=1, \cdots\cdots, m \qquad (1)$$

where subscripts i and j indicate the vector or matrix related to the i-th or j-th node; subscripts k and l in the parenthesis ( ) denote the variable corresponding to the k-th or l-th layer ; m expresses the number of layers. The matrix $K_{i(k)j(l)}$ is the three-by-three sub-matrix of the non-linear (tangent) stiffness matrix ; $F_{i(k)}$ is the three-dimensional nonlinear vector, f is the loading parameter ; $f_{i(k)}$ is the three-dimensional, nodal-load vector. The summation convention applies to dummy variables j and l. We employed a special way to identify nodes, that is, each node is represented by two types of numbers, including : (1) the layer number and (2) the node number in a layer.

Equation (1) is covariant with a group if the following relationship is satisfied for all the transformation T_g initiated by elements of the group :

$$H_{i(k)}(f, T_g x) = T_g H_{i(k)}(f, x) \qquad (2)$$

where g denotes the element of the group.

A dihedral group of degree n (D_n), used extensively to express the geometric symmetry of regular polygons in mathematics, is utilized here to describe the geometric symmetry of the regular-polygonal domes. This group consists of the following 2 n elements :

$$\sigma_j \text{ and } \tau\sigma_j \qquad j=1, 2, \cdots\cdots, n \qquad (3)$$

where σ_j denotes the 360 $(j-1)/n$ degree rotation in the $x-y$ plane about the z-axis and τ is the reflection in the y-axis. The element $\tau\sigma_j$ causes a 360 x $(j-1)/n$ degree rotation about the origin followed by the reflection in the y-axis; this element consequently denotes the reflection in the straight line intersecting with the y-axis at the origin at an angle of -180 x $(j-1)/n$ degrees. The group D_n made up of these elements, denote rotational symmetry regarding 360 x $(j-1)/n$ degree rotations ($j=1, 2, \cdots\cdots, n$) and line symmetry regarding n straight lines. For this group, the transformation T_g equals either $T_{\sigma j}$ or $T_{\tau\sigma j}$ ($j=1, 2, \cdots\cdots, n$). The transformation $T_{\sigma j}$, for example, permutes the nodal coordinate vectors $x_{1(k)}, x_{2(k)}, \cdots\cdots, x_{n(k)}$ as in what follows :

$$T\sigma_j(x_{1(k)}, x_{2(k)}, \cdots\cdots, x_{n(k)})^t = \begin{cases} (x_{1(k)}, x_{2(k)}, \cdots\cdots, x_{n(k)})^t & \text{for } j=1 \\ (x_{n-j+1(k)}, x_{n-j+2(k)}, \cdots\cdots, x_{n(k)}, x_{1(k)}, x_{2(k)}, \cdots\cdots, x_{n-j(k)})^t & \text{for } j \geqq 2 \end{cases} \qquad (4)$$

in which the superscript t denotes the transpose of a vector.

The transformation T_g, representing either such a rotation or reflection, merely permutes those nodes. This transformation consequently denotes the following permutation of node numbers :

$$i(k) \rightarrow a_i(k) \qquad i=1, 2, \cdots\cdots, \text{or } n \qquad (5)$$

where the index a_i takes a value either $1, 2, \cdots\cdots$, or n.

Such permutation of node number does not alter the stiffness distribution but merely reallocates the tangent stiffness matrix components. Hence one can arrive at the following condition of stiffness reallocation :

$$K_{i(k)j(l)}(T_g(x)) = K_{a_i(k)a_j(l)} \qquad (6)$$

For axisymmetric loadings, all the nodal load vectors can be preserved by each transformation T_g, that is,

$$T_g f_{i(k)} = f_{a_i(k)} \qquad i=1, 2, \cdots\cdots, n; \quad k=1, 2, \cdots\cdots, m \qquad (7)$$

with the use of these relationships, the left hand side of Eq. (1) for the truss dome structures becomes :

$$H_{i(k)}(f,\ T_g\, x)=K_{a_i(k)a_j(l)}x_{a_j(l)}-F_{a_i(k)}-f\cdot f_{a_i(k)} \qquad (8)$$

Dummy variable a_j in this equation can be replaced by a variable j and a vector $f_{a_i(k)}$ is equal to $T_g f_{i(k)}$ as indicated in Eq. (7). On considering these relationships in Eq. (8), one can show that the condition for covariancy, Eq. (2), is satisfied for all the transformation initiated by elements of group D_n. Hence those regular-n-gonal domes under axisymmetric loadings are D_n-covariant. The authors will employ this group in the remainder of the paper as a mathematical tool to describe the D_n-covariant domes.

The verification of D_n-covariancy of polygonal domes performed herein is applicable to other truss domes under axisymmetric loadings, for which the condition of realocation of stiffness matrix Eq. (6) is satisfied. The verification of covariancy of truss domes under a symmetry group, in general, reduces to the proof of this equation. Such verification can be performed with ease by observing the domes' configurations and investigating if their stiffness distribution can be preserved by the transformation caused by each element of the symmetry group.

3. BIFURCATION MODES OF REGULAR-HEXAGONAL TRUSS DOMES

As we have seen in the previous section, the regular-polygonal truss domes under axisymmetric loadings are covariant with a dihedral group so that the bifurcation behavior of the domes can be characterized by its subgroups[9),13)~17)]. As an application of this group, we study the bifurcation modes of regular-hexagonal domes (see Fig. 3) with the aid of a dihedral group of degree six (D_6). This group possesses the following 16 subgroups[13)] :

$$D_6=\langle\sigma_1,\ \sigma_2,\ \sigma_3,\ \sigma_4,\ \sigma_5,\ \sigma_6,\ \tau\sigma_1,\ \tau\sigma_2,\ \tau\sigma_3,\ \tau\sigma_4,\ \tau\sigma_5,\ \tau\sigma_6\rangle$$

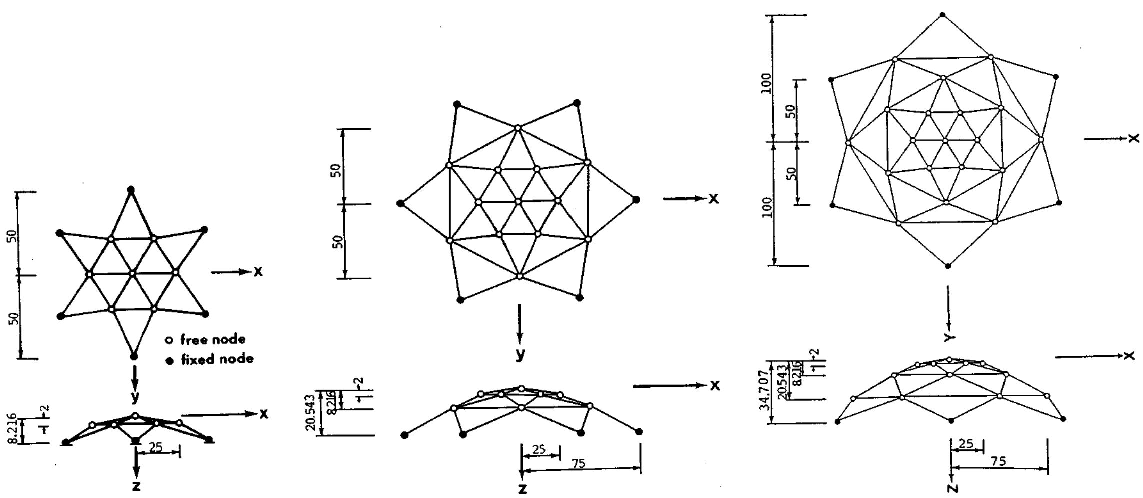
$$C_6=\langle\sigma_1,\ \sigma_2,\ \sigma_3,\ \sigma_4,\ \sigma_5,\ \sigma_6\rangle \qquad C_3=\langle\sigma_1,\ \sigma_3,\ \sigma_5\rangle \qquad C_2=\langle\sigma_1,\ \sigma_4\rangle \qquad E=\langle\sigma_1\rangle$$

$$D'_{6/2}=\langle\sigma_1,\ \sigma_3,\ \sigma_5,\ \tau\sigma_1,\ \tau\sigma_3,\ \tau\sigma_5\rangle \qquad D_{6/2}=\langle\sigma_1,\ \sigma_3,\ \sigma_5,\ \tau\sigma_2,\ \tau\sigma_4,\ \tau\sigma_6\rangle$$

$$D_2^j=\langle\sigma_1,\ \sigma_4,\ \tau\sigma_j,\ \tau\sigma_{j+3}\rangle \quad j=1,2,3 \qquad D_1^j=\langle\sigma_1,\ \tau\sigma_j\rangle \quad j=1,2,\cdots\cdots,6 \qquad (9)$$

where the right hand side of these equations denote the elements of the relevant group and σ_j denotes a $60\cdot(j-1)$ degree rotation and $\tau\sigma_j$ a reflection in the straight line crossing with the y-axis at an angle of $-30\cdot(j-1)$ degrees.

The nodal displacements of the i-th nodes were expressed in terms of the cylindrical coordinates (r_i, θ_i, and z_i) in order to express rotatory symmetry of the domes in a methodical manner. Following a method for determining the interrelationships among the domes' nodal displacements related to those subgroups[17)], we arrived at the nodal displacement interrelationships for the hexagonal truss domes shown in Table 1. As



(a) Single-hexagonal Dome (b) Double-hexagonal Dome (c) Triple-hexagonal Dome

Fig. 3 Regular-Hexagonal Truss Domes.

Table 1 Nodal Displacement Interrelationships for Bifurcation Modes of the Regular-Hexagonal Domes.

(a) Odd-Numbered Layers

Bifurcation Modes	Displacement Interrelationships of the Nodes: $u_i = r_i$ or z_i	Displacement Interrelationships of the Nodes: θ_i
D_6	$u_1 = u_2 = u_3 = u_4 = u_5 = u_6$	$\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = \theta_6 = 0$
C_6	$u_1 = u_2 = u_3 = u_4 = u_5 = u_6$	$\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = \theta_6$
$D_{\eta/2}$	$u_1 = u_3 = u_5$; $u_2 = u_4 = u_6$	$\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = \theta_6 = 0$
$D_{n/2}$	$u_1 = u_2 = u_3 = u_4 = u_5 = u_6$	$\theta_1 = -\theta_2 = \theta_3 = -\theta_4 = \theta_5 = -\theta_6$
D_2^1	$u_2 = u_5$; $u_1 = u_3 = u_4 = u_6$	$\theta_2 = \theta_5 = 0$; $\theta_1 = -\theta_3 = \theta_4 = -\theta_6$
D_2^2	$u_3 = u_6$; $u_1 = u_2 = u_4 = u_5$	$\theta_3 = \theta_6 = 0$; $\theta_1 = -\theta_2 = \theta_4 = -\theta_5$
D_2^3	$u_1 = u_4$; $u_2 = u_3 = u_5 = u_6$	$\theta_1 = \theta_4 = 0$; $\theta_2 = -\theta_3 = \theta_5 = -\theta_6$
C_3	$u_1 = u_3 = u_5$; $u_2 = u_4 = u_6$	$\theta_1 = \theta_3 = \theta_5$; $\theta_2 = \theta_4 = \theta_6$
C_2	$u_1 = u_4$; $u_2 = u_5$; $u_3 = u_6$	$\theta_1 = \theta_4$; $\theta_2 = \theta_5$; $\theta_3 = \theta_6$
D_1^1	$u_1 = u_6$; $u_2 = u_5$; $u_3 = u_4$	$\theta_1 = -\theta_6$; $\theta_2 = -\theta_5$; $\theta_3 = -\theta_4$
D_1^2	$u_1 = u_5$; $u_2 = u_4$	$\theta_3 = \theta_6 = 0$; $\theta_1 = -\theta_5$; $\theta_2 = -\theta_4$
D_1^3	$u_1 = u_4$; $u_2 = u_3$; $u_5 = u_6$	$\theta_1 = -\theta_4$; $\theta_2 = -\theta_3$; $\theta_5 = -\theta_6$
D_1^4	$u_1 = u_3$; $u_4 = u_6$	$\theta_2 = \theta_5 = 0$; $\theta_1 = -\theta_3$; $\theta_4 = -\theta_6$
D_1^5	$u_1 = u_2$; $u_3 = u_6$; $u_4 = u_5$	$\theta_1 = -\theta_2$; $\theta_3 = -\theta_6$; $\theta_4 = -\theta_5$
D_1^6	$u_2 = u_6$; $u_3 = u_5$	$\theta_1 = \theta_4 = 0$; $\theta_2 = -\theta_6$; $\theta_3 = -\theta_5$
E	arbitrary	arbitrary

(b) Even-Numbered Layers

Bifurcation Modes	Displacement Interrelationships of the Nodes: $u_i = r_i$ or z_i	Displacement Interrelationships of the Nodes: θ_i
D_6	$u_1 = u_2 = u_3 = u_4 = u_5 = u_6$	$\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = \theta_6 = 0$
C_6	$u_1 = u_2 = u_3 = u_4 = u_5 = u_6$	$\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = \theta_6$
$D_{\eta/2}$	$u_1 = u_2 = u_3 = u_4 = u_5 = u_6$	$\theta_1 = -\theta_2 = \theta_3 = -\theta_4 = \theta_5 = -\theta_6$
$D_{n/2}$	$u_1 = u_3 = u_5$; $u_2 = u_4 = u_6$	$\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = \theta_6 = 0$
D_2^1	$u_1 = u_4$; $u_2 = u_3 = u_5 = u_6$	$\theta_1 = \theta_4 = 0$; $\theta_2 = -\theta_3 = \theta_5 = -\theta_6$
D_2^2	$u_2 = u_5$; $u_1 = u_3 = u_4 = u_6$	$\theta_2 = \theta_5 = 0$; $\theta_1 = -\theta_3 = \theta_4 = -\theta_6$
D_2^3	$u_3 = u_6$; $u_1 = u_2 = u_4 = u_5$	$\theta_3 = \theta_6 = 0$; $\theta_1 = -\theta_2 = \theta_4 = -\theta_5$
C_3	$u_1 = u_3 = u_5$; $u_2 = u_4 = u_6$	$\theta_1 = \theta_3 = \theta_5$; $\theta_2 = \theta_4 = \theta_6$
C_2	$u_1 = u_4$; $u_2 = u_5$; $u_3 = u_6$	$\theta_1 = \theta_4$; $\theta_2 = \theta_5$; $\theta_3 = \theta_6$
D_1^1	$u_2 = u_6$; $u_3 = u_5$	$\theta_1 = \theta_4 = 0$; $\theta_2 = -\theta_6$; $\theta_3 = -\theta_5$
D_1^2	$u_1 = u_6$; $u_2 = u_5$; $u_3 = u_4$	$\theta_1 = -\theta_6$; $\theta_2 = -\theta_5$; $\theta_3 = -\theta_4$
D_1^3	$u_1 = u_5$; $u_2 = u_4$	$\theta_3 = \theta_6 = 0$; $\theta_1 = -\theta_5$; $\theta_2 = -\theta_4$
D_1^4	$u_1 = u_4$; $u_2 = u_3$; $u_5 = u_6$	$\theta_1 = -\theta_4$; $\theta_2 = -\theta_3$; $\theta_5 = -\theta_6$
D_1^5	$u_1 = u_3$; $u_4 = u_6$	$\theta_2 = \theta_5 = 0$; $\theta_1 = -\theta_3$; $\theta_4 = -\theta_6$
D_1^6	$u_1 = u_2$; $u_3 = u_6$; $u_4 = u_5$	$\theta_1 = -\theta_2$; $\theta_3 = -\theta_6$; $\theta_4 = -\theta_5$
E	arbitrary	arbitrary

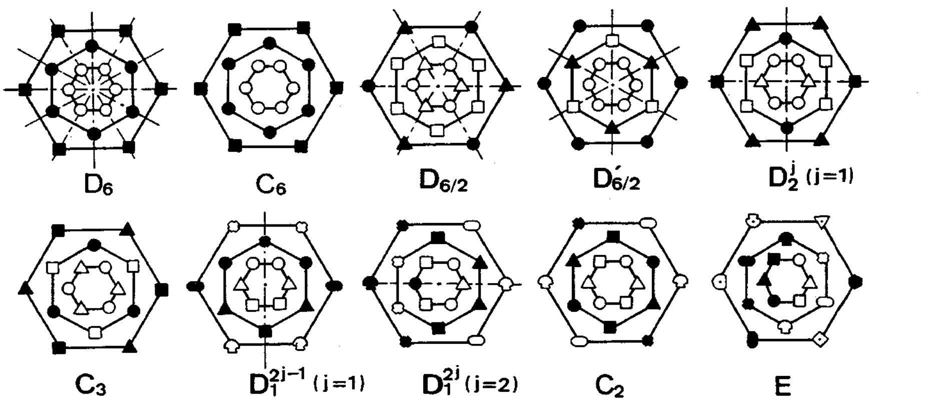


Fig. 4 Deformation Modes of the Triple-Hexagonal Truss Dome Expressed in Terms of Relative Displacements of the Free Nodes.

can be seen from this table, the nodes in all the odd-numbered layers possess the same types of deformation patterns for each subgroup and so do those in all the even-numbered layers. The bifurcation modes for D_2^1, D_2^2, and D_2^3 are rotatory symmetric and correspond to the same physical behavior. Such is also the case for D_1^1, D_1^3, and D_1^5 and D_1^2, D_1^4, and D_1^6. These modes, therefore, are represented by D_2^j, D_1^{2j-1}, and D_1^{2j}, respectively. A more detailed discussion regarding rotatory symmetry among subgroups (bifurcation modes) can be found in Ref. 16.

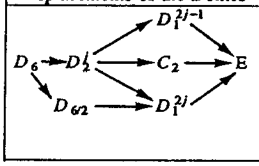
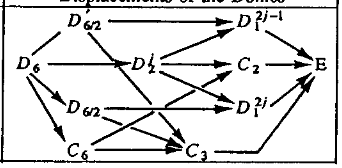
We illustrated schematically the deformation modes of the triple-hexagonal dome in Fig. 4 in terms of relative displacements of the dome's free nodes. The symbols (□), (▲), ······, and (○) indicate that nodes with the same symbol have the same vertical and radial displacements. The solid-dash lines denote the axes of line symmetry. The nodes on the axes of symmetry cannot rotate, whereas nodes elsewhere can rotate in such a manner that the nodes after rotation satisfy the required line and point symmetries.

The group D_6 denotes a mode characterized by identical displacements for all the nodes belonging to the same layer. The group $D^{*}_{6/2}$ expresses a three-axes symmetric mode exhibiting the same vertical and radial displacements for the nodes in all the even layers and for every other node in all the odd layers. The group $D_{6/2}$ expresses another three-axes symmetric mode for which all the nodes in even layers displace symmetrically and so do every other node in each of the other layers. The group D_2^j is a two-axes symmetric mode that is line symmetric in two perpendicular axes. Groups D_1^{2j-1} and D_1^{2j} denote bifurcation modes with an axis of line symmetry. Point symmetric modes are represented by groups C_6, C_3 and C_2; for these rotational symmetric modes, nodes with the same symbols must rotate toward the same direction with the same angle so as to satisfy the point symmetry required. The completely asymmetric mode is expressed by the trivial group E.

Considering only vertical displacements, we had noted in Ref. 14 for the single-hexagonal dome that groups D_6, C_6, and $D^{*}_{6/2}$ degenerated into a mode and so did groups $D_{6/2}$ and C_3. We had interpreted such cases that multiple modes degenerated into the same mode (mode degeneration)[14]. Mode degeneration, by contrast, did not occur for any of the hexagonal domes when radial and rotational displacements were included.

As reported in Ref. 14, hierarchial frameworks of bifurcation paths of domes can be represented by subgroup structures of relevant dihedral groups (D_6 for this case). Based on the method for obtaining subgroup structures of dihedral groups[10),17)], we obtained hierarchial frameworks of bifurcation modes of the hexagonal domes. Table 2 compares the hierarchy obtained by considering only vertical displacements and that by all the displacement components. The symbol $S \rightarrow T \rightarrow U$ in this table denotes that the group T is a subgroup of S and U is of T; furthermore, one can bypass group T and interpret that group U is a subgroup of S. The hierarchy obtained by considering all the components was significantly more complex than that arrived at by using only vertical displacements in that the former possessed the modes related to groups C_6, C_3, and $D^{*}_{6/2}$, while the latter did not owing to mode degeneration. Radial and rotational displacements, to be precise, must be considered in describing bifurcation modes of truss domes.

Table 2 Bifurcation Modes Hierarchies for Hexagonal Domes.

Considering only Vertical Displacements of the Domes	Considering All the Displacements of the Domes
	

S → T : group T is a subgroup of group S.

As we have seen, the consideration of radial and rotatory displacements resulted in more diversified types of bifurcation modes and hierarchies than those obtained by considering only vertical displacements. All the hexagonal domes possessed the same bifurcation modes and hierarchies and that the distinction of the number of layers was immaterial. The validity and usability of these data will be verified through bifurcation path tracing analyses of these dome structures in the next section.

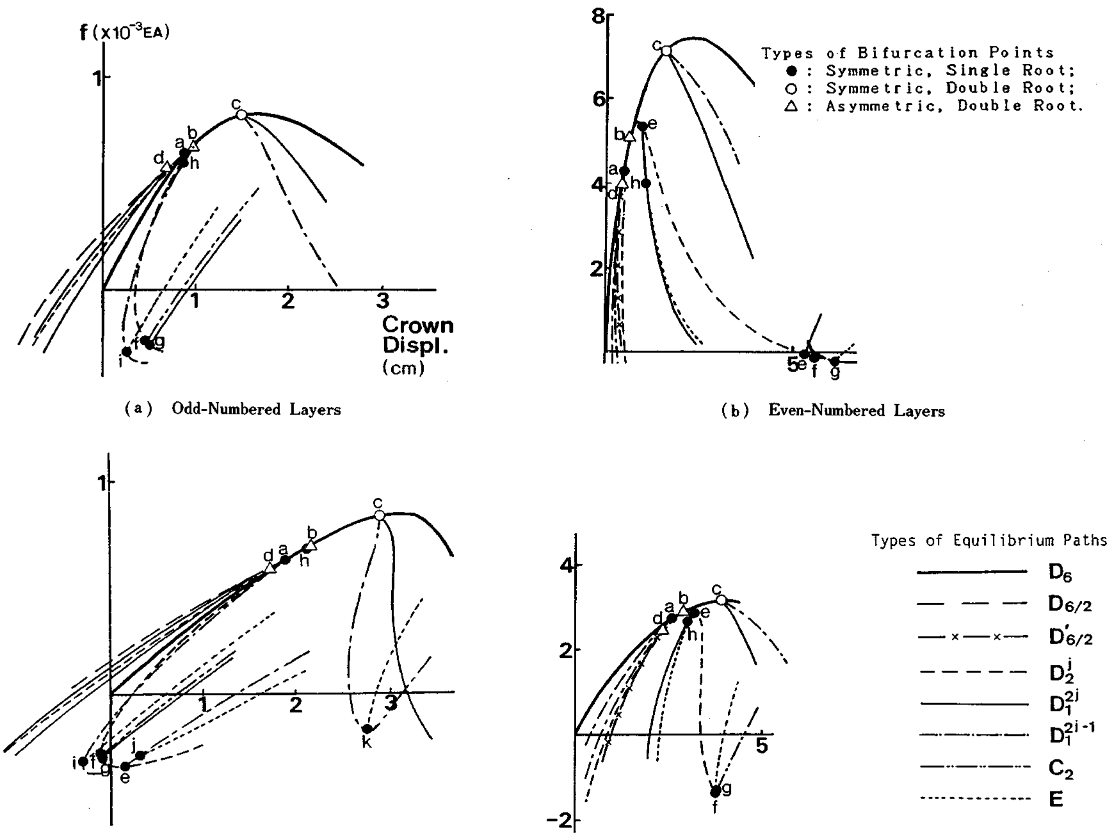
4. BIFURCATION BEHAVIOR OF REGULAR-HEXAGONAL TRUSS DOMES

We have advanced a series of data regarding the bifurcation modes of hexagonal truss domes in the previous section. In order to assess the usefulness and usability of these data, they are adopted in this section to describe the bifurcation behavior of the double and triple-hexagonal domes subjected to the axisymmetric vertical loading pattern (a) or (b) listed in Table 3. The behavior of single-haxagonal dome is not studied here but its behavior has already been found in Ref. 13 to satisfy those data. Figure 5 contains equilibrium paths of the domes obtained with the use of a finite displacement analysis technique[7]. During the course of this analysis, we investigated deformation modes of the domes to ascertain that each path can be represented by a subgroup of D_6. The types of paths are indicated in this figure by means of various kinds of lines, while the types of bifurcation points by the symbols (●), (○), and (△). Although these equilibrium paths had various types of bifurcation paths, each type matched perfectly with what advanced in Fig. 4

Table 3 Vertical Loading Patterns Applied to the Double and Triple-Hexagonal Domes.

Layer Number	Node Number	Loading Pattern (a)	Loading Pattern (b)
0	0	1/2	0
1	1	1	0
	2	1	0
	.	.	.
	.	.	.
	n	1	0
2	1	1	1
	2	1	1
	.	.	.
	.	.	.
	n	1	1
3	1	1	0
	2	1	0
	.	.	.
	.	.	.
	n	1	0

The loads on the third layer must be ignored for the double hexagonal dome.



(a) Odd-Numbered Layers

(b) Even-Numbered Layers

(c) Tripe-Hexagonal Dome : Loading Pattern (a)

(d) Triple-Hexagonal Dome : Loading Pattern (b)

Fig. 5 Equilibrium Paths Computed for the Double and Triple-hexagonal Domes under the Loading Patterns (a) and (b).

and every bifurcation process followed perfectly the bifurcation mode hierarchy advanced in Table 2. Furthermore, bifurcation paths represented by the group $D^{*}_{6/2}$ existed in the equilibrium paths for those domes for the loading pattern (b). This validates the aforementioned prediction that the hexagonal domes can possess a mode relevant with group $D^{*}_{6/2}$, which was not identified by considering only vertical displacements of the domes. Although we could not find a path relevant with groups C_6 and C_3, these types of paths are theoretically feasible and are expected to exist in some other cases.

As we have seen, the bifurcation behavior of the polygonal domes occurred exactly as predicted in the previous section. To be precise, the potential bifurcation modes advanced in Fig. 4 covered all the bifurcation modes found in those equilibrium pahts, while the bifurcation path hierarchy advanced in Table 2 explained well every branching process encountered. This insures the applicability of the group theoretic method to the study of complex regular-polygonal dome structures with realistic configurations.

5. BIFURCATION MODES OF REGULAR-POLYGONAL TRUSS DOMES

The study of bifurcation modes of a series of regular-hexagonal domes performed in previous sections revealed that the mode degeneration did not occur when radial and rotational displacements of the domes were considered. These components, therefore, must be considered in describing the bifurcation modes. This implies the inadequacy of the information regarding bifurcation modes derived on the basis of single-polygonal domes[14] by considering only vertical displacements, for which several bifurcation modes were lost owing to the mode degeneration. To enhance the generality of the information, the authors obtained bifurcation modes of rgular-polygonal dome structures with reference to all the displacement components, (that is, by arriving at the modes expressed by all the subgroups of relevant dihedral groups).

The bifurcation modes of regular-n-gonal domes with degrees three through 16 were obtained by examining all the subgroups of corresponding dihedral groups by means of the method for obtaining the subgroups of dihedral groups advanced in Refs. 10 and 17. The subgroups of D_n consist of :

$$D_n,\ C_n,\ D_{gcd(n,m)} \text{ and } C_{gcd(n,m)} \qquad m=1,2,\cdots\cdots,(n-1)/2 \qquad \text{for odd } n;$$
$$D_n,\ C_n,\ D_{n/2},\ C_{n/2},\ D_{gcd(n,m)} \text{ and } C_{gcd(n,m)} \qquad m=1,2,\cdots\cdots,n/2-1 \qquad \text{for even } n. \quad \cdots\cdots\cdots (10)$$

where $gcd(n, m)$ denotes the greatest common divisor of the numbers n and m and $D_{gcd(n,m)}$ denotes the following groups :

$$D_{gcd(n,m)}=\begin{cases} D^{j}_{gcd(n,m)} & j=1,2,\cdots\cdots,n/gcd(n,m) \quad \text{when } n/gcd(n,m) \text{ is odd;} \\ D^{2j-1}_{gcd(n,m)} \text{ and } D^{2j}_{gcd(n,m)} & j=1,2,\cdots\cdots,n/(2\cdot gcd(n,m)) \quad \text{when } n/gcd(n,m) \text{ is even.} \end{cases} \quad \cdots\cdots (11)$$

The groups in Eqs. (10) and (11) can be defined as :

$$E=C_1=\langle\sigma_1\rangle$$
$$C_i=\langle\sigma_1,\ \sigma_{1+n/i},\ \sigma_{1+n(i-1)/i}\rangle \qquad i=gcd(n,m)$$
$$D^{j}_{i}=\langle\sigma_1,\ \sigma_{1+n/i},\cdots\cdots,\sigma_{1+n(i-1)/i},\ \tau\sigma_j,\ \tau\sigma_{j+n/i},\cdots\cdots,\tau\sigma_{j+n(i-1)/i}\rangle$$
$$D^{2j-1}_{i}=\langle\sigma_1,\ \sigma_{1+n/i},\cdots\cdots,\sigma_{1+n(i-1)/i},\ \tau\sigma_{2j-1},\ \tau\sigma_{2j-1+n/i},\cdots\cdots,\tau\sigma_{2j-1+n(i-1)/i}\rangle$$
$$D^{2j}_{i}=\langle\sigma_1,\ \sigma_{1+n/i},\cdots\cdots,\sigma_{1+n(i-1)/i},\ \tau\sigma_{2j},\ \tau\sigma_{2j+n/i},\cdots\cdots,\tau\sigma_{2j+n(i-1)/i}\rangle$$
$$D^{*}_{n/2}=\langle\sigma_1,\ \sigma_3,\cdots\cdots,\sigma_{n-1},\ \tau\sigma_1,\ \tau\sigma_3,\cdots\cdots,\tau\sigma_{n-1}\rangle$$
$$D_{n/2}=\langle\sigma_1,\ \sigma_3,\cdots\cdots,\sigma_{n-1},\ \tau\sigma_2,\ \tau\sigma_4,\cdots\cdots,\tau\sigma_n\rangle$$
$$D_n=\langle\sigma_1,\ \sigma_2,\cdots\cdots,\sigma_n,\ \tau\sigma_1,\ \tau\sigma_2,\cdots\cdots,\tau\sigma_n\rangle \quad \cdots\cdots (12)$$

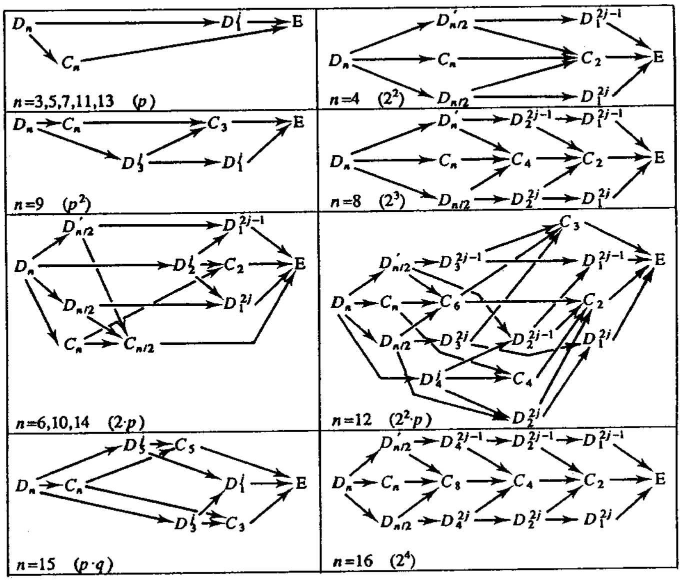
Table 4 contains renewed data regarding the existence of bifurcation modes. The symbol (○) in this table expresses the mode obtained by considering only vertical displacements; (△) denotes the mode which were identified by considering radial and rotational displacements, in addition to the radial ones. As can be seen, groups C_n, $D_{n/2}$ and $C_{n/2}$ did not undergo the degeneration for the formulation considering all the displacement components but did for the other formulation (we investigated the domes' deformation modes related to these groups to insure their existence when all the components were considered). Such

Table 4 Potential Bifurcation Modes of Regular-Polygonal Domes (n=3 through 16).

Modes	Degrees													
	3	4	5	6	7	8	9	10	11	12	13	14	15	16
E	○	○	○	○	○	○	○	○	○	○	○	○	○	○
C_2		Δ		○		○		○		○		○		○
C_3	Δ			Δ			○			○			○	
C_4		Δ				Δ				○				○
C_5			Δ					Δ					○	
C_6				Δ						Δ				
C_7					Δ							Δ		
C_8						Δ								Δ
D_1^j	○		○		○		○		○		○		○	
D_2^j				○				○				○		
D_3^j							○						○	
D_4^j										○				
D_5^j													○	
D_1^{2j-1}		○		○		○		○		○		○		○
D_1^{2j}		○		○		○		○		○		○		○
D_2^{2j-1}						○				○				○
D_2^{2j}						○				○				○
D_3^{2j-1}										○				
D_3^{2j}										○				
D_4^{2j-1}														○
D_4^{2j}														○
C_n	Δ	Δ	Δ	Δ	Δ	Δ	Δ	Δ	Δ	Δ	Δ	Δ	Δ	Δ
$D'_{n/2}$		Δ		Δ		Δ		Δ		Δ		Δ		Δ
$D_{n/2}$		○		○		○		○		○		○		○
D_n	○	○	○	○	○	○	○	○	○	○	○	○	○	○

○ : bifurcation modes identified by considering only vertical displacements of the domes;
Δ : bifurcation modes identified by considering all the displacement components of the domes.

Table 5 Hierarchal Frameworks of Bifurcation Modes.



( ) : factorization of degree;
p,q : prime numbers.

avoidance of degeneration for the former is attributable to its greater ability to represent symmetric nature of bifurcation modes.

The existence of bifurcation modes was highly periodic and systematic with regard to the degree of polygons. For example, a mode related to group D_1^j existed solely for odd degrees, while modes associated with groups D_1^{2j-1}, D_1^{2j}, $D_{n/2}$ and $D_{n/2}^{\cdot}$ did only for even degrees. Such periodic nature arises from dependency of subgroup types on degree n, as can be noted from Eq. (10).

Table 5 shows updated bifurcation mode hierarchies, which include more bifurcation modes than those advanced in Ref. 14. The consideration of radial and rotational displacements made the hierarchies much more theoretically complete and systematic. The data regarding bifurcation modes obtained herein will be of great assistance in studying bifurcation behavior of regular-polygonal-shaped dome structures subjected to axisymmetric loadings. For another type of loading, the subgroup representing the symmetry of the loading becomes its symmetry group. Then, the relevant bifurcation phenomena can be described by this symmetry group and its subgroups. This is identical with the application of dihedral groups to truss domes with imperfections advanced in Ref. 15.

6. CONCLUDING REMARKS

The group theoretic method has come to be used extensively to describe bifurcation behavior in last decades, especially in applied mathematics[9)~12)]. The authers extended and refined this method to be compatible with conventional bifurcation buckling analyses of dome structures[13)~17)]. While capable of describing well bifurcation behavior of the simple polygonal truss dome structures (see Fig. 1), the method had limited sense of generality as these domes had very simple geometric configurations.

This research was undertaken so as to extend the applicability of the method to dome structures with more realistic configurations with greater number of degree of freedoms. For this purpose, bifurcation buckling behavior of the general polygonal domes shown in Fig. 2 was investigated on the basis of a group theoretic standpoint. We proved these domes under axisymmetric loadings to be covariant with dihedral groups, thus insuring the usability of the groups in describing the domes' bifurcation behavior. As an application of these groups, we studied bifurcation behavior of the regular-hexagonal domes. Potential bifurcation modes of the domes and their bifurcation mode hierarchies were expressed in terms of the subgroups of a dihedral group of degree six (D_6). An investigation of actual equilibrium paths obtained through bifurcation path tracing analyses insured the validity and usability of the information advanced herein.

The group theoretic method, whose validity has been assessed herein and in previous papers[13)~17)], was used to obtain the potential bifurcation modes and hierarchies of regular-polygonal domes with degrees ranging from three through 16. Because of avoidance of mode degeneration, the use of radial and rotational displacements of the trusses resulted in more diversified and theoretically complete bifurcation mode hierarchies than those obtained in Ref. 14 by considering only vertical ones. All the displacement components should be included in describing bifurcation modes. These bifurcation modes and hierarchies obtained herein were suggested for use in studying bifurcation behavior of dome structures.

As we have seen, the group theoretic approach described well bifurcation behavior of regular-polygonal domes, whereas retaining desired simplicity and theoretical completeness. In this manner, the bifurcation behavior may be understood not as a result of highly complex nonlinear phenomena but as a natural consequence of highly systematic and organized bifurcation hierarchy. Most of the qualitative bifurcation behavioral characteristics of domes can be determined from their geometric configurations, while the loading condition and detailed aspects of the domes (their height, radii, etc.) can determine quantitative aspects. Provided with a number of group theoretic information regarding bifurcation phenomena, one can trace such phenomena with much more confidence and hopefully with fewer mistakes. Of course, care should be taken regarding the fact that the domes selected here as examples can represent only a structural type of domes among numerous alternatives so that one needs to investigate bifurcation behavioral features of various types of domes prior to arriving at general rules. The information regarding those polygonal domes obtained in this paper could form a basis in studying the other types of domes.

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