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# MECHANICAL PROPERTIES OF SOLIDS CONTAINING A DOUBLY PERIODIC RECTANGULAR ARRAY OF CRACKS

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In the present study, the problem of an infinitely extended elastic solid containing a doubly periodic rectangular array of cracks is considered. The analysis is made based on the method of pseudo-tractions, and a first-order approximate but explicit solution is obtained. The study reveals the fundamental difficulty of the problems of doubly periodic cracks which gives rise to the discrepancies between the results of the previous works. The stress intensity factors of mode I and mode II and the overall compliance are derived as functions of the crack density and geometry of the crack array. The overall compliance of solids with randomly distributed, unidirectional cracks is evaluated by the self-consistent method. The validity of the self-consistent method is discussed in terms of the results of the doubly periodic cracks.

*Keywords*: *cracked solids, overall moduli, fracture mechanics*

## 1. INTRODUCTION

The behavior of solids containing multiple cracks has long been the subject of interest of numerous researches, especially those in the field of geomechanics. The mechanical response of the cracked solids depends not only on the properties of the matrix surrounding the cracks, but also on the size, shape, orientation, and distribution of the cracks. There have been considerable efforts directed at estimating the overall response of solids containing cracks of various shapes and arrangements; for example, Walsh[1),2)], Vakulenko and Kachanov[3)], Garbin and Knopoll[4)], Salganik[5)], Vavakin and Salganik[6)], Hudson[7)], Budiansky and O'Connell[8)], Eimer[9),10)], Hoenig[11)], Kachanov[12)], Leguillon and Sanchez-Palencia[13)], Horii and Nemat-Nasser[14)], Oda[15),16)], Oda, Suzuki, and Maeshibu[17)].

In general, it is difficult to establish a boundary value problem if the orientation and the distribution of cracks are allowed to be arbitrary. Therefore, some restrictions have to be made regarding the arrangement and the scale of the cracks to render the problem mathematically tractable. The investigation of the behavior of solids containing multiple cracks is then directed to those special problems concerning cracks arranged in regular patterns, among which the simplest form is the doubly periodic rectangular array. Although the problem may lose generalities, such simplified models can disclose, at least, some general trends, and in fact they may be of some interest on their own account.

In connection with the problem of a doubly periodic array of cracks, Delameter et al.[18)] investigated the stress intensity factors for a doubly periodic rectangular array of cracks under mode I and mode II

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deformations. Representing each crack by a certain distribution of dislocations, they obtained singular integral equations for the dislocation distribution function through satisfaction of the boundary conditions at the traction-free crack faces. Extending the work of Delameter et al.[18], Karihaloo[19),20] examined the stress relaxation process from the tips of the cracks in doubly periodic (rectangular and diamond-shaped) arrays in all modes of deformation. Isida[21] and Isida et al.[22] made extensive analyses on a single crack in a rectangular finite plate by employing the boundary collocation method. By choosing appropriate boundary conditions, they obtained numerical solutions to the problem of doubly periodic rectangular array of cracks under mode I. The numerical results reported by Isida[21] and Isida et al.[22] are seen to be different from those given for mode I by Delameter et al.[18] The inconsistency between these results needs explanation and it is one of the aims of the present work to make clear what contributes to this difference.

Provided that the statistical distribution of the cracks is random, use can be made of the so-called "self-consistent method" to include the interaction effects in estimating the overall properties of the body. This method was originally proposed for aggregates of crystals and was later applied to solids with randomly distributed inhomogeneities, such as solid inclusions or voids; see Willis[23] and Walpole[24] who give extensive lists of the literature. Application of the method to crack problems was pioneered by Budiansky and O'Connel[8]. They considered a body containing randomly distributed penny-shaped cracks, and estimated analytically its effective elastic moduli. Since then, this method has been extensively exploited in the analyses of the macroscopic properties of the body with cracks of various sizes and configurations; see, for example, Hoenig[11], Kachanov[12], Horii and Nemat-Nasser[14]. Although the method has found its application in a wide variety of problems, it evaluates the effect of interaction between neighboring inhomogeneities just in an indirect manner. In the present work, we make an attempt to examine whether or not the self-consistent method reasonably estimates the interaction.

## 2. THE METHOD OF PSEUDO-TRACTIONS

An infinite elastic plane containing a doubly periodic rectangular array of cracks is shown in Fig. 1. Each crack in the rectangular array is of length $2c$, and is separated from other cracks by a distance $H$ vertically and a distance $W$ horizontally. The quantities which are used hereafter are also shown in Fig. 1.

The problem of an infinitely extended elastic solid containing a doubly periodic rectangular array of cracks under far-field stresses is referred to as the "original problem". [In the doubly periodic problems the far-field stresses are not well-defined. In this study, they are introduced through the following superposition.] The solution of the problem is given by the superposition of the solutions of a homogeneous problem and a subsidiary problem, as shown in Fig. 2. The homogeneous problem contains no cracks and is subjected to uniform stresses at infinity, being denoted by the quantities with superscript $\infty$ in Fig. 2. In the subsidiary problem, the same constant stresses are prescribed on the surfaces of doubly periodic cracks with no stresses at infinity. The boundary conditions along any crack $k$ are given by

$$\sigma_y^k=-\sigma_y^\infty,\quad \tau_{xy}^k=-\tau_{xy}^\infty,\quad -c\leq x^k\leq c,\quad y^k=0 \qquad (1)$$

Since the solution to the homogeneous problem is known, only the subsidiary problem is to be solved. To apply the method of pseudo-tractions proposed by Horii and Nemat-Nasser[25], all the cracks are numbered

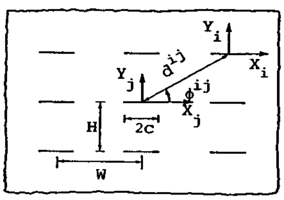


Fig. 1 A doubly periodic rectangular array of cracks.

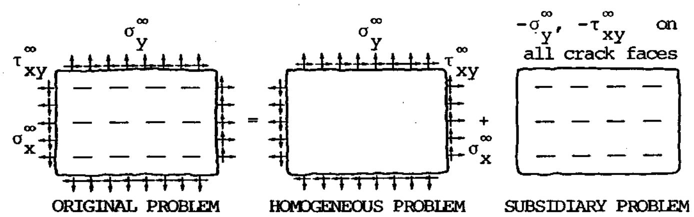


Fig. 2 Decomposition of an original problem into a homogeneous problem and a subsidiary problem.

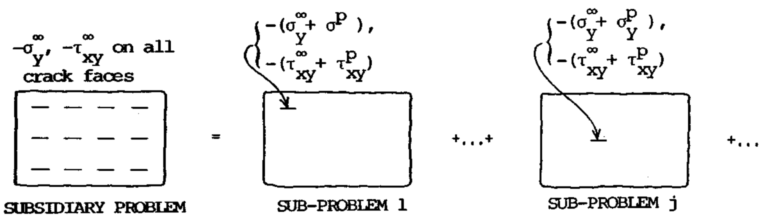


**Fig. 3** Decomposition of the subsidiary problem into sub-problems.

and the subsidiary problem is decomposed into infinite number of sub-problems, as shown in **Fig. 3**. In sub-problem $j$, the boundary conditions along the crack surfaces of crack $j$ are given by

$$\sigma_y^j=-(\sigma_y^\infty+\sigma_y^p),\ \ \tau_{xy}^j=-(\tau_{xy}^\infty+\tau_{xy}^p),\ \ -c\leqq x^j\leqq c,\ \ y^j=0 \qquad (2)$$

The quantities $\sigma_y^p$, $\tau_{xy}^p$, being called the "pseudo-tractions", are unknown tractions which must be determined such that the boundary conditions (1) are satisfied when all the sub-problems are superimposed. Note that the pseudo-tractions are the same for all cracks because of the symmetry of the problem.

In the present analysis, we employ Muskhelishvili's complex stress functions $\phi(z)$ and $\psi(z)$ (see Muskhelishvili[26]). Stress and displacement components are given in terms of these potentials by

$$\left.\begin{array}{l}\sigma_x(z)+\sigma_y(z)=2[\Phi(z)+\overline{\Phi(z)}],\ \sigma_y(z)-\sigma_x(z)+2i\tau_{xy}(z)=2[\bar{z}\Phi'(z)+\Psi(z)],\\ 2G[u(z)+iv(z)]=\varkappa\phi(z)-z\overline{\phi'(z)}-\overline{\psi(z)}\end{array}\right\} \qquad (3)$$

when $\Phi(z)=\phi'(z)$, $\Psi(z)=\psi'(z)$, $z=x+iy$, $i=\sqrt{-1}$, $G$ is the shear modulus, $\varkappa=3-4\nu$ for plane strain, $\varkappa=(3-\nu)/(1+\nu)$ for plane stress, $\nu$ is Poisson's ratio, the overbar denotes the complex conjugate, and prime stands for differentiation with respect to the argument.

For the sub-problem $j$, the stress functions are given by (see Muskhelishvili[26])

$$\left.\begin{array}{l}\Phi^j(z^j)=-\dfrac{1}{2\pi i(z^{j2}-c^2)^{1/2}}\displaystyle\int_{-c}^{c}\frac{(x^2-c^2)^{1/2}}{x-z^j}[(\sigma_y^\infty+\sigma_y^p)-i(\tau_{xy}^\infty+\tau_{xy}^p)]dx\\ \Psi^j(z^j)=\overline{\Phi^j(\overline{z^j})}-\Phi^j(z^j)-z^j\Phi^{j\prime}(z^j),\ \ z^j=x^j+iy^j,\ \ j=1,2,\cdots\end{array}\right\} \qquad (4)$$

The requirement that the sum of the sub-problems must be equivalent to the subsidiary problem leads to

$$\sigma_y^p-i\tau_{xy}^p=\sum_k^{\mathrm{ALL}}[\Phi^k(z^k)+\overline{\Phi^k(z^k)}+z^k\overline{\Phi^{k\prime}(z^k)}+\overline{\Psi^k(z^k)}],\ \ z^k=d^{jk}e^{i\phi^{jk}}+x^j,\ \ -c\leqq x^j\leqq c \qquad (5)$$

where $d^{jk}$, $\phi^{jk}$, $x^j$ are defined in **Fig. 1**, and $\sum_k^{\mathrm{ALL}}$ denotes the summation for all the cracks except for the crack $j$ under consideration. Note that $\sum_k^{\mathrm{ALL}}$ represents a doubly infinite summation in the present problem. The right-hand side of Eqn. (5) represents the sum of tractions on the crack $j$ caused by all other cracks.

Eqns. (4) and (5) form an integral equation for the pseudo-tractions which are functions of $x^j$. It is not possible to solve this integral equation explicitly. However, this integral equation can be reduced to a system of algebraic equations in the following manner.

We expand the pseudo-tractions into a Taylor series as

$$\sigma_y^p(x^j)-i\tau_{xy}^p(x^j)=\sum_{n=0}^{\infty}(P_n-iQ_n)\left[\frac{x^j}{c}\right]^n, \qquad (6)$$

Substituting Eqn. (6) into Eqns. (4) and (5), the integral equation (5) is reduced to a system of algebraic equations for $P_n$ and $Q_n$. Horii and Nemat-Nasser[25] showed that $P_n$ and $Q_n$ are of the order of $(c/d)^{N+1}$, where $d$ denotes distance between cracks. Neglecting terms of orders higher than $(c/d)^{N+1}$, we have $2N$ linear algebraic equations for $P_0$, $\cdots$, $P_{N-1}$, $Q_0$, $\cdots$, $Q_{N-1}$. The first-order approximate but explicit solutions are obtained by taking $N=1$, as follows. Eqn. (6) becomes

$$\sigma_y^p(x^j)-i\tau_{xy}^p(x^j)=P_0-iQ_0 \qquad (7)$$

implying that the pseudo-tractions are constant. Substituting Eqn. (7) into Eqn. (4), we obtain the stress function $\Phi^j(z^j)$ for the sub-problem $j$ as

$$\Phi^j(z^j)=\frac{1}{2}[(\sigma_y^\infty+P_0)-i(\tau_{xy}^\infty+Q_0)]\left\{\frac{z^j}{(z^{j2}-c^2)^{1/2}}-1\right\} \quad (8)$$

It follows from Horii and Nemat-Nasser[25] that

$$P_0=\left[\sum_k^{ALL} A_{00}^{jk}\right]\sigma_y^\infty+\left[\sum_k^{ALL} B_{00}^{jk}\right]\tau_{xy}^\infty,\quad Q_0=\left[\sum_k^{ALL} C_{00}^{jk}\right]\sigma_y^\infty+\left[\sum_k^{ALL} D_{00}^{jk}\right]\tau_{xy}^\infty \quad (9)$$

where

$$A_{00}^{jk}=\left[\cos(2\phi^{jk})-\frac{1}{2}\cos(4\phi^{jk})\right](c/d^{jk})^2,\quad B_{00}^{jk}=\frac{1}{2}[-\sin(2\phi^{jk})+\sin(4\phi^{jk})](c/d^{jk})^2$$

$$C_{00}^{jk}=B_{00}^{jk},\quad D_{00}^{jk}=\frac{1}{2}\cos(4\phi^{jk})(c/d^{jk})^2$$

The stress functions of the sub-problem $j$ are given by Eqns. (8) and (9), and the stress and displacement fields are known through Eqn. (3). The stress and displacement fields of the original problem are then obtained from the superposition. Furthermore, following Horii and Nemat-Nasser[25], the stress intensity factors for mode I and mode II are given by

$$\frac{K_I}{\sigma_y^\infty\sqrt{\pi c}}=1+\frac{P_0}{\sigma_y^\infty},\quad \frac{K_{II}}{\tau_{xy}^\infty\sqrt{\pi c}}=1+\frac{Q_0}{\tau_{xy}^\infty} \quad (10)$$

The doubly infinite series in Eqn. (9) represent the interaction effects from all other cracks on the crack $j$. These summations can be evaluated numerically by considering the partial sums for finite number of cracks and increasing the number of cracks. If the doubly infinite series in Eqn. (9) are convergent, the way to take the partial sums does not affect the values of the limits. Here, we use a rectangular array of $R$ rows by $C$ columns of cracks for the partial sums. Keeping the ratio $C/R$ constant, we increase $C$ and $R$. Then the doubly infinite series in Eqn. (9) are expressed as

$$\left.\begin{aligned}
\sum_k^{ALL} A_{00}^{jk}&=\lim_{R,C\to\infty}\left[\frac{c}{W}\right]^2\left[-\frac{3}{\left[\frac{H}{W}\right]^2}\sum_{m=1}^{\frac{R}{2}}\left(\frac{1}{m^2}\right)+\sum_{n=1}^{\frac{C}{2}}\left(\frac{1}{n^2}\right)+2\sum_{m=1}^{\frac{R}{2}}\sum_{n=1}^{\frac{C}{2}}\left\{\frac{-3m^4\left[\frac{H}{W}\right]^4+6m^2n^2\left[\frac{H}{W}\right]^2+n^4}{\left[m^2\left[\frac{H}{W}\right]^2+n^2\right]^3}\right\}\right]\\
\sum_k^{ALL} B_{00}^{jk}&=\sum_k^{ALL} C_{00}^{jk}=0\\
\sum_k^{ALL} D_{00}^{jk}&=\lim_{R,C\to\infty}\left[\frac{c}{W}\right]^2\left[\frac{1}{\left[\frac{H}{W}\right]^2}\sum_{m=1}^{\frac{R}{2}}\left(\frac{1}{m^2}\right)+\sum_{n=1}^{\frac{C}{2}}\left(\frac{1}{n^2}\right)+2\sum_{m=1}^{\frac{R}{2}}\sum_{n=1}^{\frac{C}{2}}\left\{\frac{m^4\left[\frac{H}{W}\right]^4+6m^2n^2\left[\frac{H}{W}\right]^2+n^4}{\left[m^2\left[\frac{H}{W}\right]^2+n^2\right]^3}\right\}\right]
\end{aligned}\right\} \quad (11)$$

The convergence of the infinite series in Eqn. (11) is numerically examined and found to be fast. As an example, for $H/W=1$, $c/W=0.1$ with $C/R=1$ an array of 10 by 10 cracks gives $\sum_k^{ALL} A_{00}^{jk}$ and $\sum_k^{ALL} D_{00}^{jk}$ respectively the values of $-0.01036$ and $+0.01036$, while the final values are $-0.01039$ and $+0.01039$. By varying the ratio $C/R$, the limits of the series are investigated and shown in Figs. 4 and 5 for $H/W=1$, $c/W=0.1$. The ratio $C/R$ represents the shape of the array that is used to approach the limits of the sums. The results reveal that as we increase number of the cracks under summations, the value of each

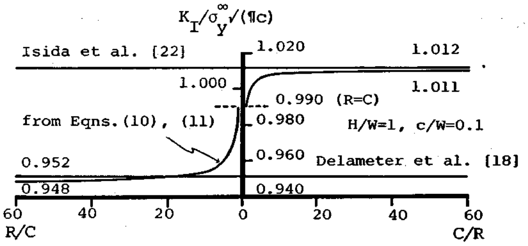


Fig. 4 Limits of $K_I/\sigma_y^\infty\sqrt{\pi c}$.

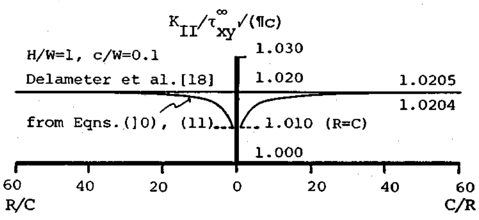


Fig. 5 Limits of $K_{II}/\tau_{xy}^\infty\sqrt{\pi c}$.

sums approaches different limit for different values of $C/R$. Therefore, it is concluded that each doubly infinite series of Eqn. (11) diverges because the limiting value depends on the way of taking the summation, which means that Eqn. (10) is not the solution of the doubly periodic problem. This is the essential difficulty of the problem of doubly periodic cracks and it always arises when we use the superposition principle to account for the influence from all the cracks on the state of stresses at a point in the cracked plane. The superposition always leads to the doubly infinite series of which limits depend on the way of taking the summation.

## 3. THE AVERAGE STRESSES

The reason for the difficulty shown in the preceding section is given as follows. We evaluate the infinite series as the limits of finite sums. This corresponds to solving a problem of an infinite plane with cracks of finite number and increasing the number of the cracks. No matter how we increase the number of the cracks in the infinite plane, we always have uncracked area surround the cracked region. For each value of $C/R$ the array has a particular shape and the stress and strain fields have different patterns.

Now we evaluate the average stresses along the middle part of the cracked portion of the plane, i. e. along line AA' in Fig. 6. By superposition, the average stresses $\hat{\sigma}_y$, $\hat{\tau}_{xy}$ are found to be

$$\hat{\sigma}_y - i\hat{\tau}_{xy} = \sigma_y^\infty - i\tau_{xy}^\infty + \frac{4}{W}\sum_{m=1}^{\frac{R}{2}}\sum_{n=1}^{\frac{C}{2}}\left[\int_{(n-1)W}^{nW}\{\sigma_y(x, y_m) - i\tau_{xy}(x, y_m)\}dx\right],\ y_m = H\left(m - \frac{1}{2}\right) \quad \cdots\cdots (12)$$

where $\sigma_y(x, y_m)$ and $\tau_{xy}(x, y_m)$ are the stresses along the segment AA' in the sub-problems.

With the help of Eqns. (3), (8), and (9), we get

$$\hat{\sigma}_y - i\hat{\tau}_{xy} = \lim_{R,C\to\infty}\left[\sigma_y^\infty\left\{1 + 2\left[\frac{c}{W}\right]^2\sum_{m=1}^{\frac{R}{2}}\sum_{n=1}^{\frac{C}{2}} C_{mn}^{\mathrm{I}}\right\} - i\tau_{xy}^\infty\left\{1 + 2\left[\frac{c}{W}\right]^2\sum_{m=1}^{\frac{R}{2}}\sum_{n=1}^{\frac{C}{2}} C_{mn}^{\mathrm{II}}\right\}\right] \quad \cdots\cdots (13)$$

where

$$C_{mn}^{\mathrm{I}} = \frac{n^3(n-1)^3 + \left[\frac{y_m}{W}\right]^2 n(n-1)(7n^2 - 7n - 3) + \left[\frac{y_m}{W}\right]^4(3n^2 - 3n - 1) - 3\left[\frac{y_m}{W}\right]^6}{\left[(n-1)^2 + \left[\frac{y_m}{W}\right]^2\right]^2\left[n^2 + \left[\frac{y_m}{W}\right]^2\right]^2}$$

$$C_{mn}^{\mathrm{II}} = \frac{n^3(n-1)^3 + \left[\frac{y_m}{W}\right]^2\left[\left[\frac{y_m}{W}\right]^2 + n(n-1)\right](5n^2 - 5n + 1) + \left[\frac{y_m}{W}\right]^6}{\left[(n-1)^2 + \left[\frac{y_m}{W}\right]^2\right]^2\left[n^2 + \left[\frac{y_m}{W}\right]^2\right]^2}$$

As an example, the results for $H/W = 1$, $c/W = 0.1$ are shown in Figs. 7 and 8. It is seen that for different $C/R$, both summations tend to different limits as number of the cracks in the solid is increased.

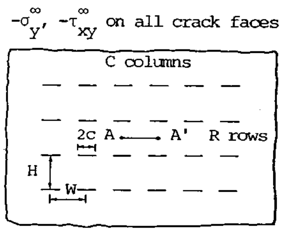


Fig. 6 A solid with an array of $R$ rows by $C$ columns of cracks.

## 4. THE STRESS INTENSITY FACTORS

In previous sections we have shown that the stress intensity

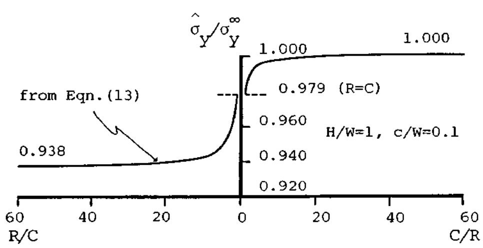


Fig. 7 Limits of $\hat{\sigma}_y/\sigma_y^\infty$.

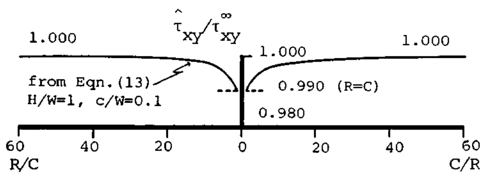


Fig. 8 Limits of $\hat{\tau}_{xy}/\tau_{xy}^\infty$.

factors and the average stresses depend on $C/R$, i.e. they depend on the way of taking the summation. In the sequel we show that being expressed in terms of the average stresses, the stress intensity factors are uniquely determined.

From Eqns. (10), (11), and (13), the stress intensity factors normalized by the average stresses instead of the stresses at infinity are given by

$$\left.\begin{aligned} \frac{K_I}{\hat{\sigma}_y\sqrt{\pi c}} &= 1+\left[\frac{c}{W}\right]^2 F_I(\phi) = 1+\left[\frac{c}{H}\right]^2 G_I(\phi) = 1+\frac{c^2}{HW} H_I(\phi) \\ \frac{K_{II}}{\hat{\tau}_{xy}\sqrt{\pi c}} &= 1+\left[\frac{c}{W}\right]^2 F_{II}(\phi) = 1+\left[\frac{c}{H}\right]^2 G_{II}(\phi) = 1+\frac{c^2}{HW} H_{II}(\phi) \end{aligned}\right\} \quad (14)$$

where

$$F_I(\phi)=\lim_{R,C\to\infty}\left[-\frac{3}{\left[\frac{H}{W}\right]^2}\sum_{m=1}^{\frac{R}{2}}\left[\frac{1}{m^2}\right]+\sum_{n=1}^{\frac{C}{2}}\left[\frac{1}{n^2}\right]+2\sum_{m=1}^{\frac{R}{2}}\sum_{n=1}^{\frac{C}{2}}\left\{\frac{-3m^4\left[\frac{H}{W}\right]^4+6m^2n^2\left[\frac{H}{W}\right]^2+n^4}{\left[m^2\left[\frac{H}{W}\right]^2+n^2\right]^3}\right.\right.$$
$$\left.\left.-\frac{n^3(n-1)^3+\left[\frac{y_m}{W}\right]^2 n(n-1)(7n^2-7n-3)+\left[\frac{y_m}{W}\right]^4(3n^2-3n-1)-3\left[\frac{y_m}{W}\right]^6}{\left[(n-1)^2+\left[\frac{y_m}{W}\right]^2\right]^2\left[n^2+\left[\frac{y_m}{W}\right]^2\right]^2}\right\}\right]$$

$$F_{II}(\phi)=\lim_{R,C\to\infty}\left[\frac{1}{\left[\frac{H}{W}\right]^2}\sum_{m=1}^{\frac{R}{2}}\left[\frac{1}{m^2}\right]+\sum_{n=1}^{\frac{C}{2}}\left[\frac{1}{n^2}\right]+2\sum_{m=1}^{\frac{R}{2}}\sum_{n=1}^{\frac{C}{2}}\left\{\frac{m^4\left[\frac{H}{W}\right]^4+6m^2n^2\left[\frac{H}{W}\right]^2+n^4}{\left[m^2\left[\frac{H}{W}\right]^2+n^2\right]^3}\right.\right.$$
$$\left.\left.-\frac{n^3(n-1)^3-\left[\frac{y_m}{W}\right]^2\left[\left[\frac{y_m}{W}\right]^2+n(n-1)\right](5n^2-5n+1)+\left[\frac{y_m}{W}\right]^6}{\left[(n-1)^2+\left[\frac{y_m}{W}\right]^2\right]^2\left[n^2+\left[\frac{y_m}{W}\right]^2\right]^2}\right\}\right]$$

$$G_I(\phi)=\left[\frac{H}{W}\right]^2 F_I(\phi),\ H_I(\phi)=\frac{H}{W}F_I(\phi),\ G_{II}(\phi)=\left[\frac{H}{W}\right]^2 F_{II}(\phi),\ H_{II}(\phi)=\frac{H}{W}F_{II}(\phi),\ \tan\phi=\frac{H}{W}.$$

The numerical results obtained from Eqn. (14) for the case of $H/W=1$, $c/W=0.1$ are shown in **Figs. 9** and **10**. It is seen that a single limit exists for each fracture mode, which means that the doubly infinite series in Eqn. (14) are convergent series and the stress intensity factors are uniquely determined.

For $H/W=1$, $c/W=0.1$, $K_I/\hat{\sigma}_y\sqrt{\pi c}$ is given by Eqn. (14) to be 1.0110 which is very close to the value of $K_I/\sigma_0\sqrt{\pi c}$ reported by Isida[21] and Isida et al.[22], being 1.012. [Note that the present solution is an approximate solution.] However, the numerical value of $K_I/\sigma\sqrt{\pi c}$ for the present case is shown by Delameter et al[18] to be 0.9517. The numerical solutions given by Isida[21] and Isida et al.[22] are considered to be the solution to the problem of doubly periodic cracks since his collocation procedure is performed on the unit cell, a rectangular plate with a single crack, with the appropriate boundary conditions of the doubly periodic array of cracks. On the other hand, the method used by Delameter et al.[18] has its basis on the superposition. In his formulation, Delameter represents each crack by a certain distribution of dislocations and employs the superposition principle to obtain the stress field due to the applied stresses and the dislocations. This leads to a term with doubly infinite summation of the contribution of all other cracks. He evaluates the doubly infinite summations in the vertical direction first and sets up a numerical scheme to solve the resulting integral equations. If the ratio of $C/R$ is set to be very small, which is equivalent to taking the summation in the vertical direction first, Eqn. (10) gives $K_I/\sigma_y^\infty\sqrt{\pi c}$ the value of 0.9482. Hence, it seems that the stress intensity factors obtained by Delameter et al.[18] corresponds to Eqn. (10), which differs from our final solution (14).

The accuracy of our first-order approximate solution is examined by comparing the values of $K_I$ given by Eqn. (14) to those reported by Isida et al.[22]. **Table 1** shows the values of $K_I/\hat{\sigma}_y\sqrt{\pi c}$ computed from

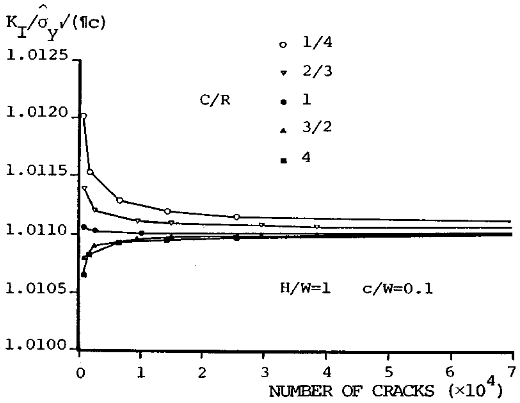


Fig. 9 Convergence of $K_I/\hat{\sigma}_y\sqrt{\pi c}$.

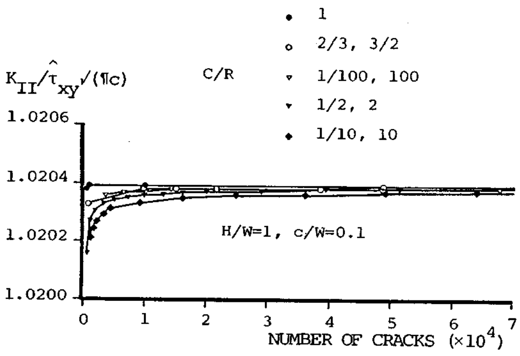


Fig. 10 Convergence of $K_{II}/\hat{\tau}_{xy}\sqrt{\pi c}$.

Table 1 $K_I/\hat{\sigma}_y\sqrt{\pi c}$ for several values of $H/W$ and $c/W$.

| H/W \ c/W | 0 | 0.05 | 0.10 | 0.15 | 0.20 | Remark |
|---|---|---|---|---|---|---|
| 0.5 | 1.0 | 0.982 | 0.928 | 0.838 | | from (14) |
| 1.0 | 1.0 | 1.003 | 1.011 | 1.025 | | |
| 1.5 | 1.0 | 1.004 | 1.016 | 1.036 | 1.064 | |
| 0.5 | 1.0 | 0.983 | 0.948 | 0.922 | | from Isida (1981) |
| 1.0 | 1.0 | 1.003 | 1.012 | 1.031 | | |
| 1.5 | 1.0 | 1.004 | 1.017 | 1.039 | 1.074 | |

Table 2 $K_{II}/\hat{\tau}_{xy}\sqrt{\pi c}$ for several values of $H/W$ and $c/\min(H/W)$.

| H/W \ c/min (H,W) | 0.100 | 0.125 | 0.167 | 0.200 | 0.250 | Remark |
|---|---|---|---|---|---|---|
| 0.0 | 1.0164 | 1.0257 | 1.0457 | | 1.1028 | from (14) |
| 0.4 | | | | | 1.1028 | |
| 0.6 | | | 1.0460 | | | |
| 0.8 | | 1.0274 | | | 1.1094 | |
| 1.0 | 1.0204 | | | | | |
| 1.2 | | | | 1.0713 | | |
| 1.6 | | | | 1.0664 | | |
| 2.0 | | | | 1.0659 | | |
| ∞ | 1.0164 | | | 1.0658 | | |
| 0.0 | 1.0160 | 1.0246 | 1.0424 | | 1.0881 | from Delameter et al. (1975) |
| 0.4 | | | | | 1.0881 | |
| 0.6 | | | 1.0429 | | | |
| 0.8 | | 1.0267 | | | 1.1022 | |
| 1.0 | 1.0205 | | | | | |
| 1.2 | | | | 1.0787 | | |
| 1.6 | | | | 1.0757 | | |
| 2.0 | | | | 1.0753 | | |
| ∞ | 1.0160 | | | 1.0753 | | |

Eqn. (14) and those obtained by Isida et al.[22] for various values of $H/W$ and $c/W$. Good agreement between both results is observed for small values of $c/W$.

For mode II, it can be shown from Eqn. (13) that as $C/R$ decreases, the value of $\hat{\tau}_{xy}/\tau_{xy}^{\infty}$ approaches 1.0, making the values of $K_{II}/\tau_{xy}^{\infty}\sqrt{\pi c}$ of Eqn. (10) approach those of $K_{II}/\hat{\tau}_{xy}\sqrt{\pi c}$ of Eqn. (14). (See also Figs. 5 and 8.) Accordingly, the numerical results for mode II reported by Delameter et al.[18] can be used to examine the accuracy of our first-order approximate solution. Table 2 shows the comparison of the values of $K_{II}/\hat{\tau}_{xy}\sqrt{\pi c}$ obtained via Eqn. (14) and those computed by Delameter et al.[18] for several values of $H/W$ and $c/\min(H, W)$. Satisfactory agreement is observed for small values of $c/\min(H, W)$.

When $H/W$ goes to infinity, Eqn. (14) yields the solution of an infinite row of collinear cracks which agrees with the expansion of the exact solution of Westergaard[27], neglecting higher order terms, i.e.

$$\lim_{H/W\to\infty}\frac{K_I}{\hat{\sigma}_y\sqrt{\pi c}}=1+\frac{\pi^2}{6}\left[\frac{c}{W}\right]^2,\quad \lim_{H/W\to\infty}\frac{K_{II}}{\hat{\tau}_{xy}\sqrt{\pi c}}=1+\frac{\pi^2}{6}\left[\frac{c}{W}\right]^2 \qquad (15)$$

The solution of a single stack of cracks is obtained from Eqn. (14) if $H/W$ goes to zero. The solutions (16) shown below compare well with Fig. 2.29 (p. 123) of Isida[28] and correspond to the first-order approximate solutions of Horii and Nemat-Nasser[25].

$$\lim_{H/W\to 0}\frac{K_I}{\hat{\sigma}_y\sqrt{\pi c}}=1-\frac{\pi^2}{2}\left[\frac{c}{H}\right]^2,\quad \lim_{H/W\to 0}\frac{K_{II}}{\hat{\tau}_{xy}\sqrt{\pi c}}=1+\frac{\pi^2}{6}\left[\frac{c}{H}\right]^2 \qquad (16)$$

In Figs. 11 and 12, the values of the stress intensity factors computed using Eqn. (14) are shown for various values of $c/\min(H, W)$ and $\phi$. Note that the angle $\phi$ defines the crack arrangement; $\tan\phi=H/W$. It is seen that for $\phi$ less than $0.214\pi$, the values of $K_I/\hat{\sigma}_y\sqrt{\pi c}$ are less than one. For mode II,

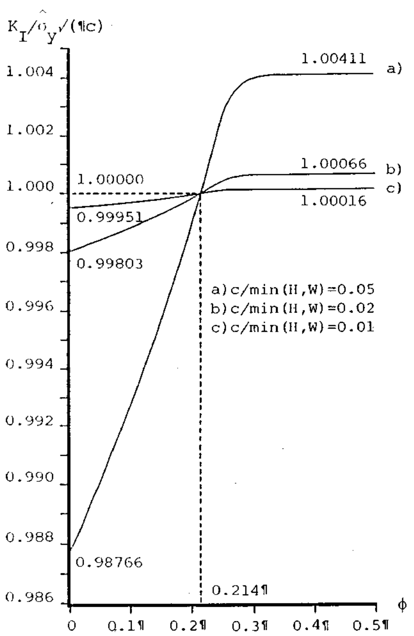


Fig. 11 $K_I/\hat{\sigma}_y\sqrt{\pi c}$ plotted against $\phi$.

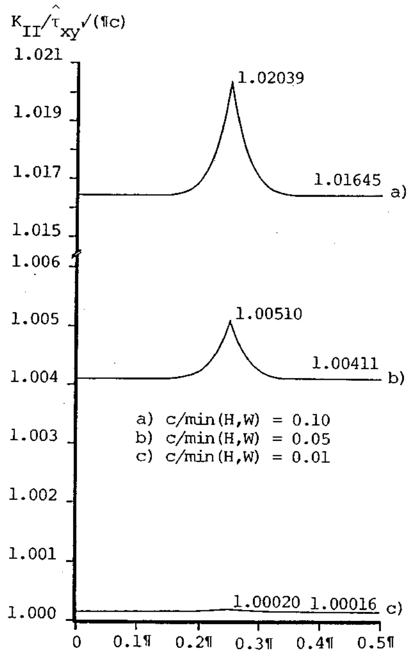


Fig. 12 $K_{II}/\hat{\tau}_{xy}\sqrt{\pi c}$ plotted against $\phi$.

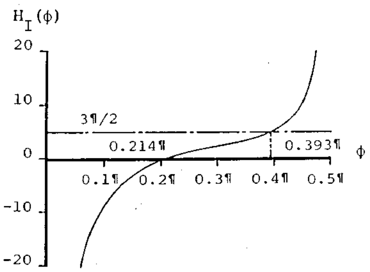


Fig. 13 Plot of $H_I(\phi)$.

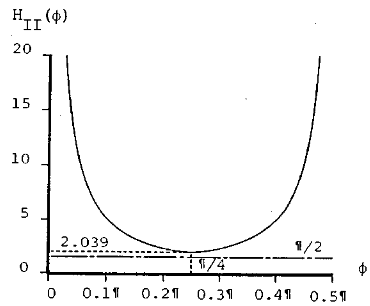


Fig. 14 Plot of $H_{II}(\phi)$.

values of $K_{II}/\hat{\tau}\sqrt{\pi c}$ are seen to be always larger than one.

From Eqn. (14) it is seen that the variation of the stress intensity factors with the crack arrangement at a constant crack density, $c^2/HW$, is represented by functions $H_I(\phi)$ and $H_{II}(\phi)$. The functions $H_I(\phi)$ and $H_{II}(\phi)$ are plotted against $\phi$ in Fig. 13 and 14, respectively.

## 5. THE OVERALL COMPLIANCE AND THE SELF-CONSISTENT METHOD

The overall compliance of an infinite elastic solid containing a doubly periodic rectangular array of cracks is evaluated by utilizing the solutions obtained in the previous chapter. The relationship between the average strain $\hat{\varepsilon}_{ij}$ and the average stress $\hat{\sigma}_{ij}$ in the cracked solid has been shown by Horii and Nemat-Nasser[14] to be

$$\hat{\varepsilon}_{ij}=D_{ijkl}\hat{\sigma}_{kl}+\frac{1}{V}\int_S\frac{1}{2}([u_i]n_j+[u_j]n_i)dS=(D_{ijkl}+H_{ijkl})\hat{\sigma}_{kl}=\overline{D}_{ijkl}\hat{\sigma}_{kl},\quad i,j,k,l=1,2,3 \cdots\cdots\cdots (17)$$

where $[u_j]$ denotes the displacement gap along the crack surface with unit normal vector $n_i$, $D_{ijkl}$ is the elastic compliance of the uncracked solid, $\overline{D}_{ijkl}$ is the overall compliance, and the integration is carried over the crack surface $S$ contained in the solid of volume $V$. Once the displacement gap is known, the tensor $H_{ijkl}$ and then $\overline{D}_{ijkl}$ are obtained.

For the present, two-dimensional problem, the overall compliance matrix is written as

$$\begin{Bmatrix} \hat{\epsilon}_x \\ \hat{\epsilon}_y \\ 2\hat{\epsilon}_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{D}_{11} & \overline{D}_{12} & \overline{D}_{16} \\ \overline{D}_{21} & \overline{D}_{22} & \overline{D}_{26} \\ \overline{D}_{61} & \overline{D}_{62} & \overline{D}_{66} \end{bmatrix} \begin{Bmatrix} \hat{\sigma}_x \\ \hat{\sigma}_y \\ \hat{\tau}_{xy} \end{Bmatrix}, \text{ with } \overline{D}_{ij}=D_{ij}+H_{ij},\ i,j=1,2,6 \quad (18)$$

With the solution obtained in the previous section, the components of the tensor $H_{ij}$ for the solid containing a doubly periodic set of cracks are calculated as

$$H_{22}=\frac{(\varkappa+1)}{4G}\frac{\pi c^2}{HW}\left[1+\frac{c^2}{HW}H_{\mathrm{I}}(\phi)\right],\ H_{66}=\frac{(\varkappa+1)}{4G}\frac{\pi c^2}{HW}\left[1+\frac{c^2}{HW}H_{\mathrm{II}}(\phi)\right], \text{ otherwise } H_{ij}=0 \quad (19)$$

in which $H_{\mathrm{I}}(\phi)$ and $H_{\mathrm{II}}(\phi)$ are given by Eqn. (14). From Eqns. (18) and (19) the overall compliance of the solid is then given by

$$[\overline{D}_{ij}]=\frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1+2\frac{\pi c^2}{HW}\left[1+\frac{c^2}{HW}H_{\mathrm{I}}(\phi)\right] & 0 \\ 0 & 0 & 2(1+\nu)+2\frac{\pi c^2}{HW}\left[1+\frac{c^2}{HW}H_{\mathrm{II}}(\phi)\right] \end{bmatrix} \quad (20)$$

for the case of plane stress; and for plane strain we have

$$[\overline{D}_{ij}]=\frac{1+\nu}{E}\begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & (1-\nu)+2(1-\nu)\frac{\pi c^2}{HW}\left[1+\frac{c^2}{HW}H_{\mathrm{I}}(\phi)\right] & 0 \\ 0 & 0 & 2+2(1-\nu)\frac{\pi c^2}{HW}\left[1+\frac{c^2}{HW}H_{\mathrm{II}}(\phi)\right] \end{bmatrix} \quad (21)$$

The same results are obtained by evaluating the difference between the strain energy of the body with and without cracks subjected to the same applied stresses. Results obtained through Eqns. (20) and (21) are shown in Fig. 15 and 16.

It can be observed from Eqns. (20) and (21) that if the cracked solid is subjected to applied stresses, the macroscopic elastic response appears to be anisotropic. Since the characteristics of the functions $H_{\mathrm{I}}(\phi)$ and $H_{\mathrm{II}}(\phi)$ are different, the influences of the geometry of the array on $\overline{D}_{22}$ and $\overline{D}_{66}$ are not the same. For the solid undergoing shear deformation, holding the crack density $c^2/HW$ as a constant, $\overline{D}_{66}$ is seen to reach its minimum value when $\phi=\pi/4$. Under tension in the direction perpendicular to the crack faces, the interaction increases the stiffness of the solid when $\phi$ is less than $0.214\pi$ ($H/W$ being 0.80).

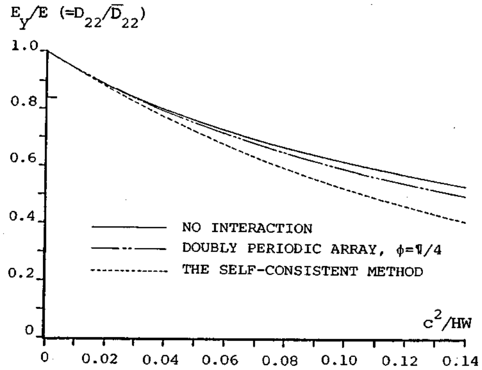


Fig. 15 $E_y/E$ as a function of the crack density.

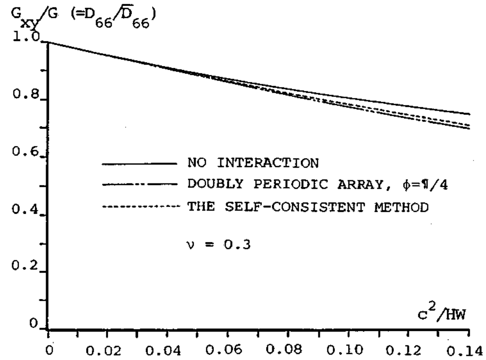


Fig. 16 $G_{xy}/G$ as a function of the crack density.

The overall elastic moduli of solids containing randomly distributed, unidirectional cracks evaluated by the self-cnsistent method are examined by comparison with Eqns. (20) and (21) as follows. Consider a single crack embeded in the infinite solid whose compliance is set to be the overall compliance $\overline{D}_{ij}$, which is the unknown to be determined. [Note that for a solid containing randomly distributed, unidirectional cracks, the overall response of the solid is anisotropic.] The displacement gap along the crack surface is expressed in terms of the unknown overall compliance by using the solution of a single crack in an anisotropic material given by Sih, Paris, and Irwin[29]. According to Eqns. (17) and (18), the displacement gap is integrated and then the overall compliance is obtained as

$$\overline{D}_{22}=D_{22}+\frac{\pi c^2}{HW}\sqrt{\{\overline{D}_{22}[2D_{12}+\overline{D}_{66}+2\sqrt{(D_{11}\overline{D}_{22})}]\}},\ \overline{D}_{66}=D_{66}+\frac{\pi c^2}{HW}\sqrt{\{D_{11}[2D_{12}+\overline{D}_{66}+2\sqrt{(D_{11}\overline{D}_{22})}]\}} \quad \cdots (22)$$

This equation is called the consistency condition for the unknown overall compliance $\overline{D}_{ij}$. An approximate solution is obtained through asymptotic expansions of the unknown variables with respect to a small parameter $c^2/HW$ as

$$\overline{D}_{22}=\frac{1}{E}+\frac{2}{E}\frac{\pi c^2}{HW}\left[1+\frac{c^2}{HW}\frac{3\pi}{2}\right],\ \overline{D}_{66}=2\frac{(1+\nu)}{E}+\frac{2}{E}\frac{\pi c^2}{HW}\left[1+\frac{c^2}{HW}\frac{\pi}{2}\right] \cdots (23)$$

for plane stress, and for the state of plane strain we have

$$\overline{D}_{22}=\frac{1-\nu^2}{E}+2\frac{(1-\nu^2)}{E}\frac{\pi c^2}{HW}\left[1+\frac{c^2}{HW}\frac{3\pi}{2}\right],\ \overline{D}_{66}=2\frac{(1+\nu)}{E}+2\frac{(1-\nu^2)}{E}\frac{\pi c^2}{HW}\left[1+\frac{c^2}{HW}\frac{\pi}{2}\right] \cdots (24)$$

Eqns. (23) and (24) are seen to be of the same form as those for the doubly periodic cracks, Eqns. (20) and (21), except that $H_{\mathrm{I}}(\phi)$ and $H_{\mathrm{II}}(\phi)$ are replaced by $3\pi/2$ and $\pi/2$, respectively. [Note that these terms account for the effects of crack interaction.]

From Fig. 14, the results by the self-consistent method for mode II are seen to be the lower bound of those for the doubly periodic cracks. The nearest value is attained when the array is of square shape; $\phi=\pi/4$ or $H=W$. This result seems to support the applicability of the self-consistent method to the problems of mode II deformation. On the other hand, Fig. 13 shows that for mode I the value of $H_{\mathrm{I}}(\phi)$ increases monotonically and takes the same value as that given by the self-consistent method when the array is far from the square; $\phi$ being $0.393\pi$ or $H/W=2.86$. This seems to indicate that the applicability of the self-consistent method to the problems of unidirectional cracks under mode I deformation is questionable.

The overall moduli for the plane stress obtained by means of the self-consistent method are shown in Figs. 15 and 16 in comparison with the results of the preceding sections. The rate of decrease of the modulus predicted by the self-consistent method are shown to be faster for the mode I deformation, comparing with the solution for the array having $\phi$ equal to $\pi/4$. Deforming under mode II does not show much differences between the rates of change of the overall properties obtained by either methods.

## 6. CONCLUSIONS

In the present study, it is revealed that the difficulty in solving the doubly periodic problems arises from the superposition principle. An approximate but explicit solution to the problem of doubly periodic cracks is derived. The stress intensity factors and the overall compliance are obtained as functions of the crack density and geometry of the crack array. The self-consistent method is applied to obtain the overall compliance of a solid containing randomly distributed, unidirectional cracks. It is shown that the solution of the doubly periodic cracks supports the validity of the self-consistent method for mode II. However, for mode I, the self-consistent method seems to be questionable.

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