投稿論文
Paper

Structural Eng. / Earthquake Eng. Vol. 4. No. 1, 1 s-9 s. April 1987
Japan Society of Civil Engineers (Proc. of JSCE No. 380/I-7)

A STUDY OF THE STABILITY AND BEHAVIOR AT THE CRITICAL POINT BY THE TAYLOR EXPANSION

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A Study on the stability and the behavior at the critical points of nonlinear discrete structural system is presented utilizing the Taylor expansion of the governing algebraic equations. The results are compared with the previous works on the same subject. This method is found to be much simpler to explain the subject than the perturbation method together with the energy function which is the technique mostly used in literature.

Three examples on plane and space elastic truss structures are presented to illustrate the analytical results.

Keywords : stability, bifurcation, nonlinear structural analysis

1. INTRODUCTION

Many structures may exhibit more than one equilibrium configuration under a given external loading. Such structures can undergo a remarkable change in the deformation that is not associated with a failure of the material but rather represents a loss of stability of the original equilibrium configuration. This behavior can only be explained by nonlinear theory of elastic stability.

The study of the general elastic stability as well as the characteristics of bifurcation points of a continuous system was presented by Koiter[1]. Thompson et al.[2] and Huseyin[3] presented the results of the similar study of a discrete nonlinear system. All of them utilized the so-called perturbation method together with the energy function. They employed as the condition of the stability of an equilibrium state an axiom that a complete relative minimum of the total potential energy with respect to the generalized coordinates is necessary and sufficient for a stable state. It is worth to note that this criterion for the stability has only been proved on single degree of freedom system[2),4)]. Britvec[5] analyzed the same subjects by utilizing the same energy function but employing Taylor expansion directly without going through perturbation technique as Refs. 2 and 3. In his analysis the derivatives with respect to a loading parameter are present in addition to the derivatives with respect to the components of displacement vector as in Refs. 2 and 3. The same method of Taylor expansion is used to study the behavior at the critical points[5]. Recently, Nishino et al.[6] studied the behavior at critical points by examining the solution of linear singular

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incremental equations at those points. Similar work is also presented by Pecknold et al.[7] on single critical points, both of them gave numerical examples on truss structures.

This paper presents a study on the stability of equilibrium state as well as the behavior at the critical points of nonlinear static discrete structural system utilizing Taylor expansion of the governing algebraic equations. The use of this governing equation contributes to delete derivatives with respect to loading parameter which were present in the analyses based on the energy function[2),3),5)] and hence to simplify the theoretical developments.

In this study any general loading system is presented without losing any generality by a sequence of piecewise linearly varying loading within a short interval[8)].

2. STABILITY OF EQUILIBRIUM STATE

The governing equilibrium equations of a discrete structural system under static and conservative loading can be written employing Cartesian coordinates for simplicity as

$$F_i=K_i(\boldsymbol{x}) \qquad (i=1 \text{ to } n) \tag{1}$$

where F_i= i-th component of the external force vector ; $\boldsymbol{x}=\langle x_1 \ x_2 \ \cdots \ x_n\rangle^T$=displacements or position vector ; K_i= i-th component of internal force vector ; and n=number of degrees of freedom of the system.

The external force F_i is expressed in a general loading case as[8)]

$$F_i=\int_0^f f_i(t)dt \qquad (i=1 \text{ to } n) \tag{2}$$

where f is a parameter to specify the loading path ; and $f_i(f)$= i-th component of the loading pattern vector. The loading pattern f_i is considered in this study as a given quantity as part of the problem statement. To simplify the formulation of the problem without losing generality, the external force F_i can be idealized as a piecewise linear function of f. With this idealization, $f_i(f)$ becomes constant at each piecewise linear part whereas the parameter f could represent the loading intensity or loading history. In this paper, f is defined as to represent loading intensity.

The principal coordinate $\boldsymbol{y}=<y_1 \ y_2 \ \cdots \ y_n>^T$ at a point p on an equilibrium path is defined as the coordinate obtained by orthogonal transformation of x which diagonalizes $K_{i,j}(\boldsymbol{x}^p)$, the first derivative of K_i with respect to x_j at the point p[8)].

Consider a point p and a nearby point, $(f^p, \boldsymbol{y}^p)$ and $(f^p+\Delta f, \boldsymbol{y}^p+\Delta\boldsymbol{y})$, respectively, on a piecewise linear part of the equilibrium path. Expressing the external force vector in the principal coordinate $\boldsymbol{y}$ as $G_i(f)$, the governing equations at both points can be written respectively as

$$G_i^p=D_i(\boldsymbol{y}^p), \quad G_i^p+\Delta G_i=D_i(\boldsymbol{y}^p+\Delta\boldsymbol{y}) \tag{3·a, b}$$

Equation 3·b can be rewritten as

$$-\Delta G_i=G_i^p-D(\boldsymbol{y}^p+\Delta\boldsymbol{y}) \tag{4}$$

If ΔG_i is considered as a sole infinitesimally small disturbing force in the i-th direction of the principal coordinate, exerting this disturbing force into an equilibrium position $(\boldsymbol{y}^p, f^p)$ results in a change of position of $\Delta\boldsymbol{y}$. If this force is removed, the left hand side of Eq. 4 indicates the unbalanced force at the position $\boldsymbol{y}^p+\Delta\boldsymbol{y}$ under the force G_i^p. Because of this unbalanced force, the system starts to move. If the direction of the i-th component of $\Delta\boldsymbol{y}$ and the unbalanced force $-\Delta G_i$ present at the disturbed position is not the same, i. e. $(\Delta y_i)(-\Delta G_i)<0$ or rewriting

$$\Delta y_i \Delta G_i>0 \qquad (i \text{ not summed}) \tag{5}$$

the system will start to move back and resume its original equilibrium position due to damping.

In the following presentation, the summation convention is employed for all subscript indices unless otherwise stated. Summation signs, however, are used at some equations to emphasize summation. Expanding the right hand side of Eq. 3·b into the Taylor series and taking the difference with Eq. 3·a result in

$$\Delta G_i=D_{i,j}^p\Delta y_j+D_{i,jk}^p\Delta y_j\Delta y_k/2!+D_{i,jkl}^p\Delta y_j\Delta y_k\Delta y_l/3!+\cdots \tag{6}$$

where $(\ )_{,j}=\partial(\ )/\partial y_j$. Substituting this result into Eq. 5 leads to the stability condition of a structural system with n degrees of freedom for the disturbance in the i-th direction as

$$D^p_{i,j}\Delta y_i\Delta y_j+D^p_{i,jk}\Delta y_i\Delta y_j\Delta y_k/2!\cdots>0 \qquad \text{(i not summed)} \quad (7)$$

When the first term does not vanish, the higher terms can be neglected due to the smaller quantity compared with the first term. Noting that $D^p_{i,j}$ is a diagonal matrix due to the definition of principal coordinate y_i, the condition for a stable state in the i-th direction is given as

$$D^p_{i,i}(\Delta y_i)^2>0 \qquad (i \text{ not summed}) \quad (8)$$

where $D^p_{i,i}$ is the i-th diagonal element which also represents the eigenvalue of the first order term. For a system to be stable, it has to resume the original position for disturbance in any combinations of principal directions and hence all $D^p_{i,i}$'s have to be positive. The stability condition, then, can be expressed as

$$\sum_{i=1}^{n} D^p_{i,i}(\Delta y_i)^2>0 \quad (9)$$

This expression is known as the quadratic form of $D^p_{i,j}$ which is also of $K^p_{i,j}$, the first order term expressed in the original Cartesian coordinate or mostly known as the tangential stiffness matrix. As all $D^p_{i,i}$'s are positive for a stable state, it indicates that $K^p_{i,j}$ is a positive definite matrix. Hence a system is stable when Eq. 9 holds, i. e. when $K^p_{i,j}$ is positive definite. This stability condition, however, has not been proved except for one degree of freedom system, but accepted as an axiom for multi-degrees of freedom system[2),4)]. When at least one of $D^p_{i,i}$'s is negative while the rest are positive, the structure is unstable and this condition is equivalent to indefiniteness of $K^p_{i,j}$.

If $K^p_{i,j}$ is positive semi-definite, i. e., if there are s zero values of $D^p_{i,i}$ while the other $(n-s)$ are positive, the next order term in Eq. 7 should be considered for the corresponding s equations of Eq. 8. This condition happens at the critical points where the first order term $K^p_{i,j}$ is singular. In the following expressions, the superscript p is replaced with c to indicate that the point is a critical point. In this case it is not proved that the system is stable when Eq. 7 holds with the summation convention being applied for all subscripts but this condition is adopted in this paper as an axiom as adopted in the literature[2),3)] and hence the stability condition when the tangential stiffness matrix is positive semi-definite is expressed as

$$D^c_{i,j}\Delta y_i\Delta y_j+D^c_{i,jk}\Delta y_i\Delta y_j\Delta y_k/2!+D^c_{i,jkl}\Delta y_i\Delta y_j\Delta y_k\Delta y_l/3!+\cdots>0 \quad (10)$$

Assuming the first s eigenvalues are zero, the value of Δy_i which makes the quadratic form of Eq. 9 vanish is given as

$$\Delta y_i=\langle C^1 \quad C^2 \quad \cdots \quad C^s \quad 0 \quad 0 \quad \cdots \quad 0\rangle^T \quad (11)$$

where $C^i(i=1$ to $s)$ are arbitrary constants. Substituting Eq. 11 into Eq. 10 and considering only the second order terms yields the condition of the stable state as

$$\frac{1}{2}\sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{k=1}^{s} D^c_{i,jk}C^iC^jC^k>0 \quad (12)$$

The sense of C^i and hence of $C^iC^jC^k$ take both positive and negative values depending on the sense of disturbing forces. Because of this the structure is in an unstable state unless all $D^c_{i,jk}$ are zero. If all of them are zero, the next higher order terms should be considered, which yields

$$\frac{1}{6}\sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{k=1}^{s}\sum_{l=1}^{s} D^c_{i,jkl}C^iC^jC^kC^l>0 \quad (13)$$

If the $D^c_{i,jkl}$ is positive definite, it gives a sufficient but not necessary condition for a stable state. This is due to the fact that Eq. 13 can be regarded as a quadratic form if C^iC^j and C^kC^l are considered as one variable, but in that case there exist terms such as $(C^i)^2$ which take only positive value. If $D^c_{i,jkl}$ is positive semi-definite, a further higher order terms have to be considered and the same procedure is repeated.

The method presented here gives identical results with the one studied by the perturbation method of energy function[2),3)] up to the cubic order term, i. e. Eqs. 9 and 12. But the condition given in Eq. 13 is different with those reported in Refs. 2 and 3 which includes an additional term given as

$$\sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{k=1}^{s}\sum_{l=1}^{s}\sum_{p=s+1}^{n}[D^c_{i,jkl}-3\,D^c_{p,ij}D^c_{p,kl}/D^c_{p,p}]C^iC^jC^kC^l>0 \quad (14)$$

3. BEHAVIOR AT CRITICAL POINTS

In view of Eq. 6 and replacing p by c to indicate a critical point, the incremental equations considering a linear piecewise part of G_i within a small neighborhood of the point and neglecting higher order terms can be written as

$$g^c_i\Delta f=D^c_{i,i}\Delta y_i \quad (i \text{ not summed}) \quad (15)$$

where g_i= i-th component of the loading pattern vector in the principal coordinate and $g_i\Delta f=\Delta G_i$. At a critical point, one or more eigenvalues are equal to zero, which is referred to a single critical point or a coincidental critical point, respectively[2),3)]. The incremental equations at these critical points with first s zero eigenvalues are, then, expressed as[8)]

$$g^c_i\Delta f=D^c_{i,jk}\Delta y_j\Delta y_k/2! \quad (i=1 \text{ to } s\,;\, j,\,k=1 \text{ to } n) \quad (16\cdot a)$$

$$g^c_i\Delta f=D^c_{i,i}\Delta y_i \quad (i=s+1 \text{ to } n\,;\, i \text{ not summed}) \quad (16\cdot b)$$

(1) Single Critical Point

Assuming the first diagonal element of $D^p_{i,i}$ is zero, and noting that if $g_1\neq 0$, a solution of Eq. 15 exists only when $\Delta f=0$, and it is given as

$$\Delta y_i=C^1\langle 1 \;\; 0 \;\; 0 \;\; \cdots \;\; 0\rangle^T \quad (17)$$

Though the magnitude of C^1 is not determined, the existence of a unique solution shows that the critical point cannot be a bifurcation point, but it is a stationary point since Δf is zero. In order to check the behavior at this stationary point, the incremental equation of Eq. 16·a has to be examined. After dividing $D^c_{1,ij}$ into three parts, Eq. 16·a becomes

$$g^c_1\Delta f=D^c_{1,11}(\Delta y_1)^2/2+D^c_{1,i1}\Delta y_i\Delta y_1+D^c_{1,ij}\Delta y_i\Delta y_j/2 \quad (i,\,j=2 \text{ to } n) \quad (18)$$

In view of Eq. 17, $|\Delta y_1|\gg|\Delta y_i|$ $(i\neq 1)$ at a point very close to this stationary point, the second and third terms of the right hand side can be neglected and Eq. 18 becomes

$$g^c_1\Delta f=D^c_{1,11}(\Delta y_1)^2/2 \quad (19)$$

This indicates that the critical point is either the local maximum or minimum point with respect to $\boldsymbol{y}$, when $D^c_{1,11}/g^c_1$ is negative or positive, respectively. If $D^c_{1,11}$ is zero then the next higher order term should be considered in a similar way which results in

$$g^c_1\Delta f=D^c_{1,111}(\Delta y_1)^3/6 \quad (20)$$

If $D^c_{1,111}$ is not zero, the point is a saddle point with respect to y_1, otherwise the next higher order term should be considered and the same procedure is repeated.

If $g^c_1=0$, the first equation of Eq. 15 does not give any information. A solution has to be obtained through Eq. 16 which takes the following form after substituting the $(n-1)$ linear equation into the first equation

$$0=D^c_{1,11}(\Delta y_1)^2/2+A\Delta y_1\Delta f+B(\Delta f)^2/2 \quad (21)$$

where

$$A=\sum_{i=2}^{n}D^c_{i,11}g^c_i/D^c_{i,i}\,;\text{ and }\; B=\sum_{C=2}^{n}\sum_{j=2}^{n}D^c_{1,ij}g^c_ig^c_j/(D^c_{i,i}D^c_{j,j}) \quad (22\cdot a,\, b)$$

Equation 21 is a quadratic equation in terms of Δy_1 and Δf and hence solving this equation yields two distinct Δf for one Δy_1 provided B is not equal to zero as

$$\Delta f=[-A\pm\sqrt{(A^2-D^c_{1,11}B)}]\Delta y_1/B \quad (23)$$

If the discriminant is positive, this critical point is a bifurcation point where two equilibrium paths intersect.

If $D^c_{1,11}$ is not equal to zero, Eq. 23 yields two distinct non-zero roots of Δf, and hence two equilibrium paths intersect at the critical point with non-zero slopes. The critical point is referred to an asymmetric bifurcation point[2),3)]. Due to non-zero values of $D^c_{1,11}$ this point is unstable.

If $D^c_{1,11}$ is zero, one of the roots will be zero, and hence one of the equilibrium paths intersecting at the

Table 1

$D_{1,1}=0$	$g_1 \neq 0$	$D^c_{1,11} \neq 0$	Limit Point	
		$D^c_{1,11}=0$	$D^c_{1,111} \neq 0$	Saddle Point
			$D^c_{1,111}=0$	The Same Procedure to be Repeated
	$g_1=0$	$D^c_{1,11} \neq 0$	Asymmetric Bifurcation Point	
		$D^c_{1,11}=0$	Symmetric Bifurcation Point when Bifurcation Path being Stationary	

critical point is stationary with respect to $\boldsymbol{y}$. This point is called symmetric bifurcation point, when the stationary path is the bifurcation path. The stability of this point is determined from the sign of $D^c_{1,111}$. When this term does not vanish, this point is stable or unstable, depending on $D^c_{1,111}$ being positive or negative.

This study on a single critical point gives the same results as the those studied by the perturbation method together with energy function[2),3)], which are summarized in Table 1.

(2) Coincidental Critical Point

Assuming the first s diagonal elements of $D^c_{i,i}$'s are zero, Eq. 15 indicates that if at least one of g^c_i ($i=1$ to s) is not equal to zero, then Δf must be equal to zero. This shows that the critical point is a stationary point with respect to $\boldsymbol{y}$ for the main and all the intersecting bifurcation paths if they are present.

When all g^c_i ($i=1$ to s) are zero, the first s equations of Eq. 15 does not give any information, but it has to be obtained through Eq. 16. Consider a special case where $D^c_{i,jk}=0$ for i, j, $k=1$ to s. In this case, substituting the $(n-s)$ linear equations into the first s nonlinear equations yields

$$\sum_{j=1}^{s}\sum_{k=s+1}^{n}(D^c_{i,jk}\Delta y_j g^c_k/D^c_{k,k})\Delta f+\sum_{k=s+1}^{n}\sum_{l=s+1}^{n}D^c_{i,kl}(g^c_k/D^c_{k,k})(g^c_l/D^c_{l,l})(\Delta f)^2=0 \cdots\cdots (24)$$

The solutions of Eq. 24 is given as

$$\Delta f=0, \qquad \sum_{j=1}^{s}\sum_{k=s+1}^{n}(D^c_{i,jk}\Delta y_j g^c_k/D^c_{k,k})+\sum_{k=s+1}^{n}\sum_{l=s+1}^{n}D^c_{i,kl}(g^c_k/D^c_{k,k})(g^c_l/D^c_{l,l})\Delta f=0 \cdots\cdots (25\cdot\text{a, b})$$

Equation 25・a indicates a number of symmetric paths with respect to $\boldsymbol{y}$ intersecting at the critical point but the definite values of Δy_i ($i=1$ to s) must be determined by considering the next third order term since the left hand side of Eq. 24 does not give any information on Δy_i when Δf is equal to zero, eventhough all the coefficients of the second order terms do not vanish. On the other hand, Eq. 25・b is a linear simultaneous equations, which can be solved for $\Delta y_i/\Delta f$ and give solutions on a path, in general, with non-zero slopes with respect to $\boldsymbol{y}$.

(3) Number of Bifurcation Paths

The maximum number of paths from one critical point has been reported[2),3)] as 2^s-1 where s is the number of zero eigenvalues of the tangential stiffness matrix. This formula is valid only when the Jacobian of Eq. 16 does not vanish. The Jacobian can vanish with or without vanishing of all coefficients of the second order terms in those incremental equations. Ref. 5 reported the maximum number as 3^{s-1}, which was derived by considering the second order terms not only at s equations where the eigenvalues of the tangentical matrix vanish but in all n equations. Because of this, it is obvious that the formula is over-estimating the maximum number of bifurcation paths. When the Jacobian vanishes, the next higher order term must be considered for the first s equations until the system of equations has a non-zero value of Jacobian[8)]. The maximum number of bifurcation paths then depends on the minimum order of the Taylor series expansion which must be considered to have non-zero value of Jacobian and can be computed based on Bezout's theorem[9),10)] as $2^{s_2}\times 3^{s_3}\times\cdots\times i^{s_i}-1$ where s_i=number of equations of which the maximum i-th order terms of the Taylor series expansion must be considered to get non-zero value of the Jacobian and s_2

$+s_3+\cdots+s_i=s$.

Example 1 in Ref. 8 shows two bifurcation paths emanating from a single critical point due to the vanishing of all coefficients of the second order terms in the incremental equations and at the critical point C of the second example of the same reference shows six bifurcation paths emanating from a double coincidental critical point due to the condition given in Eq. 25·a. In these two cases, the Jacobian of the incremental equations considering only the second order term becomes zero, and consequently, the third order terms need to be considered. It is found that they satisfy the above formula proposed in this study. In the case of axially loaded circular column, it is obvious that the bifurcation takes any direction yielding a bifurcation surface. Hence it can be expected that the Jacobian remains zero for infinite order of the Taylor series expansion.

4. NUMERICAL EXAMPLES

Three numerical examples are presented to demonstrate the theoretical results on elastic trusses of which the governing equations as well as the incremental equations are presented in Ref. 8. The bifurcation paths are traced utilizing the method presented in the same reference.

(1) Two Bar Plane Truss

The main equilibrium path and the eigenvalue curves of a two bar truss under a vertical loading is shown in Fig. 1 in two dimensional space with non-dimensionalized loading intensity of node 1 and eigenvalues as vertical axis and the vertical position of node 1 as horizontal axis, where E=Young's modules; A=cross sectional area; and x_3^1=vertical displaced position of loading point 1 with the origin at the same node 1 at the initial loading free configuration. The detail of the numerical calculation is given in Ref. 8. This example is presented to show the case of symmetric bifurcation and stationary point and to check the number of bifurcation paths proposed in this study. Six single critical points are obtained by eigenvalue analysis and marked as points A to F. At critical points A, B, E and F, $D_{1,1}^c$ and g_1^c are zero and hence they are bifurcation points. Considering the second order term, it is found that $D_{1,11}^c$ is zero and hence one of the equilibrium paths at each point is symmetric. The stability of these critical points are checked by considering the sign of $D_{1,111}^c$. It is found that these points are unstable. At critical points C and D, $D_{2,2}^c$ is zero but g_2^c is not zero and hence they are stationary points. All of the results agree with the numerically traced solution curves.

It is shown[8] that at height to projected length ratio γ equal to $\sqrt{5.75}$, the critical points A and B, and E and F coincide but they are still single critical points. It is found that two bifurcation paths are emanating from these critical points which exceeds the maximum number of $2^1-1=1$ as given in Refs. 2 and 3. The incremental equations in this case show that the third order terms must be considered in the first incremental equation as all coefficients of the second order terms vanish. This increases the number of possible bifurcation paths. The formula for maximum number of bifurcation paths proposed in this paper is $3^1-1=2$ which satisfies this case.

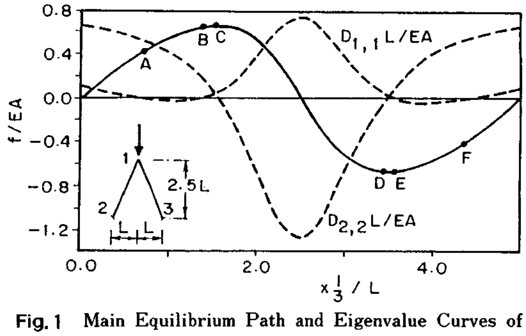

Fig. 1 Main Equilibrium Path and Eigenvalue Curves of Two Bar Truss.

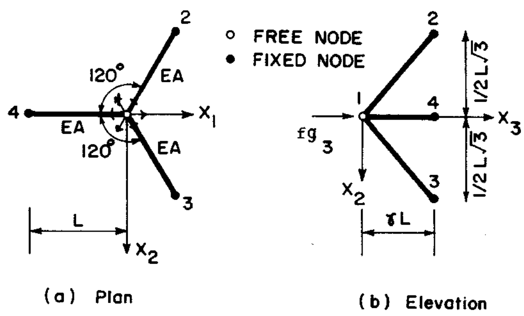

Fig. 2 Three Bar Space Truss.

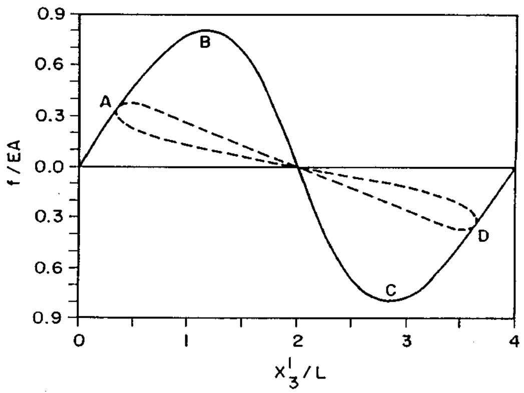

Fig. 3 Main and Bifurcation Paths of Example 2.

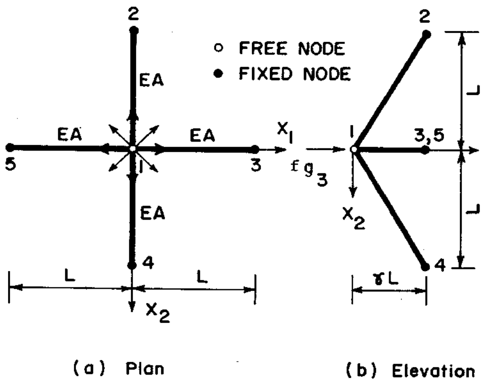

Fig. 4 Four Bar Space Truss.

(2) Three Bar Space Truss

A three bar space truss with the initial configuration as shown in Fig. 2 is analyzed as the example of asymmetric bifurcation. The truss has three degrees of freedom and is subjected to a vertical loading at node 1, i. e. $\langle g_1 \quad g_2 \quad g_3 \rangle^T = \langle 0 \quad 0 \quad 1 \rangle^T$. Computing the first order term yields a diagonal matrix, the elements of which are

$$D_{1,1}=D_{2,2}=3\,EA(1/\eta-1/\xi+0.5/\xi^3)/L\,; \qquad D_{3,3}=3\,EA(1/\eta-1/\xi+(\gamma-\bar{x}_3^1)^2/\xi^3)/L \quad \cdots\cdots (26\cdot a,\ b)$$

in which

$$\eta=\sqrt{(1+\gamma^2)} \text{ and } \xi=\sqrt{\{1+(\gamma-\bar{x}_3^1)^2\}} \quad \cdots\cdots (27\cdot a,\ b)$$

where $\bar{x}_3^1=x_3^1/L$; and γ=ratio between height and projected length L; and upperscript to coordinates (x_1, x_2, x_3) denotes node for the coordinates.

The main equilibrium path is shown in Fig. 3 for $\gamma=2$ by the solid line in two dimensional space with non-dimensionalized loading intensity and the vertical position of node 1 as vertical and horizontal axes, respectively. Also shown in the same figure are four critical points of which A and D are double critical points and B and C are single critical points. Critical points B and C where $D_{3,3}^c$ is zero are stationary points since g_3^c is not zero. The incremental equations, Eq. 16, at critical points A and D, where $D_{1,1}^c$ and $D_{2,2}^c$ are zero, take the following forms

$$D_{1,12}^c \Delta x_1^1 \Delta x_2^1 + D_{1,13}^c \Delta x_1^1 \Delta x_3^1 = 0\,; \qquad D_{2,11}^c (\Delta x_1^1)^2/2 + D_{2,22}^c (\Delta x_2^1)^2/2 + D_{2,23}^c \Delta x_2^1 \Delta x_3^1 = 0\,;$$
$$D_{3,3}^c \Delta x_3^1 = \Delta f \quad \cdots\cdots (28\cdot a\sim c)$$

where

$$D_{1,12}^c = D_{2,11}^c = -D_{2,22}^c = 2.25\,EA/(L^2\xi^5); \qquad D_{1,13}^c = D_{2,23}^c = 3\,EA(\gamma-\bar{x}_3^1)(-1/\xi^3+1.5/\xi^5)/L^2 \cdots (29\cdot a,\ b)$$

Since g_1^c, g_2^c are zero but $D_{1,12}^c$ is not equal to zero, the bifurcation paths are asymmetric which are shown as dashed line in Fig. 3. The bifurcation points are unstable due to non-zero value of $D_{1,12}^c$. All of these results agree with the numerically traced solution curves, which consist of three bifurcation paths of which directions are shown schematically by arrows in Fig. 2.

For $\gamma=\sqrt{2.375}$, all three eigenvalues become zero at the same point, hence it is a triple coincidental critical point. At this point, the nonlinear incremental equations become

$$D_{1,12}^c \Delta x_1^1 \Delta x_2^1 = 0, \qquad D_{2,11}^c (\Delta x_1^1)^2/2 + D_{2,22}^c (\Delta x_2^1)^2/2 = 0, \qquad D_{3,33}^c (\Delta x_3^1)^2 = \Delta f \quad \cdots\cdots (30\cdot a\sim c)$$

where

$$D_{3,33}^c = 9\,EA(\gamma-\bar{x}_3^1)[-1/\xi^3+(\gamma-\bar{x}_3^1)^2/\xi^5]/L^2 \quad \cdots\cdots (31)$$

Since g_3^c is not zero, all the intersecting paths must be stationary but due to non-zero value of $D_{1,12}^c$ the bifurcation paths must be asymmetric which is in contrary if they exist. It can be seen that at this critical point no bifurcation paths will emanate as the only possible solution of Eq. 30 is given as

$$\Delta x_1^1 = \Delta x_2^1 = 0.0, \qquad \Delta x_3^1 = \pm\sqrt{(\Delta f / D_{3,33}^c)} \quad \cdots\cdots (32)$$

Since there is only one solution, it must be the solution on the main path.

(3) Four Bar Space Truss

A four bar space truss with the initial configuration shown in Fig. 4 is analyzed to show the correlation with the proposed number of paths. The truss has three degrees of freedom and is subjected to a vertical force at node 1, i. e. $\langle g_1 \ \ g_2 \ \ g_3\rangle^T=\langle 0 \ \ 0 \ \ 1\rangle^T$. Computing the first order term with the same notations as in the previous examples yields a diagonal matrix, the elements of which are

$$D_{1,1}=D_{2,2}=2\,EA(2/\eta-2/\xi+1/\xi^3)/L \quad (33\cdot a)$$

$$D_{3,3}=4\,EA[1/\eta-1/\xi+(\gamma-\bar{x}_3^1)^2/\xi^3]/L \quad (33\cdot b)$$

The main equilibrium path for $\gamma=2.5$ is the same as in Fig. 1, with the ordinate multiplied by two. Four critical points A, B, C and D are obtained as for the three bar truss example, of which two A and D are double coincidental critical points. At these points where $D^c_{1,1}$ and $D^c_{2,2}$ are zero, the incremental equations take the following form

$$D^c_{1,13}\Delta x_1^1\Delta x_3^1=0, \qquad D^c_{2,23}\Delta x_2^1\Delta x_3^1=0, \qquad D^c_{3,3}\Delta x_3^1=\Delta f \quad (34\cdot a\sim c)$$

where

$$D^c_{1,13}=D^c_{2,23}=2\,EA(\gamma-\bar{x}_3^1)(-2/\xi^3+3/\xi^5)/L^2 \quad (35)$$

Since g^c_1, g^c_2 and $D^c_{i,jk}$ $(i,\ j,\ k=1$ to $2)$ are zero, the bifurcation paths are symmetric, however, as shown in Eq. 25·a, the next third order terms must be considered in the incremental equations at these critical points which will take the following form

$$D^c_{1,13}\Delta x_1^1\Delta x_3^1+D^c_{1,111}(\Delta x_1^1)^3/6+D^c_{1,122}\Delta x_1^1(\Delta x_2^1)^2/2+D^c_{1,133}\Delta x_1^1(\Delta x_3^1)^2/2=0 \quad (36\cdot a)$$

$$D^c_{2,23}\Delta x_2^1\Delta x_3^1+D^c_{2,222}(\Delta x_2^1)^3/6+D^c_{2,112}(\Delta x_1^1)^2\Delta x_2^1/2+D^c_{2,233}\Delta x_2^1(\Delta x_3^1)^2/2=0 \quad (36\cdot b)$$

where

$$D^c_{1,111}=D^c_{2,222}=6\,EA(2/\xi^3-6/\xi^5+5/\xi^7)/L^3\,; \qquad D^c_{1,122}=D^c_{2,112}=4\,EA(1/\xi^3-3/\xi^5)/L^3 \quad (37\cdot a,\ b)$$

$$D^c_{1,133}=D^c_{2,233}=2\,EA[2/\xi^3-3/\xi^5-6/(\gamma-\bar{x}_3^1)^2/\xi^5+15(\gamma-\bar{x}_3^1)^2/\xi^7]/L^3 \quad (37\cdot c)$$

Four symmetric bifurcation paths are obtained by solving Eqs. 36 and 34·c of which directions are shown schematically by arrows in Fig. 4. The number of bifurcation paths of 4 for $\gamma=2.5$ exceeds the maximum number given as $2^2-1=3$ proposed in Refs. 2 and 3 but satisfies the proposed formula in this paper which is given as $3^2-1=8$. When $\gamma=\sqrt{2.375}$, critical points A and B and also C and D coincide and the critical point is a triple coincidental critical point as that in the Example 2. Four symmetric bifurcation paths emanate from this point, which is also a symmetric, i. e., stationary point of the main equilibrium path.

5. SUMMARY AND CONCLUDING REMARKS

There are numerous works on the study of elastic stability and behavior at critical points but most of them are based on the potential energy theorem and utilize the perturbation technique.

During the work of tracing bifurcation paths as reported in Ref. 8, it was found that Taylor expansion is a simple but sufficiently powerful tool to study the state of stability of equilibrium and the behavior at the critical points.

In this study, only the Taylor expansion is utilized to investigate the stability of equilibrium states and the behavior at critical points. This procedure is much simpler compared with the previous works by the perturbation method together with the potential energy theorem[2),3)]. On the other hand, Ref. 5 employs the Taylor expansion directly on the energy function, of which result is not so simple as reported in this study but more or less the same as in Refs. 2 and 3. In this paper, the nonlinear governing equations are used directly as the basic governing equations rather than utilizing the potential energy theorem. These equations so used also contributed to simplify the explanation. One of the examples of this simplicity is due to the expression of the loading in the form of Eq. 2, which may be proposed for the first time in this sort of study. In this study the loading pattern f_i is treated as a given quantity and in addition it is treated as piecewise constant function of the loading parameter. This treatment does not loose any generality. Because of this, no derivative of variables with respect to the loading parameter appear in this study. On the other hand, derivatives of not only the first order but also higher orders with respect to loading parameter appear in Refs. 2, 3 and 5. This expression of the loading could also cover the so-called multi-parameter loading system studied in Refs. 2, 3 and 5 as a separate subject. To derive definite conclusion on this multi-parameter system, a further study is needed.

It was found that the results of this study of the state of stability of equilibrium mostly agree with those analyzed by the perturbation method[2),3)], except the result expressed in Eq. 13. The reason for this difference has not yet been clarified, but left for the future study.

Not only the behavior at single critical points but also of coincidental critical points are studied by the same method. For single critical points, the results of this study agree with those reported in Rels. 2 and 3 based on the perturbation method. On the same subject, Ref. 5 takes into account the second order terms in all of n incremental equations at the critical points, which is not necessary and makes the formulation complicated. The study of the behavior at the coincidental critical points is very primitive and indicating further study is needed. Nevertheless it has not previously been reported.

The stability condition of Eq. 9 for multi-degrees of freedom system has been proved. This condition of stability has been proved only for single degree of freedom system, but the result has been extended without proof and used as an axiom for multi-degrees of freedom system[2),4)]. The stability condition, when the tangential stiffness matrix is positive semi-definite, is yet to be proved. This paper has employed the stability condition of this case as an axiom as employed in all other references.

The maximum number of bifurcation paths reported in Refs. 2 and 3 has been shown not to be sufficient by simple examples while the one reported in Ref. 5 is overestimated. The maximum number of bifurcation paths proposed in this paper depends on the degree of the Taylor expansion that should be considered to get non-zero value of Jacobian in the incremental equations at the bifurcation point.

6. ACKNOWLEDGEMENTS

This paper is based on the master's thesis of the second author at the Asian Institute of Technology, Bangkok, Thailand, who received the scholarship donated by The Royal Netherland Government. The work is also financed by the Japan International Cooperation Agency.

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(Received May 10 1986)