Structural Eng. / Earthquake Eng. Vol. 4. No. 2, 401s-411s. October 1987
Japan Society of Civil Engineers (Proc. of JSCE No. 386/I-8)

GENERALIZATION OF ELLIOTT'S SOLUTION TO TRANSVERSELY ISOTROPIC SOLIDS AND ITS APPLICATION

*By Isamu A. OKUMURA**

A generalized solution of Elliott's solution to transversely isotropic solids is proposed in cylindrical coordinates. The solution consists of five potential functions and includes some new terms corresponding to equal roots of a quadratic equation. When elastic constants of transversely isotropic solids are replaced with those of isotropic solids, the solution is exactly coincident with a general solution to isotropic solids. Expressions for the potential functions are presented in referring to non-axially symmetric problems of finite, hollow cylinders. As an application of the solution, an axially symmetric problem of a finite cylinder subjected to a band load is analyzed.
Keywords : elasticity solution, transverse isotropy, finite cylinder

1. INTRODUCTION

With the advance of three-dimensional elasticity problems of isotropic solids, several studies on two-dimensional or three-dimensional elasticity problems of anisotropic solids have been appeared so far. Though there are various classes of anisotropy as stated in Love's book[1], anisotropy with a practical interest seems to be transverse isotropy and orthotropy. Because the number of elastic constants of transverse isotropy is few, elasticity solutions seem to be obtained with comparative facility and so some solutions have been found by some researchers. Lekhnitskii[2] has found stress functions for two-dimensional problems and axially symmetric problems. Michell[3] has found a solution as the development of a solution to isotropic solids and has analyzed a semi-infinite solid. His solution seems, however, to have some difficulty with practical applications to general boundary-value problems. Elliott[4] has found a solution by making use of two potential functions. His solution is an ingenious solution obtained by a method using two roots of a quadratic equation composed of four elastic constants. When the roots become equal roots, two potential functions are, however, reduced to one potential function. Then, his solution becomes inapplicable to general boundary-value problems. Lodge[5] has found a solution by making use of one potential function. His solution is very important to non-axially symmetric problems. Hata[6] has obtained Elliott's solution and Lodge's solution by a method using operators in rectangular Cartesian coordinates and has discussed a certain justice in their solutions. Though Elliott's solution has been applied to various boundary-value problems[7)~12)], it seems that this solution has not been applied to three-dimensional problems of finite solids, for instance, rectangular prisms, finite plates with moderate thicknesses or finite cylinders. Though that reason is not clear, the number of potential functions in Elliott's solution

* Member of JSCE, Dr. Eng., Professor, Department of Civil Engineering, Kitami Institute of Technology (Koen-cho, Kitami).

appears to be seemingly insufficient for analyses of finite solids. It is evident from the comparison of Elliott's solution accompanied by Lodge's solution with a general solution to isotropic solids. The author does, however, not insist on a defect of Elliott's solution from this point. The author has, however, a notion that all solutions to anisotropic solids have to yield general solutions to isotropic solids when they are specialized to isotropic solids, if they are completely general solutions. Elliott's solution yields simply the first basic solution proposed by Boussinesq when it is specialized to isotropic solids referring to a special case of equal roots. If attension is riveted on it, Elliott's solution appears to be not a completely general solution and to be short of other solutions. From this point of view, this paper proposes new solutions to be not included in Elliott's solution and generalizes Elliott's solution to a more general solution in cylindrical coordinates. A generalized solution presented in this paper includes five potential functions and yields exactly a solution in a previous paper[13] when it is specialized to isotropic solids. As a special case of that solution, a solution omitted Lodge's solution from the generalized solution is reduced to Elliott's solution when it is restricted to the case of distinct roots, or two potential functions in it are omitted. That solution is extensively appricable to non-axially symmetric or axially symmetric problems of finite solids. As an application of that solution, a finite cylinder subjected to a band load is analyzed.

2. DISPLACEMENT EQUATIONS OF EQUILIBRIUM

If we use cylidrical coordinates $(r,\ \theta,\ z)$ such that the axis of z is taken parallel to the axis of elastic symmetry, we may obtain the following generalized Hooke's law :

$$\sigma_{rr}=C_{11}\varepsilon_{rr}+C_{12}\varepsilon_{\theta\theta}+C_{13}\varepsilon_{zz},\quad \sigma_{\theta\theta}=C_{12}\varepsilon_{rr}+C_{11}\varepsilon_{\theta\theta}+C_{13}\varepsilon_{zz};$$
$$\sigma_{zz}=C_{13}\varepsilon_{rr}+C_{13}\varepsilon_{\theta\theta}+C_{33}\varepsilon_{zz},\quad \sigma_{\theta z}=2C_{44}\varepsilon_{\theta z},\quad \sigma_{zr}=2C_{44}\varepsilon_{zr},\quad \sigma_{r\theta}=2C_{66}\varepsilon_{r\theta},\cdots\cdots \tag{1 · a~f}$$

in which

$$C_{66}=(C_{11}-C_{12})/2,\cdots\cdots \tag{2}$$

and $C_{\alpha\beta}$, σ_{ij} and ε_{ij} are the elastic constants of transversely isotropic solids, stress tensor and strain tensor, respectively. The number of independent elastic constants is five. Components of strain are expressed as

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r},\quad \varepsilon_{\theta\theta}=\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta},\quad \varepsilon_{zz}=\frac{\partial u_z}{\partial z},\quad \varepsilon_{\theta z}=\frac{1}{2}\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial \theta}\right);$$
$$\varepsilon_{zr}=\frac{1}{2}\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right),\quad \varepsilon_{r\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right),\cdots\cdots \tag{3 · a~f}$$

in which u_r, u_θ and u_z are components of displacement, respectively. If we substitute Eqs. (1 · a~f) and (3 · a~f) into the equations of equilibrium and exclude body forces, we obtain displacement equations of equilibrium in the form

$$C_{11}\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}\right)+C_{66}\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}+C_{44}\frac{\partial^2 u_r}{\partial z^2}+(C_{12}+C_{66})\frac{1}{r}\frac{\partial^2 u_\theta}{\partial r\partial\theta}$$
$$-(C_{11}+C_{66})\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}+(C_{13}+C_{44})\frac{\partial^2 u_z}{\partial r\partial z}=0,\cdots\cdots \tag{4 · a}$$

$$C_{66}\left(\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}\right)+C_{11}\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+C_{44}\frac{\partial^2 u_\theta}{\partial z^2}+(C_{12}+C_{66})\frac{1}{r}\frac{\partial^2 u_r}{\partial r\partial\theta}$$
$$+(C_{11}+C_{66})\frac{1}{r^2}\frac{\partial u_r}{\partial\theta}+(C_{13}+C_{44})\frac{1}{r}\frac{\partial^2 u_z}{\partial \theta\partial z}=0,\cdots\cdots \tag{4 · b}$$

$$C_{44}\nabla_1^2 u_z+C_{33}\frac{\partial^2 u_z}{\partial z^2}+(C_{13}+C_{44})\frac{\partial}{\partial z}\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)=0,\cdots\cdots \tag{4 · c}$$

in which

$$\nabla_1^2\equiv\partial^2/\partial r^2+(1/r)\partial/\partial r+(1/r^2)\partial^2/\partial\theta^2.\cdots\cdots \tag{5}$$

3. GENERALIZATION OF ELLIOTT'S SOLUTION

We now define the following quadratic equation composed of four elastic constants as shown in Elliott's solution[4]:

$$C_{11}C_{44}\nu^2+[C_{13}(C_{13}+2C_{44})-C_{11}C_{33}]\nu+C_{33}C_{44}=0, \quad (6)$$

and denote the roots in ν by ν_1 and ν_2. We define also the following parameter expressed in the roots and three elastic constants:

$$k_\alpha=\frac{C_{11}\nu_\alpha-C_{44}}{C_{13}+C_{44}} \quad (\alpha=1,\ 2). \quad (7)$$

Then, we obtain the following relation from Eqs. (6) and (7):

$$\nu_1\nu_2=\frac{C_{33}}{C_{11}},\quad k_1k_2=1,\quad \frac{C_{33}k_\alpha}{C_{44}k_\alpha+(C_{13}+C_{44})}=\nu_\alpha \quad (\alpha=1,\ 2). \quad (8\cdot \text{a}\sim\text{c})$$

Hereafter, we shall derive solutions to Eqs. (4 · a~c) in dividing them into two forms.

(1) A solution of the first form

We assume a solution corresponding to that to isotropic solids in the previous paper[13] in the form

$$u_r=\frac{\partial}{\partial r}\left(r\frac{\partial\phi_1}{\partial r}\right)+\xi_1,\quad u_\theta=\frac{1}{r}\frac{\partial}{\partial\theta}\left(r\frac{\partial\phi_1}{\partial r}\right)+\eta_1,\quad u_z=k_2\frac{\partial}{\partial z}\left(r\frac{\partial\phi_1}{\partial r}\right)+\zeta_1. \quad (9\cdot \text{a}\sim\text{c})$$

If we substitute Eqs. (9 · a~c) into Eqs. (4 · a~c) and take ϕ_1 as a potential function satisfying the following equation:

$$\nabla_1^2\phi_1+\nu_2\frac{\partial^2\phi_1}{\partial z^2}=0, \quad (10)$$

we obtain a system of partial differential equations with three unknowns in the form

$$C_{11}\left(\frac{\partial^2\xi_1}{\partial r^2}+\frac{1}{r}\frac{\partial\xi_1}{\partial r}-\frac{\xi_1}{r^2}\right)+C_{66}\frac{1}{r^2}\frac{\partial^2\xi_1}{\partial\theta^2}+C_{44}\frac{\partial^2\xi_1}{\partial z^2}+(C_{12}+C_{66})\frac{1}{r}\frac{\partial^2\eta_1}{\partial r\partial\theta}$$

$$-(C_{11}+C_{66})\frac{1}{r^2}\frac{\partial\eta_1}{\partial\theta}+(C_{13}+C_{44})\frac{\partial^2\zeta_1}{\partial r\partial z}=2C_{11}\nu_2\frac{\partial^3\phi_1}{\partial r\partial z^2}, \quad (11\cdot \text{a})$$

$$C_{66}\left(\frac{\partial^2\eta_1}{\partial r^2}+\frac{1}{r}\frac{\partial\eta_1}{\partial r}-\frac{\eta_1}{r^2}\right)+C_{11}\frac{1}{r^2}\frac{\partial^2\eta_1}{\partial\theta^2}+C_{44}\frac{\partial^2\eta_1}{\partial z^2}+(C_{12}+C_{66})\frac{1}{r}\frac{\partial^2\xi_1}{\partial r\partial\theta}$$

$$+(C_{11}+C_{66})\frac{1}{r^2}\frac{\partial\xi_1}{\partial\theta}+(C_{13}+C_{44})\frac{1}{r}\frac{\partial^2\zeta_1}{\partial\theta\partial z}=2C_{11}\nu_2\frac{1}{r}\frac{\partial^3\phi_1}{\partial\theta\partial z^2}, \quad (11\cdot \text{b})$$

$$C_{44}\nabla_1^2\zeta_1+C_{33}\frac{\partial^2\zeta_1}{\partial z^2}+(C_{13}+C_{44})\frac{\partial}{\partial z}\left(\frac{\partial\xi_1}{\partial r}+\frac{\xi_1}{r}+\frac{1}{r}\frac{\partial\eta_1}{\partial\theta}\right)=2k_2C_{33}\frac{\partial^3\phi_1}{\partial z^3}. \quad (11\cdot \text{c})$$

Putting that

$$\xi_1=\frac{\partial\phi_{01}}{\partial r}-\frac{2C_{11}\nu_2}{C_{11}\nu_2-C_{44}}\frac{\partial\phi_1}{\partial r},\quad \eta_1=\frac{1}{r}\frac{\partial\phi_{01}}{\partial\theta}-\frac{2C_{11}\nu_2}{C_{11}\nu_2-C_{44}}\frac{1}{r}\frac{\partial\phi_1}{\partial\theta};$$

$$\zeta_1=k_1\frac{\partial\phi_{01}}{\partial z}+\frac{\partial\zeta}{\partial z}, \quad (12\cdot \text{a}\sim\text{c})$$

and substituting these into Eqs. (11 · a~c), we obtain

$$\nabla_1^2\phi_{01}+\nu_1\frac{\partial^2\phi_{01}}{\partial z^2}=-\frac{C_{13}+C_{44}}{C_{11}}\frac{\partial^2\zeta}{\partial z^2}, \quad (13\cdot \text{a})$$

$$\left[C_{44}k_1+(C_{13}+C_{44})\right]\left(\nabla_1^2\phi_{01}+\nu_1\frac{\partial^2\phi_{01}}{\partial z^2}\right)+C_{44}\nabla_1^2\zeta+C_{33}\frac{\partial^2\zeta}{\partial z^2}$$

$$=-2\nu_2C_{44}(k_1-k_2)\frac{\partial^2\phi_1}{\partial z^2}. \quad (13\cdot \text{b})$$

Furthermore, substituting Eq. (13 · a) into Eq. (13 · b) and using Eq. (8 · a), we obtain

$$\nabla_1^2\zeta+\nu_2\frac{\partial^2\zeta}{\partial z^2}=-2\nu_2(k_1-k_2)\frac{\partial^2\phi_1}{\partial z^2}. \qquad (14)$$

The general solution to Eq. (14) is expressed in the sum of complemental function ζ_c and particular integral ζ_p. Complemental function ζ_c shall, however, be excluded here, because it is dependent upon function ϕ_{03} in a solution of the second form which shall be stated later. Particular integral ζ_p is

$$\zeta_p=(k_1-k_2)r\frac{\partial\phi_1}{\partial r}. \qquad (15)$$

Substituting this solution into the right-hand side of Eq. (13・a), we obtain

$$\nabla_1^2\phi_{01}+\nu_1\frac{\partial^2\phi_{01}}{\partial z^2}=-(\nu_1-\nu_2)r\frac{\partial^3\phi_1}{\partial r\partial z^2}. \qquad (16)$$

The general solution to Eq. (16) is expressed in the sum of complemental function $\phi_{01,c}$ and particular integral $\phi_{01,p}$. If we notice the right-hand side of Eq. (16), we find that particular integral $\phi_{01,p}$ is divided into two expressions corresponding to equal roots ($\nu_1=\nu_2$) or distinct roots ($\nu_1\neq\nu_2$). In case of equal roots, the particular integral becomes

$$\phi_{01,p}=0 \quad \text{for} \quad \nu_1=\nu_2. \qquad (17)$$

In case of distinct roots, we shall use the method of undetermined coefficients. Then, we may assume $\phi_{01,p}$ in the form

$$\phi_{01,p}=a_1\phi_1+a_2r\frac{\partial\phi_1}{\partial r}. \qquad (18)$$

Substituting Eq. (18) into the left-hand side of Eq. (16), we obtain

$$a_1=-\frac{2\nu_2}{\nu_1-\nu_2}, \quad a_2=-1. \qquad (19\cdot\text{a, b})$$

Furthermore, substituting the above values into Eq. (18), we obtain

$$\phi_{01,p}=-\frac{2\nu_2}{\nu_1-\nu_2}\phi_1-r\frac{\partial\phi_1}{\partial r} \quad \text{for} \quad \nu_1\neq\nu_2. \qquad (20)$$

By making use of Eq. (15), equations (12・a~c) are therefore expressed in the form

$$\xi_1=\frac{\partial}{\partial r}\left(\phi_{01,c}+\phi_{01,p}-\frac{2C_{11}\nu_2}{C_{11}\nu_2-C_{44}}\phi_1\right), \qquad (21\cdot\text{a})$$

$$\eta_1=\frac{1}{r}\frac{\partial}{\partial\theta}\left(\phi_{01,c}+\phi_{01,p}-\frac{2C_{11}\nu_2}{C_{11}\nu_2-C_{44}}\phi_1\right), \qquad (21\cdot\text{b})$$

$$\zeta_1=\frac{\partial}{\partial z}\left[k_1(\phi_{01,c}+\phi_{01,p})+(k_1-k_2)r\frac{\partial\phi_1}{\partial r}\right]. \qquad (21\cdot\text{c})$$

Lastly, substituting the above functions into Eqs. (9・a~c) and using Eqs. (17) and (20), we obtain the following solution :

$$u_r=\frac{\partial}{\partial r}\left(\phi_{01}+\gamma_1r\frac{\partial\phi_1}{\partial r}-\gamma_2\phi_1\right), \quad u_\theta=\frac{1}{r}\frac{\partial}{\partial\theta}\left(\phi_{01}+\gamma_1r\frac{\partial\phi_1}{\partial r}-\gamma_2\phi_1\right);$$

$$u_z=k_1\frac{\partial}{\partial z}\left(\phi_{01}+\gamma_1r\frac{\partial\phi_1}{\partial r}-\gamma_3\phi_1\right), \qquad (22\cdot\text{a}\sim\text{c})$$

in which ϕ_{01} denotes $\phi_{01,c}$, and

$$\nabla_1^2\phi_{01}+\nu_1\frac{\partial^2\phi_{01}}{\partial z^2}=0, \quad \nabla_1^2\phi_1+\nu_2\frac{\partial^2\phi_1}{\partial z^2}=0, \qquad (23\cdot\text{a, b})$$

$$\gamma_1=\begin{cases}1 & [\nu_1=\nu_2]\\ 0 & [\nu_1\neq\nu_2]\end{cases}, \quad \gamma_2=\begin{cases}\dfrac{2C_{11}\nu_2}{C_{11}\nu_2-C_{44}} & [\nu_1=\nu_2]\\ \dfrac{2\nu_2}{\nu_1-\nu_2}\cdot\dfrac{C_{11}\nu_1-C_{44}}{C_{11}\nu_2-C_{44}} & [\nu_1\neq\nu_2]\end{cases};$$

$$\gamma_3=\begin{cases}0 & [\nu_1=\nu_2]\\ \dfrac{2\nu_2}{\nu_1-\nu_2} & [\nu_1\neq\nu_2].\end{cases} \qquad (24\cdot\text{a}\sim\text{c})$$

The first term in the right-hand side of Eqs. (22 · a~c) is Elliott's solution which is the complemental function of Eq. (16). The second and third terms are new solutions obtained by the present method. They correspond to solutions for equal roots and distinct roots.

(2) A solution of the second form

We assume again a solution corresponding to that in the previous paper in the form

$$u_r=\frac{\partial}{\partial r}\left(z\frac{\partial \phi_3'}{\partial z}\right)+\xi_3,\quad u_\theta=\frac{1}{r}\frac{\partial}{\partial \theta}\left(z\frac{\partial \phi_3'}{\partial z}\right)+\eta_3,\quad u_z=k_1\frac{\partial}{\partial z}\left(z\frac{\partial \phi_3'}{\partial z}\right)+\zeta_3. \qquad (25\cdot a\sim c)$$

If we substitute Eqs. (25 · a~c) into Eqs. (4 · a~c) and take ϕ_3' as a potential function satisfying the following equation :

$$\nabla_1^2\phi_3'+\nu_1\frac{\partial^2\phi_3'}{\partial z^2}=0, \qquad (26)$$

we obtain a system of partial differential equations with three unknowns in the form

$$C_{11}\left(\frac{\partial^2\xi_3}{\partial r^2}+\frac{1}{r}\frac{\partial \xi_3}{\partial r}-\frac{\xi_3}{r^2}\right)+C_{66}\frac{1}{r^2}\frac{\partial^2\xi_3}{\partial\theta^2}+C_{44}\frac{\partial^2\xi_3}{\partial z^2}+(C_{12}+C_{66})\frac{1}{r}\frac{\partial^2\eta_3}{\partial r\partial\theta}$$
$$-(C_{11}+C_{66})\frac{1}{r^2}\frac{\partial\eta_3}{\partial\theta}+(C_{13}+C_{44})\frac{\partial^2\zeta_3}{\partial r\partial z}=-2C_{11}\nu_1\frac{\partial^3\phi_3'}{\partial r\partial z^2}, \qquad (27\cdot a)$$

$$C_{66}\left(\frac{\partial^2\eta_3}{\partial r^2}+\frac{1}{r}\frac{\partial \eta_3}{\partial r}-\frac{\eta_3}{r^2}\right)+C_{11}\frac{1}{r^2}\frac{\partial^2\eta_3}{\partial\theta^2}+C_{44}\frac{\partial^2\eta_3}{\partial z^2}+(C_{12}+C_{66})\frac{1}{r}\frac{\partial^2\xi_3}{\partial r\partial\theta}$$
$$+(C_{11}+C_{66})\frac{1}{r^2}\frac{\partial\xi_3}{\partial\theta}+(C_{13}+C_{44})\frac{1}{r}\frac{\partial^2\zeta_3}{\partial\theta\partial z}=-2C_{11}\nu_1\frac{1}{r}\frac{\partial^3\phi_3'}{\partial\theta\partial z^2}, \qquad (27\cdot b)$$

$$C_{44}\nabla_1^2\zeta_3+C_{33}\frac{\partial^2\zeta_3}{\partial z^2}+(C_{13}+C_{44})\frac{\partial}{\partial z}\left(\frac{\partial\xi_3}{\partial r}+\frac{\xi_3}{r}+\frac{1}{r}\frac{\partial\eta_3}{\partial\theta}\right)=-2k_1C_{33}\frac{\partial^3\phi_3'}{\partial z^3}. \qquad (27\cdot c)$$

Putting that

$$\xi_3=\frac{\partial\phi_{03}}{\partial r}+\frac{\partial\xi}{\partial r},\quad \eta_3=\frac{1}{r}\frac{\partial\phi_{03}}{\partial\theta}+\frac{1}{r}\frac{\partial\xi}{\partial\theta},\quad \zeta_3=k_2\frac{\partial\phi_{03}}{\partial z}-\frac{2k_1C_{11}\nu_2}{C_{11}\nu_2-C_{44}}\frac{\partial\phi_3'}{\partial z}, \qquad (28\cdot a\sim c)$$

and substituting these into Eqs. (27 · a~c), we obtain

$$\nabla_1^2\phi_{03}+\nu_2\frac{\partial^2\phi_{03}}{\partial z^2}+\nabla_1^2\xi+\frac{C_{44}}{C_{11}}\frac{\partial^2\xi}{\partial z^2}=\frac{2C_{44}(\nu_1-\nu_2)}{C_{11}\nu_2-C_{44}}\frac{\partial^2\phi_3'}{\partial z^2}, \qquad (29\cdot a)$$

$$\nabla_1^2\phi_{03}+\nu_2\frac{\partial^2\phi_{03}}{\partial z^2}=-\frac{\nu_2(C_{13}+C_{44})}{C_{33}k_2}\nabla_1^2\xi. \qquad (29\cdot b)$$

Furthermore, substituting Eq. (29 · b) into Eq. (29 · a) and using Eq. (8 · a), we obtain

$$\nabla_1^2\xi+\nu_1\frac{\partial^2\xi}{\partial z^2}=\frac{2C_{11}\nu_1(\nu_1-\nu_2)}{C_{11}\nu_2-C_{44}}\frac{\partial^2\phi_3'}{\partial z^2}. \qquad (30)$$

The general solution to Eq. (30) is expressed in the sum of complemental function ξ_c and particular integral ξ_p. Complemental function ξ_c shall, however, be excluded, because it is dependent upon function ϕ_{01} in the solution of the first form stated before. Particular integral ξ_p is

$$\xi_p=\frac{C_{11}(\nu_1-\nu_2)}{C_{11}\nu_2-C_{44}}z\frac{\partial\phi_3'}{\partial z}. \qquad (31)$$

Substituting this solution into the right-hand side of Eq. (29 · b) and using Eq. (26), we obtain

$$\nabla_1^2\phi_{03}+\nu_2\frac{\partial^2\phi_{03}}{\partial z^2}=\frac{\nu_1-\nu_2}{k_2^2}z\frac{\partial^3\phi_3'}{\partial z^3}. \qquad (32)$$

The general solution to Eq. (32) is expressed in the sum of complemental function $\phi_{03,c}$ and particular integral $\phi_{03,p}$. If we use the method of undetermined coefficients as stated before, we obtain the following particular integral :

$$\phi_{03,p}=\begin{cases}0 & [\nu_1=\nu_2]\\ -\dfrac{2\nu_2}{\nu_1-\nu_2}\phi_3-z\dfrac{\partial\phi_3}{\partial z} & [\nu_1\neq\nu_2]\end{cases}, \quad (33)$$

in which

$$\phi_3=\phi_3'/k_2^2. \quad (34)$$

By making use of Eqs. (31) and (34), equations (28・a~c) are therefore expressed in the form

$$\xi_3=\frac{\partial}{\partial r}\left[\phi_{03,c}+\phi_{03,p}+\frac{C_{11}(\nu_1-\nu_2)k_2^2}{C_{11}\nu_2-C_{44}}z\frac{\partial\phi_3}{\partial z}\right], \quad (35・a)$$

$$\eta_3=\frac{1}{r}\frac{\partial}{\partial\theta}\left[\phi_{03,c}+\phi_{03,p}+\frac{C_{11}(\nu_1-\nu_2)k_2^2}{C_{11}\nu_2-C_{44}}z\frac{\partial\phi_3}{\partial z}\right], \quad (35・b)$$

$$\zeta_3=k_2\frac{\partial}{\partial z}\left[\phi_{03,c}+\phi_{03,p}-\frac{2C_{11}\nu_2k_1k_2}{C_{11}\nu_2-C_{44}}\phi_3\right]. \quad (35・c)$$

Lastly, substituting the above functions and Eq. (34) into Eqs. (25・a~c) and using Eqs. (8・b) and (33), we obtain the following solution:

$$u_r=\frac{\partial}{\partial r}\left(\phi_{03}+\gamma_1z\frac{\partial\phi_3}{\partial z}-\gamma_3\phi_3\right),\quad u_\theta=\frac{1}{r}\frac{\partial}{\partial\theta}\left(\phi_{03}+\gamma_1z\frac{\partial\phi_3}{\partial z}-\gamma_3\phi_3\right);$$

$$u_z=k_2\frac{\partial}{\partial z}\left(\phi_{03}+\gamma_1z\frac{\partial\phi_3}{\partial z}-\gamma_2\phi_3\right), \quad (36・a\sim c)$$

in which ϕ_{03} denotes $\phi_{03,c}$, and

$$\nabla_1^2\phi_{03}+\nu_2\frac{\partial^2\phi_{03}}{\partial z^2}=0,\quad \nabla_1^2\phi_3+\nu_1\frac{\partial^2\phi_3}{\partial z^2}=0 \quad (37・a, b)$$

Coefficients γ_1, γ_2 and γ_3 are also expressed in Eqs. (24・a~c). The first term in the right-hand side of Eqs. (36・a~c) is Elliott's solution which is the complemental function of Eq. (32). The second and third terms are new solutions obtained by the present method. They correspond again to solutions for equal roots and distinct roots.

(3) A generalized solution of Elliott's solution

Now, Lodge's solution[5] to non-axially symmetric problems is as follows:

$$u_r=\frac{1}{r}\frac{\partial\psi}{\partial\theta},\quad u_\theta=-\frac{\partial\psi}{\partial r},\quad u_z=0, \quad (38・a\sim c)$$

in which

$$\nabla_1^2\psi+\nu_3\frac{\partial^2\psi}{\partial z^2}=0,\quad \nu_3=\frac{C_{44}}{C_{66}}=\frac{2C_{44}}{C_{11}-C_{12}}. \quad (39・a, b)$$

If we make a linear combination of this solution and two solutions of the first form and of the second form, we obtain a generalized solution of Elliott's solution in the form

$$u_r=\frac{\partial}{\partial r}\left[\phi_{01}+\phi_{03}+\gamma_1\left(r\frac{\partial\phi_1}{\partial r}+z\frac{\partial\phi_3}{\partial z}\right)-\gamma_2\phi_1-\gamma_3\phi_3\right]+\frac{1}{r}\frac{\partial\psi}{\partial\theta}, \quad (40・a)$$

$$u_\theta=\frac{1}{r}\frac{\partial}{\partial\theta}\left[\phi_{01}+\phi_{03}+\gamma_1\left(r\frac{\partial\phi_1}{\partial r}+z\frac{\partial\phi_3}{\partial z}\right)-\gamma_2\phi_1-\gamma_3\phi_3\right]-\frac{\partial\psi}{\partial r}, \quad (40・b)$$

$$u_z=\frac{\partial}{\partial z}\left[k_1\phi_{01}+k_2\phi_{03}+\gamma_1\left(k_1r\frac{\partial\phi_1}{\partial r}+k_2z\frac{\partial\phi_3}{\partial z}\right)-k_1\gamma_3\phi_1-k_2\gamma_2\phi_3\right], \quad (40・c)$$

in which ϕ_{01}, ϕ_1, ϕ_{03}, ϕ_3 and ψ are the potential functions governed by Eqs. (23・a, b), (37・a, b) and (39・a), respectively, and k_1, k_2, γ_1, γ_2 and γ_3 are the parameters in Eq. (7) and the coefficients in Eqs. (24・a~c), respectively. This solution consists of five potential functions.

In the first place, we specialize the generalized solution to the case of distinct roots. Then, equations (24・a~c) become

$$\gamma_1=0,\quad \gamma_2=\frac{2\nu_2}{\nu_1-\nu_2}\cdot\frac{C_{11}\nu_1-C_{44}}{C_{11}\nu_2-C_{44}},\quad \gamma_3=\frac{2\nu_2}{\nu_1-\nu_2}. \quad (41・a\sim c)$$

If we exclude Lodge's solution from the generalized solution of Eqs. (40 · a~c) and use Eqs. (41 · a~c), we obtain

$$u_r=\frac{\partial}{\partial r}[(\phi_{01}-\gamma_3\phi_3)+(\phi_{03}-\gamma_2\phi_1)],\quad u_\theta=\frac{1}{r}\frac{\partial}{\partial\theta}[(\phi_{01}-\gamma_3\phi_3)+(\phi_{03}-\gamma_2\phi_1)];$$

$$u_z=\frac{\partial}{\partial z}[k_1(\phi_{01}-\gamma_3\phi_3)+k_2(\phi_{03}-\gamma_2\phi_1)+(k_1\gamma_3-k_2\gamma_2)(\phi_3-\phi_1)].\qquad(42\cdot\mathrm{a}\sim\mathrm{c})$$

Because the following relation is held in Eq. (42 · c) :

$$k_1\gamma_3-k_2\gamma_2=0,\qquad(43)$$

we may put that

$$\phi_{01}-\gamma_3\phi_3=\phi_1',\quad \phi_{03}-\gamma_2\phi_1=\phi_2.\qquad(44\cdot\mathrm{a,b})$$

Then, equations (42 · a~c) are exactly reduced to Elliott's solution[4] in the form

$$u_r=\frac{\partial}{\partial r}\left(\phi_1'+\phi_2\right),\quad u_\theta=\frac{1}{r}\frac{\partial}{\partial\theta}\left(\phi_1'+\phi_2\right),\quad u_z=\frac{\partial}{\partial z}\left(k_1\phi_1'+k_2\phi_2\right),\qquad(45\cdot\mathrm{a}\sim\mathrm{c})$$

in which

$$\nabla_1^2\phi_1'+\nu_1\frac{\partial^2\phi_1'}{\partial z^2}=0,\quad \nabla_1^2\phi_2+\nu_2\frac{\partial^2\phi_2}{\partial z^2}=0.\qquad(46\cdot\mathrm{a,b})$$

There is no doubt that a solution omitting ϕ_1, ϕ_3 and ψ from Eqs. (40 · a~c) is reduced to Elliott's solution. However, it seems from noticing the process shown in Eqs. (41 · a~c) to Eqs. (46 · a, b) that Elliott's solution is derived from the implicit restriction to be distinct roots.

In the second place, we specialize the generalized solution to the case of such equal roots as occur in isotropic solids. The elastic constants of transversely isotropic solids are replaced with those of isotropic solids as

$$C_{33}=C_{11},\quad C_{13}=C_{12},\quad C_{44}=C_{66}=\frac{1}{2}(C_{11}-C_{12}).\qquad(47\cdot\mathrm{a}\sim\mathrm{c})$$

Substituting the above constants into Eqs. (6), (39 · b) and (7), we obtain the following values :

$$\nu_1=\nu_2=\nu_3=1,\quad k_1=k_2=1.\qquad(48\cdot\mathrm{a,b})$$

Then, equations (24 · a~c) become

$$\gamma_1=1,\quad \gamma_2=\frac{2C_{11}}{C_{11}-C_{44}}=\frac{4C_{11}}{C_{11}+C_{12}}=4(1-\nu),\quad \gamma_3=0,\qquad(49\cdot\mathrm{a}\sim\mathrm{c})$$

in which ν denotes Poisson's ratio of isotropic solids. If we use Eqs. (48 · a, b) and (49 · a~c), and put that

$$\phi_{01}+\phi_{03}=\frac{\phi_0'}{2G},\quad \phi_1=\frac{\phi_1'}{2G},\quad \phi_3=\frac{\phi_3'}{2G},\quad \psi=\frac{\vartheta_z}{G},\qquad(50\cdot\mathrm{a}\sim\mathrm{d})$$

in which G denotes the shear modulus of isotropic solids, we obtain the following solution to isotropic solids :

$$2Gu_r=\frac{\partial}{\partial r}\left[\phi_0'+r\frac{\partial\phi_1'}{\partial r}+z\frac{\partial\phi_3'}{\partial z}-4(1-\nu)\phi_1'\right]+\frac{2}{r}\frac{\partial\vartheta_z}{\partial\theta},\qquad(51\cdot\mathrm{a})$$

$$2Gu_\theta=\frac{1}{r}\frac{\partial}{\partial\theta}\left[\phi_0'+r\frac{\partial\phi_1'}{\partial r}+z\frac{\partial\phi_3'}{\partial z}-4(1-\nu)\phi_1'\right]-2\frac{\partial\vartheta_z}{\partial r},\qquad(51\cdot\mathrm{b})$$

$$2Gu_z=\frac{\partial}{\partial z}\left[\phi_0'+r\frac{\partial\phi_1'}{\partial r}+z\frac{\partial\phi_3'}{\partial z}-4(1-\nu)\phi_3'\right],\qquad(51\cdot\mathrm{c})$$

in which

$$\nabla^2\phi_0'=\nabla^2\phi_1'=\nabla^2\phi_3'=\nabla^2\vartheta_z=0,\quad \nabla^2\equiv\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}.\qquad(52\cdot\mathrm{a,b})$$

The solution of Eqs. (51 · a~c) is exactly coincident with the solution in the previous paper[13].

4. EXPRESSIONS FOR THE POTENTIAL FUNCTIONS TO NON-AXIALLY SYMMETRIC PROBLEMS

If we solve Eqs. (23 · a, b), (37 · a, b) and (39 · a) by the method of separation of variables, we obtain expressions for the potential functions in Eqs. (40 · a~c) in the form

$$\phi_{03}=\sum_{m=0}^{\infty}\sum_{s=1}^{\infty}\cos m\theta\Big[J_m(\alpha_{ms}r)\Big(A^{(1)}_{ms}\cosh\frac{\alpha_{ms}}{\sqrt{\nu_2}}z+L^{(1)}_{ms}\sinh\frac{\alpha_{ms}}{\sqrt{\nu_2}}z\Big)+Y_m(\alpha_{ms}r)\Big(A^{(2)}_{ms}\cosh\frac{\alpha_{ms}}{\sqrt{\nu_2}}z+L^{(2)}_{ms}\sinh\frac{\alpha_{ms}}{\sqrt{\nu_2}}z\Big)\Big], \quad (53\cdot a)$$

$$\phi_{3}=\sum_{m=0}^{\infty}\sum_{s=1}^{\infty}\cos m\theta\Big[J_m(\alpha_{ms}r)\Big(C^{(1)}_{ms}\cosh\frac{\alpha_{ms}}{\sqrt{\nu_1}}z+M^{(1)}_{ms}\sinh\frac{\alpha_{ms}}{\sqrt{\nu_1}}z\Big)+Y_m(\alpha_{ms}r)\Big(C^{(2)}_{ms}\cosh\frac{\alpha_{ms}}{\sqrt{\nu_1}}z+M^{(2)}_{ms}\sinh\frac{\alpha_{ms}}{\sqrt{\nu_1}}z\Big)\Big], \quad (53\cdot b)$$

$$\phi_{01}=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\cos m\theta\cos\beta_n z[D^{(1)}_{mn}I_m(\sqrt{\nu_1}\beta_n r)+D^{(2)}_{mn}K_m(\sqrt{\nu_1}\beta_n r)], \quad (53\cdot c)$$

$$\phi_{1}=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\cos m\theta\cos\beta_n z[F^{(1)}_{mn}I_m(\sqrt{\nu_2}\beta_n r)+F^{(2)}_{mn}K_m(\sqrt{\nu_2}\beta_n r)], \quad (53\cdot d)$$

$$\psi=\sum_{m=0}^{\infty}\sum_{s=1}^{\infty}\sin m\theta\Big[J_m(\alpha_{ms}r)\Big(B^{(1)}_{ms}\cosh\frac{\alpha_{ms}}{\sqrt{\nu_3}}z+G^{(1)}_{ms}\sinh\frac{\alpha_{ms}}{\sqrt{\nu_3}}z\Big)+Y_m(\alpha_{ms}r)\Big(B^{(2)}_{ms}\cosh\frac{\alpha_{ms}}{\sqrt{\nu_3}}z+G^{(2)}_{ms}\sinh\frac{\alpha_{ms}}{\sqrt{\nu_3}}z\Big)\Big]+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sin m\theta\cos\beta_n z[E^{(1)}_{mn}I_m(\sqrt{\nu_3}\beta_n r)+E^{(2)}_{mn}K_m(\sqrt{\nu_3}\beta_n r)], \quad (53\cdot e)$$

in which $J_m(\alpha_{ms}r)$ and $Y_m(\alpha_{ms}r)$ are Bessel functions of the first kind and the second kind, respectively. Also, $I_m(\sqrt{\nu_1}\beta_n r)$ and $K_m(\sqrt{\nu_1}\beta_n r)$ are the modified Bessel functions of the first kind and the second kind, respectively. Furthermore, α_{ms} and β_n are characteristic values to be chosen according to given boundary conditions, and $A^{(1)}_{ms}$ to $E^{(2)}_{mn}$ are arbitrary constants to be determined from given boundary conditions. We shall notice that linear combinations of ϕ_{01} and ϕ_3 and of ϕ_{03} and ϕ_1 are not made for the solutions, because their potential functions are independent each other. If we substitute Eqs. (53 · a~e) into Eqs. (40 · a~c), we may obtain components of displacement. We may also obtain components of stress from those components of displacement by making use of Eqs. (3 · a~f) and (1 · a~f).

5. AN APPLICATION OF THE PRESENT SOLUTIONS TO A FINITE CYLINDER

As an application of the solutions shown in Eqs. (40 · a~c) and (53 · a~e), we analyze an axially symmetric problem of a finite cylinder subjected to a band load as shown in **Fig. 1**. Expressions for potential functions are obtained from excluding Bessel functions of the second kind and the modified Bessel

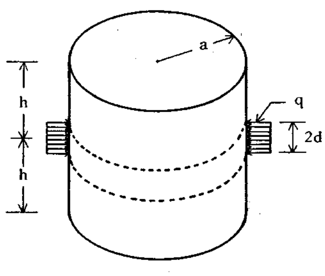

Fig. 1 Finite cylinder.

functions of the second kind in Eqs. (53 · a~e) and from putting $m=0$, in the form

$$\phi_{03}=\sum_{s=1}^{\infty}A_sJ_0(\alpha_s r)\cosh\frac{\alpha_s}{\sqrt{\nu_2}}z,\quad \phi_3=\sum_{s=1}^{\infty}C_sJ_0(\alpha_s r)\cosh\frac{\alpha_s}{\sqrt{\nu_1}}z\,;$$

$$\phi_{01}=\sum_{n=1}^{\infty}D_n\cos\beta_n zI_0(\sqrt{\nu_1}\beta_n r),\quad \phi_1=\sum_{n=1}^{\infty}F_n\cos\beta_n zI_0(\sqrt{\nu_2}\beta_n r),\qquad (54\cdot \text{a}\sim\text{d})$$

in which A_s to F_n are arbitrary constants to be determined from given boundary conditions. Also, the following expressions for the potential functions are needed as an additional solution :

$$\phi_{3,0}=C_0\left(\frac{r^2}{2}-\frac{z^2}{\nu_1}\right),\quad \phi_{1,0}=F_0\left(\frac{r^2}{2}-\frac{z^2}{\nu_2}\right),\qquad (55\cdot \text{a, b})$$

in which C_0 and F_0 are arbitrary constants. It is convenient to the analysis to take the following values for characteristic values α_s and β_n in Eqs. (54 · a~d) :

$$\alpha_s=\frac{\lambda_s}{a}\quad (s=1,\ 2,\ \cdots\cdots),\quad \beta_n=\frac{n\pi}{h}\quad (n=1,\ 2,\ \cdots\cdots),\qquad (56\cdot \text{a, b})$$

in which

$$J_1(\lambda_s)=0.\qquad (57)$$

If we substitute Eqs. (54 · a~d) and (55 · a, b) into Eqs. (40 · a, c), we may obtain components of displacement u_r and u_z. If we substitute those components into Eqs. (3 · a~f), we may obtain components of strain. We may also obtain components of stress σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz} and σ_{zr} from those components of strain and Eqs. (1 · a~f). Boundary conditions of the finite cylinder considered here are as follows :

$$\sigma_{rr}=-p(z),\quad \sigma_{zr}=0\quad \text{on } r=a,\qquad (58\cdot \text{a, b})$$

$$\sigma_{zz}=0,\quad \sigma_{zr}=0\quad \text{on } z=\pm h,\qquad (58\cdot \text{c, d})$$

in which

$$p(z)=\begin{cases} q & [-d<z<d] \\ 0 & [d<|z|] \end{cases}.\qquad (59)$$

If we impose the boundary conditions of Eqs. (58 · a~d) on the components of stress σ_{rr}, σ_{zz} and σ_{zr}, we

Table 1 Elastic constants $c_{\alpha\beta}$ (in units of 10^6 N/cm²).

$c_{\alpha\beta}$	c_{44}	c_{11}	c_{33}	c_{12}	c_{13}
Magnesium	1.64	5.97	6.17	2.62	2.17
Cadmium	1.56	11.0	4.69	4.04	3.83
Isotropy	1.0	3.0	3.0	1.0	1.0

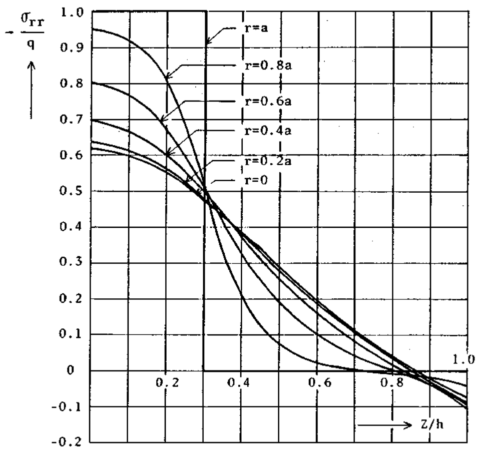

Fig. 2 Stress distribution of σ_{rr}.

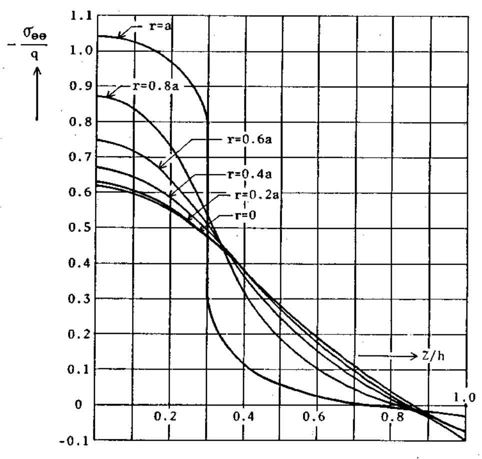

Fig. 3 Stress distribution of $\sigma_{\theta\theta}$.

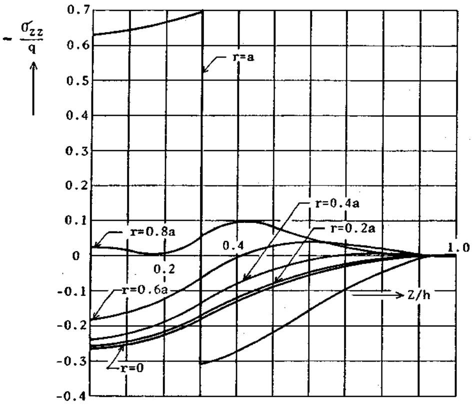

Fig. 4 Stress distribution of σ_{zz}.

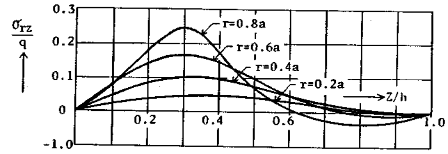

Fig. 5 Stress distribution of σ_{rz}.

may lastly obtain a system of linear algebraic equations with two unknowns, that is, C_s and F_n. That system may be numerically solved by an iterative method. Numerical calculations have been made for the short cylinder with $h/a=1.0$ and $d/h=0.3$, referring to two kinds of transverse isotropy and to isotropy. The elastic constants of magnesium crystal, cadmium crystal and an isotropic material with $\nu=0.25$ are presented in Table 1[8].

As an example of numerical results, stress distributions for magnesium crystal are shown in Fig. 2 to Fig. 5. These results have been obtained from taking the first 38 terms for s and n in a Fourier series.

6. CONCLUDING REMARKS

When quadratic equation (6) gives equal roots, Elliott's solution heretofore in use becomes inapplicable to general boundary-value problems because of the contraction as stated in Chap. 1. From paying attention to it, this paper proposed the generalized solution (40 · a~c) including new terms corresponding to equal roots, as generalization of Elliott's solution. This solution has high validity to transversely isotropic solids and is extensively applicable to non-axially symmetric problems without distinction of finite solids or infinite solids. It was shown in Eqs. (41 · a~c) to Eqs. (46 · a, b) that this solution was reduced to Elliott's solution by placing the restriction of distinct roots. It was also shown in Eqs. (47 · a~c) to Eqs. (52 · a, b) that this solution was exactly coincident with the solution in the previous paper, when this solution was specialized to isotropic solids. It confirms that the author's notion stated in Chap. 1 is satisfied by this solution. This solution was derived in dividing it into two forms for brevity's sake. The derivation of the solution of the first form may, however, yield simultaneously the solution of the second form, if complemental function ζ_c of Eq. (14) is also taken. The axially symmetric problem of the finite cylinder was analyzed in order to show that this solution is applicable to boundary-value problems of finite solids without any difficulty. In case of applications of Elliott's solution to isotropic solids, the means of replacing isotropy with approximate isotropy has been used so far. This solution does, however, not need this means and is applicable to exactly isotropic solids. In consideration of the premises, the author may conclude that the solution presented in this paper is fully available for three-dimensional elasticity problems of transversely isotropic solids.

REFERENCES

1) Love, A. E. H. : A Treatise on the Mathematical Theory of Elasticity, 4 th ed., Chap. 6, Dover Publications, 1944.
2) Lekhnitskii, S. G. : Theory of Elasticity of an Anisotropic Body, Chaps. 3 and 9, Mir Publishers, 1981.
3) Michell, J. H. : The stress in an aeolotropic elastic solid with an infinite plane boundary, Proc. of the London Mathematical Society, Vol. 32, pp. 247~258, 1900.
4) Elliott, H. A. : Three-dimensional stress distributions in hexagonal aeolotropic crystals, Proc. of the Cambridge Philosophical

Society, Vol. 44, pp. 522~533, 1948.

5) Lodge, A. S. : The transformation to isotropic form of the equilibrium equations for a class of anisotropic elastic solids, Quart. J. of Mech. and Appl. Math., Vol. 8, pp. 211~225, 1955.

6) Hata, K. : Some remarks on the three-dimensional problems concerned with the isotropic and anisotropic elastic solids, Mem. of Faculty of Eng., Hokkaido University, Vol. 10, No. 2, pp. 129~177, 1956.

7) Levine, H. S. and Klosner, J. M. : Transversally isotropic cylinders under band loads, Proc. of ASCE, Vol. 93, No. EM 3, pp. 157~174, 1967.

8) Atsumi, A. and Itou, S. : Stresses in a transversely isotropic slab having a spherical cavity, J. of Appl. Mech., Vol. 40, No. 3, pp. 752~758, 1973.

9) Atsumi, A. and Itou, S. : Stresses in a transversely isotropic circular cylinder having a spherical cavity, J. of Appl. Mech., Vol. 41, No. 2, pp. 507~511, 1974.

10) Kasano, H., Tsuchiya, H., Matsumoto, H. and Nakahara, I. : A transversely isotropic, hollow cylinder subjected to a line load along the circumferential direction, Trans. of JSME, Series A, Vol. 47, No. 418, pp. 635~641, 1981 (in Japanese).

11) Mirsky, I. : Wave propagation in transversely isotropic circular cylinders, Part I, J. of the Acoust. Soc. of America, Vol. 37, No. 6, pp. 1016~1021, 1965.

12) Chen, W. T. : On some problems in transversely isotropic elastic materials, J. of Appl. Mech., Vol. 33, No. 2, pp. 347~355, 1966.

13) Okumura, I. A. and Onaka, T. : An expression for solutions to three-dimensional elasticity problems in cylindrical and spherical coordinates, Proc. of JSCE, No. 374/I-6, Structural Engineering/Earthquake Engineering, Vol. 3, No. 2, pp. 185~194, 1986.

(Received April 17 1987)