Structural Eng. / Earthquake Eng. Vol. 4. No. 1, 11s-17s. April 1987
Japan Society of Civil Engineers (Proc. of JSCE No. 380/I-7)

ON TRACING BIFURCATION EQUILIBRIUM PATHS OF GEOMETRICALLY NONLINEAR STRUCTURES

By Wibisono HARTONO, Fumio NISHINO**, Okitsugu FUJIWARA*** and Pisidhi KARASUDHI*****

This paper presents a method to trace the bifurcation equilibrium paths utilizing the Newton-Raphson method. The initial guess solutions on the bifurcation paths are obtained by considering the higher order terms of the Taylor series expansion of the incremental equations at the critical points.
Keywords: *bifurcation, nonlinear structure*

1. INTRODUCTION

The geometrically nonlinear behavior of structures has received considerable attention in the last few decades[1)~5)]. When a structure is discretized with n degrees of freedom in displacement or position, the nonlinear response of a structure under a given loading path is presented by adding loading parameter as additional component, which may represent loading intensity or loading history, as a set of equilibrium paths in $(n+1)$ dimensional space. The paths consist of a main path which starts from the stress free state and a number of bifurcation paths which intersect the main path at the critical points. Other paths which are isolated from the main path may also exist[6)]. The existence of the critical points on the equilibrium paths can be detected by the singularity of the tangential stiffness matrix[2)~4)].

A large number of works are already reported to trace nonlinear main equilibrium paths. On the other hand, tracing bifurcation paths received comparatively little attention in the literature on relatively simple structures such as symmetric shallow arches, simple space trusses and frames under symmetric loading[4),5)]. Most of the method to trace bifurcation paths as well as to study the behavior at the critical points and the stability of equilibrium state utilized the perturbation method on the potential energy function[2),3),5)]. Nishino et al.[4)] presented a method in which singular linear incremental equations at the critical points are solved, of which solution consists of a particular solution and a homogenous solution with unknown constants which have to be determined on trial basis.

* M. Eng., Graduate Student, Department of Civil Engineering, University of Tokyo, Tokyo, 113 Japan. Formerly, Graduate Student at the Asian Institute of Technology, Bangkok, Thailand.
** Member of JSCE, Ph. D., Professor, Department of Civil Engineering, University of Tokyo (Tokyo 113). Formerly, Professor and Vice President for Academic Affairs, Asian Institute of Technology, Bangkok, Thailand.
*** Ph. D., Associate Professor of Industrial Engineering and Management, AIT, Bangkok, Thailand.
**** Ph. D., Professor of Structural Engineering and Construction, AIT, Bangkok, Thailand.

This study presents a method to determine the directions of the bifurcation paths to be used as the initial guess solutions for the Newton-Raphson method utilizing the Taylor expansion of the governing simultaneous nonlinear equations.

2. METHOD OF ANALYSIS

(1) Governing Equations

Employing Cartesian coordinate for simplicity, the nonlinear discretized governing equations of a structural problem can generally be expressed in the following system of n simultaneous nonlinear equations

$$K_i(x)-F_i=0 \qquad (i=1,2,\cdots,n) \cdots\cdots (1)$$

where $F_i=i$-th component of the external force vector; $x=\langle x_1 \ x_2 \ \cdots \ x_n\rangle^T$ which is position vector; $K_i=i$-th component of internal force vector corresponding to the external force F_i; and $n=$ number of degrees of freedom of the system.

The external force F_i is expressed in a general loading case as

$$F_i=\int_0^f f_i(t)dt \qquad (i=1,2,\cdots,n) \cdots\cdots (2)$$

where f is a parameter to specify the loading path and $f_i(f)=i$-th component of loading pattern vector. The loading pattern f_i is considered in this study as a given quantity as part of the problem statement. To simplify the formulation of the problem without losing generality, the external force F_i can be idealized as a piecewise linear function of f. With this idealization, $f_i(f)$ becomes constant at each piecewise linear part whereas parameter f could represent the loading intensity or loading history. In the following development of this paper, the parameter f is used as to represent loading intensity. Since Eq. 1 represents a system of n nonlinear simultaneous equations with $(n+1)$ unknowns in terms of x and f, an additional condition must be introduced for a solution to be found. The so-called chord length control from a known point p is expressed as[4]

$$\sum_{i=1}^{n}(\alpha_i)^2(x_i-x_i^p)^2+(\alpha_f)^2(f-f^p)^2-R^2=\sum_{i=1}^{n}(\alpha_i)^2(\Delta x_i)^2+(\alpha_f)^2(\Delta f)^2-R^2=0 \cdots\cdots (3)$$

where α_i and α_f=constants, equating the dimensions of x_i and f preferably with the dimension such that Eq. 3 becomes non-dimensionalized; x_i^p, f^p=known solution of Eq. 1 at p; Δx_i and Δf=increment of x_i and f from the known solution at p; and R=chord radius along the equilibrium path from the known point p which is a given value. Equations 1 and 3 form the governing equations for the equilibrium paths in the $(n+1)$ dimensional space.

(2) Tracing Bifurcation Paths

Equilibrium equations at a critical point denoted with superscript c as (x^c, f^c) and at a nearby point denoted by $(x^c+\Delta x, f^c+\Delta f)$, selecting both points inside a piecewise linear portion of F_i can be written, respectively as

$$\int_0^{f^c} f_i df=K_i(x^c), \quad \int_0^{f^c} f_i df+f_i^c\Delta f=K_i(x^c+\Delta x) \cdots\cdots (4\cdot a, b)$$

Expanding the right hand side of Eq. 4·b into the Taylor series and taking the difference from Eq. 4·a result in the incremental equation as

$$f_i^c\Delta f=K_{i,j}^c\Delta x_j+K_{i,jk}^c\Delta x_j\Delta x_k/2!+K_{i,jkl}^c\Delta x_j\Delta x_k\Delta x_l/3!+\ldots\ldots \cdots\cdots (5)$$

where $(\ )_{,j}=\partial(\ )/\partial x_j$ and summation convention is used for repeated indices in subscripts unless stated otherwise. Introducing orthogonal transformation which diagonalizes $K_{i,j}^c$, Eq. 5 can take the following form

$$g_i^c\Delta f=D_{i,j}^c\Delta y_j+D_{i,jk}^c\Delta y_j\Delta y_k/2!+D_{i,jkl}^c\Delta y_j\Delta y_k\Delta y_l/3!+\ldots\ldots \cdots\cdots (6)$$

where $D_{i,j}^c=0$ for $i\neq j$; and g_i^c=transformed loading pattern vector in the coordinate y_i which is called as principal coordinate in this study. Considering small increments, the second and higher order terms may be neglected in Eq. 6 and hence

$g_i^c \Delta f = D_{i,j}^c \Delta y_i$ (i not summed) ·· (7)

where $D_{i,j}^c$ = i-th diagonal term of the diagonal matrix $D_{i,j}^c$ which also represents the eigenvalue of $D_{i,j}$ and hence of $K_{i,j}$.

At a critical point c, at least one of $D_{i,i}^c$'s is equal to zero. When only one eigenvalue is zero, it is referred to single critical point whereas zero eigenvalue with multiplicity of more than one is referred to coincidental critical point[2),3)]. At a coincidental critical point with s number of zero eigenvalues, the second order terms of Eq. 6 corresponding to s equations of Eq. 7, where the eigenvalues are zero, must be considered in order to get definite value of Δy_i. Assuming they correspond to the first s equations of Eq. 7, the first s incremental equations at a critical point become

$g_i^c \Delta f = D_{i,jk}^c \Delta y_j \Delta y_k / 2!$ $(i=1,2,\cdots,s,\ j,k=1,2,\cdots,n)$ ································· (8)

A set of s second order nonlinear and $(n-s)$ linear equations as given in Eqs. 8 and 7 is obtained with $(n+1)$ unknowns in terms of Δy_i and Δf. Solving these equations together with Eq. 3 written in the same form in the principal coordinate under a given chord radius, and transforming back into the original coordinates, the incremental vectors Δx_i and Δf can be determined, provided that the Jacobian of the system of the equations does not vanish. If the Jacobian is zero with or without vanishing all terms of $D_{i,jk}$, the next higher order terms in the Taylor series expansion have to be considered until the Jacobian has non-zero value. A critical point can be either a stationary, a bifurcation point or a combinatoin of both. At a bifurcation point, solving those equations will yield at least four real roots for solutions on the main and a bifurcation paths, two solutions on each path due to the nature of chord length control. No bifurcation paths emanating from a coincidental critical point are also possible[7),8)]. More than four real roots indicate more than one bifurcation paths at the critical point.

Selecting the increment vectors Δx_i and Δf for a solution on a bifurcation path and adding it to the known solution at the critical point c, the initial value for tracing a bifurcation path can be obtained and the path can be traced by simple and effective method such as the Newton-Raphson method.

In practice, it is difficult to get the precise location of a critical point, and hence this numerical analysis has to be made at a point close to a critical point c. The selection of chord radius R needs attention. The smaller value of R is preferred as it leads to more accurate values of the incremental vectors Δx_i and Δf due to reduced error in neglecting higher order terms of the Taylor series expansion and this will lead to a faster convergence of the iteration procedure. On the other hand it has to be much larger than the distance from the true critical point to the point where the solution closest to the critical point is obtained and treated as the critical point.

3. NUMERICAL EXAMPLES

(1) Governing Equations of Elastic Trusses

Some of the theoretical developments are demonstrated in numerical examples on elastic trusses, because the exact governing equations can be easily derived. Since a complex loading sequence can be idealized without losing any generality as a piecewise linear function of f in the small interval including a critical point within the interval, only proportional loading is considered. Fig. 1 shows displaced equilibrium state of a bar element pq with the initial undeformed length l. By considering the equilibrium in the direction of the base vectors i_1, i_2 and i_3 at both end nodes p and q of the element, the equilibrium equations are expressed as

$$f\begin{Bmatrix} f_i^p \\ f_i^q \end{Bmatrix} = \frac{N}{\hat{l}}\begin{Bmatrix} x_i^p - x_i^q \\ x_i^q - x_i^p \end{Bmatrix} \qquad (i=1 \text{ to } nd) \cdots\cdots (9)$$

where N=internal axial force in the bar, positive for tension; x^p=position vector of node p; nd=number of dimension (2 or 3 for the plane or space truss); and $\hat{l}$=the

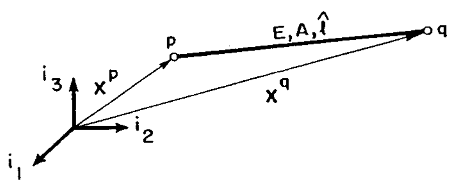

Fig. 1 A Bar in Equilibrium at Displaced State.

deformed length, which is expressed as

$$\hat{l}=|x^q-x^p| \quad (10)$$

Assuming linear elastic material, the stress-strain relationship can be expressed as

$$N=EA\varepsilon \quad (11)$$

where E=modulus of Elasticity; A=cross-sectional area of bar element; and ε=strain of bar. The strain-position relation is expressed using the deformed length $\hat{l}$ as a parameter as

$$\varepsilon=(\hat{l}-l)/l \quad (12)$$

Substituting Eqs. 12 and 11 into Eq. 9, the governing equations of the element are obtained in terms of the position vector as the main unknown as

$$f\begin{Bmatrix} f_i^p \\ f_i^q \end{Bmatrix}=EA\left(\frac{1}{l}-\frac{1}{\hat{l}}\right)\begin{Bmatrix} x_i^p-x_i^q \\ x_i^q-x_i^p \end{Bmatrix}=\begin{Bmatrix} K_i \\ K_{i+nd} \end{Bmatrix} \qquad (i=1 \text{ to } nd\,;\, K_i=-K_{i+nd}) \quad (13)$$

The first and second order derivatives of the member's governing equation, Eq. 13, are given in the Appendix.

(2) Two Bar Plane Truss

As the simplest example, a two bar plane truss with two degrees of freedom and the initial configuration as shown in Fig. 2 is investigated under a vertical force at node 1, $\langle g_1 \ \ g_2\rangle^T=\langle 0 \ \ 1\rangle^T$. The superscript to the components of forces and coordinates indicates node. The same example is solved analytically and reported in Ref. 8 utilizing the Green's strain tensor instead of Eq. 12. Britvec[9] and Kondoh and Atluri[10] also used this example but considering the bifurcation due the the flexural buckling which is outside the scope of this numerical study. Computing the first order term of the Taylor series expansion yields a diagonal matrix, the elements of which are

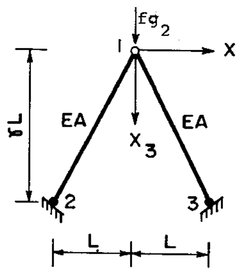

Fig. 2 Two Bar Plane Truss.

$$D_{1,1}=2\,EA(1/\eta-1/\xi+1/\xi^3)/L, \quad D_{2,2}=2\,EA[(1/\eta-1/\xi+(\gamma-\overline{x}_3^1)^3/\xi^3]/L \quad (14)$$

where $\overline{x}_3^1=x_3^1/L$; x_3^1=displaced position of node 1 in i_3 direction; γ=ratio between height and projected length L; and η and ξ are defined, respectively, as

$$\eta=\frac{L}{l}, \qquad \xi=\frac{L}{\hat{l}} \quad (15\cdot\text{a, b})$$

which are expressed as

$$\eta=\sqrt{(1+\gamma^2)}, \qquad \xi=\sqrt{[1+(\gamma-\overline{x}_3^1)^2]} \quad (16\cdot\text{a, b})$$

The main equilibrium path for $\gamma=2.5$ is shown in Fig. 3 by the solid line, where the ordinate and abscissa are non-dimensionalized intensity of the force and vertical position of node 1, respectively. Also shown by the dashed lines are the changes of the nondimensionalized magnitudes of two eigenvalues $D_{1,1}$ and $D_{2,2}$ of which zero values detect the location of critical points. Six critical points are found and marked as points A to F and all of them are single critical points. The figure shows that the critical points A and B, and also E and F are bifurcation points and the critical points C and D are stationary points. This classification can be easily made analytically[7]. At the single critical points A and B, the incremental equations, Eqs. 7 and 8 together with Eq. 3, take the following form

$$D_{1,12}^c\Delta x_1^1\Delta x_3^1=0, \quad D_{2,2}^c\Delta x_3^1=\Delta f, \quad (\alpha_1)^2(\Delta x_1^1)^2+(\alpha_2)^2(\Delta x_3^1)^2+(\alpha_f)^2(\Delta f)^2=R^2 \quad (17\cdot\text{a, b, c})$$

where

$$D_{1,12}^c=2\,EA(\gamma-\overline{x}_3^1)(-1/\xi^3+3/\xi^5)/L^2 \quad (18)$$

Selecting $\alpha_1=\alpha_2=1/L$ and $\alpha_f=1/EA$, the initial values for the first iterative solutions on bifurcation paths can be computed from Eq. 17 for a given R. The first equilibrium points were obtained by solving Eqs. 1 and 3 by the Newton-Raphson method with the initial values so obtained. Then the bifurcation equilibrium paths were extended in the same way as for the main equilibrium path starting from these first points on the bifurcation paths following the established standard procedure by using the Newton-Raphson

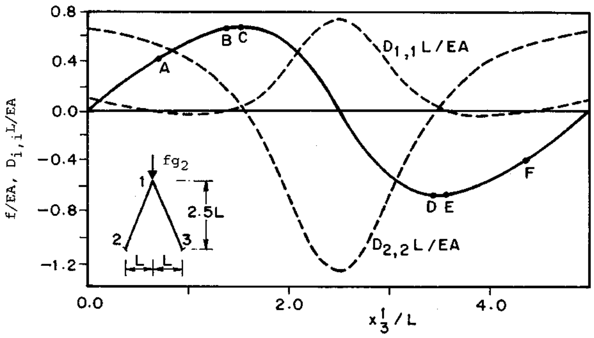

Fig. 3 Main Equilibrium Path and Eigenvalue Curves of Example 1.

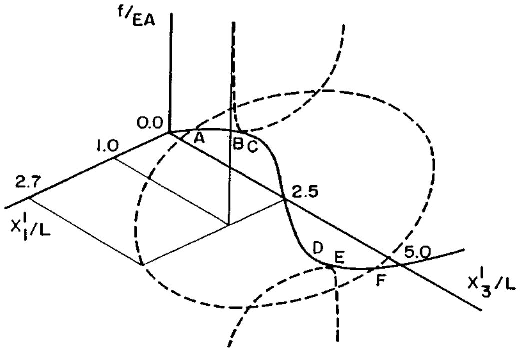

Fig. 4 Main and Bifurcation Paths of Example 1.

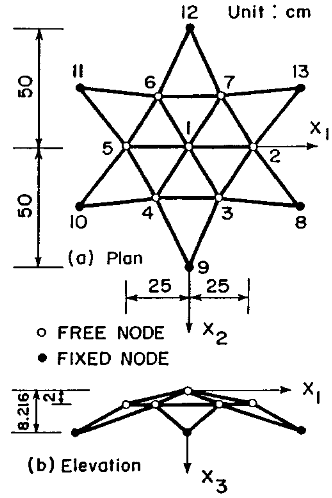

Fig. 5 Reticulated Space Truss.

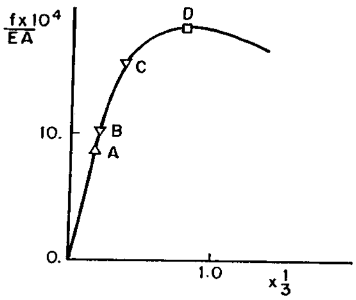

Fig. 6 Part of Main Equilibrium Path with Critical Points of Example 2.

method. The bifurcation paths so obtained are shown by dashed lines in Fig. 4. Employing the Green's strain tensor instead of Eq. 12, Pecknold et al.[8] did not obtain the bifurcation points B and E, and the bifurcation paths from points A and F are obtained by solving the governing equations directly.

The critical points A and B and also E and F coincide at $\gamma=\sqrt{5.75}$, and critical points B and C and also D and E coincide at $\gamma=\sqrt{7}$. The former case needs special attention since $D^c_{1,12}$ vanishes and the next third order terms in the first incremental equation must be considered to obtain definite values of Δx_i which yields

$$D^c_{1,111}(\Delta x^1_1)^3/6+D^c_{1,122}(\Delta x^1_1)(\Delta x^1_3)^2/2=0 \quad (19)$$

where

$$D^c_{1,111}=6\,EA(1/\xi^3-6/\xi^5+5/\xi^7)/L^3 \quad (20\cdot a)$$

$$D^c_{1,122}=2\,EA\,[1/\xi^3-3/\xi^5-3(\gamma-\bar{x}^1_3)^2/\xi^5+15(\gamma-\bar{x}^1_3)^2/\xi^7]/L^3 \quad (20\cdot b)$$

Equation 19 together with Eqs. 17·b and c yields solutions on the main and two bifurcation paths. The number of bifurcation paths in this case exceeds the maximum number given in Ref. 2 and 3 but agrees with the one given in Ref. 7. Since both eigenvalues are zero in the latter case, the two incremental equations become nonlinear as

$$D^c_{1,12}\Delta x^1_1\Delta x^1_3=0,\quad D^c_{2,11}(\Delta x^1_1)^2/2+D^c_{2,22}(\Delta x^1_3)^2/2=\Delta f \quad (21)$$

where

$$D^c_{2,22}=6\,EA\,(\gamma-\bar{x}^1_3)[-1/\xi^3+(\gamma-\bar{x}^1_3)^2/\xi^5]/L^2 \quad (22)$$

Solving Eqs. 21 and 17·c yields initial guess solutions on the main and one bifurcation paths.

Table 1 Increment Vectors ($\times 10^{-4}$) at Bifurcation Point C of Example 2 for $R=0.001$.

No	1	2	3	4	5	6	7	8	9	10	11	12
Δx_1^1	-0.11	0.11	-0.06	0.06	0.06	-0.06	0.00	0.00	-0.10	0.10	-0.10	0.10
Δx_2^1	0.00	0.00	-0.10	0.10	-0.10	0.10	0.11	-0.11	0.06	-0.06	-0.06	-0.06
Δx_3^1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Δx_1^2	-0.82	0.82	-0.41	0.41	0.41	-0.41	0.00	0.00	-0.71	0.71	-0.71	0.71
Δx_2^2	0.00	0.00	-0.30	0.30	-0.30	0.30	0.35	-0.35	0.18	-0.18	-0.18	0.18
Δx_3^2	5.70	-5.70	2.85	-2.85	-2.85	2.85	0.00	0.00	4.94	-4.94	4.94	-4.94
Δx_1^3	-0.47	0.47	-0.41	0.41	0.06	-0.06	0.20	-0.20	-0.30	0.30	-0.51	0.51
Δx_2^3	-0.20	0.20	-0.71	0.71	-0.51	0.51	0.70	-0.70	0.18	-0.18	-0.53	0.53
Δx_3^3	2.85	-2.85	5.70	-5.70	2.85	-2.85	-4.94	4.94	0.00	0.00	4.94	-4.94
Δx_1^4	-0.47	0.47	-0.06	0.06	0.41	-0.41	-0.20	0.20	-0.51	0.51	-0.30	0.30
Δx_2^4	0.20	-0.20	-0.51	0.51	-0.72	0.72	0.70	-0.70	0.53	-0.53	-0.18	0.18
Δx_3^4	-2.85	2.85	2.85	-2.85	5.70	-5.70	-4.94	4.94	-4.94	4.94	0.00	0.00
Δx_1^5	-0.82	0.82	-0.41	0.41	0.41	-0.41	0.00	0.00	-0.71	0.71	-0.71	0.71
Δx_2^5	0.00	0.00	-0.30	0.30	-0.30	0.30	0.35	-0.35	0.18	-0.18	-0.18	0.18
Δx_3^5	-5.70	5.70	-2.85	2.85	2.85	-2.85	0.00	0.00	-4.94	4.94	-4.94	4.94
Δx_1^6	-0.47	0.47	-0.41	0.41	0.06	-0.06	0.20	-0.20	-0.30	0.30	-0.51	0.51
Δx_2^6	-0.20	0.20	-0.71	0.71	-0.51	0.51	0.70	-0.70	0.18	-0.18	-0.53	0.53
Δx_3^6	-2.85	2.85	-5.70	5.70	-2.85	2.85	4.94	-4.94	0.00	0.00	-4.94	4.94
Δx_1^7	-0.47	0.47	-0.06	0.06	0.41	-0.41	-0.20	0.20	-0.51	0.51	-0.30	0.30
Δx_2^7	0.20	-0.20	-0.51	0.51	-0.71	0.71	0.70	-0.70	0.53	-0.53	-0.18	0.18
Δx_3^7	2.85	-2.85	-2.85	2.85	-5.70	5.70	4.94	-4.94	4.94	-4.94	0.00	0.00
Δf	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

$EA = 10^4$

(3) Reticulated Truss

A reticulated truss with the initial configuration as shown in Fig. 5[4] is analyzed as a numerical example with larger degrees of freedom and hence with more complex behavior. The bifurcation paths have been traced in Ref. 4, however, the initial guess solutions were obtained on trial basis of the solution of linear singular incremental equations, Eq. 7, at the critical points. The truss is subjected to a vertical loading at each node of the same intensity, except at node 1 where reduced intensity of one half is applied. Part of the main path is shown in Fig. 6[4] where the abscissa and ordinate are the vertical position of node 1 with the unit of cm and a non-dimensionalized loading intensity f. The critical points detected by zero values of eigenvalue are marked as △, ▽ and □ which indicate single bifurcation, double bifurcation and stationary points, respectively. Employing Eqs. 3, 7 and 8 at those bifurcation points, the initial guess of the increments of positions can be computed. Values of the initial guess of the increments of positions at the bifurcation point C are tabulated in Table 1 as an example of the solution. One, three and six bifurcation paths are found to emanate from the critical points A to C and this agrees with Ref. 4. The number of bifurcation paths at the point C exceeds the maximun number given in Refs. 2 and 3 which is given as $2^2-1=3$ but agree with the one given in Ref. 7. By extending the bifurcation paths, it becomes obvious that at the critical points A and C symmetric bifurcation paths emanate whereas at the point B an asymmetric one emanates. This classification can also be determined analytically[7].

4. SUMMARY AND CONCLUDING RIMARKS

There are only few works on tracing bifurcation equilibrium paths from critical points. Most of the them are based on the potential energy theorem and utilize the perturbation technique.

The purpose of this paper is to establish a simple technique to trace the bifurcation equilibrium paths from critical points. It is a well known fact that the Newton-Raphson method is one of the most powerful techniques to solve nonlinear equations in which estimation of the initial guess solution is the most important factor.

This paper presents a method to obtain initial guess vectors for the bifurcation paths from critical points which can be utilized to obtain solutions on the paths with the Newton-Raphson method. The Taylor expansion of the nonlinear governing equations has been used to obtain the incremental relation between the loading intensity and positions at critical points. The relation is solved to get initial guess solutions. The procedure has been applied successfully for the numerical examples demonstrated in this paper, i. e. solutions on bifurcation paths are obtained without any difficulty. Once a point is obtained on the path, the extension of the path can be done in the same way as tracing the main path.

The numerical examples presented are on truss structures, of which elements of the second order terms are relatively easy to formulate. Other structural elements may need complicated analytical formulation. In this case, numerical differentiation to compute those elements may be necessary.

5. ACKNOWLEDGEMENTS

This paper is based on the master's thesis of the first author at the Asian Institute of Technology, Bangkok, Thailand who received the scholarship donated by The Royal Netherland Government. The work is also financed by the Japan International Cooperation Agency.

Appendix——The First and Second Derivatives of The Governing Equation of Elastic Trusses

$$K_{i,j}=\frac{EA}{\hat{l}^3}(x_i^q-x_i^p)(x_j^q-x_j^p)+EA(1/l-1/\hat{l})\delta_{ij}=-K_{(i+nd),j}=-K_{i,(j+nd)}=K_{(i+nd),(j+nd)} \quad \cdots\cdots (23)$$

$$K_{i,jk}=\frac{3\,EA}{\hat{l}^5}(x_i^q-x_i^p)(x_j^q-x_j^p)(x_k^q-x_k^p)+\frac{EA}{\hat{l}^3}[\delta_{jk}(x_i^q-x_i^p)+\delta_{ik}(x_j^q-x_j^p)+\delta_{ij}(x_k^q-x_k^p)] \quad \cdots\cdots (24)$$

$$=-K_{(i+nd),jk}=K_{(i+nd),(j+nd)k}=K_{(i+nd),j(k+nd)}=-K_{(i+nd),(j+nd)(k+nd)}=-K_{i,(j+nd)k}=-K_{i,j(k+nd)}$$
$$=K_{i,(j+nd)(k+nd)} \quad \cdots\cdots (25)$$

where δ_{ij}=Kronecker delta ; and i, j and $k=1$ to nd.

REFERENCES

1) Koiter, W. T. : On the Stability of Elastic Equilibrium, Dissertation, Delft, Holland, 1945.
2) Huseyin, K. : Nonlinear Theory of Elastic Stability, Noordhoff International Publishing, Leyden, 1975.
3) Thompson, J. M. T. and Hunt, G. W. : A General Theory of Elastic Stability, John Wiley and Sons, New York, 1973.
4) Nishino, F., Ikeda, K., Sakurai, T. and Hasegawa, A. : A Total Lagrangian Nonlinear Analysis of Elastic Trusses, Proc. of JSCE, Struct. Eng. /Earthquake Eng. Vol. 1, No. 1, pp. 1s~15s, April 1984.
5) Riks, E. : An Incremental Approach to the Solution of Snapping and Buckling Problems, International Journal of Solids and Structures, Vol. 15, No. 7, pp. 529~551, July 1979.
6) Chaisomphob, T., Nishino, F., Hasegawa, A. and Abdel-Shafy, A. G. A. : An Elastic Finite Displacement Analysis of Inplane Beams with and without Shear Deformation, Proc. of JSCE, Struct. Eng. /Earthquake Eng. Vol. 3, No. 1, pp. 157s~165s, April 1986.
7) Nishino, F., Hartono, W., Fujiwara, O. and Karasudhi, P. : Study of The Stability and Behavior at the Critical Point by the Taylor Expansion, Proc. of JSCE, Struct. Eng. /Earthquake Eng. Vol. 4, No. 1, pp. 1 s~9 s, April 1987.
8) Pecknold, D. A., Ghaboussi, J. and Healey, T. J. : Snap Through and Bifurcation in a Simple Structure, Journal of Engineering Mechanics, ASCE, Vol. 111, No. 7, Proc. Paper 19834, pp. 909~922, July 1985.
9) Britvec, S. J. : The Stability of Elastic Systems, Pergamon Press, New York, 1973.
10) Kondoh, K. and Atluri, S. N. : Influence of Local Buckling on Global Instability : Simplified, Large Deformation, Post Buckling Analyses of Plane Trusses, Computers and Structures, Vol. 21, No. 4, pp. 613~627, April 1985.

(Received May 10 1986)