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PROPERTIES OF COHERENCE FUNCTIONS AND MODIFICATION OF COMPUTATIONAL METHOD

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The difference of seismic waves observed on piers at both ends of a long span bridge or at locations along an underground pipe often becomes a critical factor and is taken into consideration for aseismic design of such structures. The characteristics of the effect of the difference may be described by a coherence function which represents the dependency of two time series, and it is calculated from the power and cross-spectral density functions. However, in its estimation, it is often smoothed in the frequency domain. In this paper, the effects of the smoothing on the coherence function estimate are analyzed quantitatively, in the case that a time delay is observed between the two records or that they are statistically independent. Then, a modified method of calculating coherence function estimates is proposed.
Keywords : coherence functions, spectrum analysis, waves

1. INTRODUCTION

Owing to a large increase in the size of recent structures, it is required to determine strong-motion earthquake inputs or storm wind forces not only at a single place but also at several places along the structure. This requirement is based on the fact, for example, that the seismic response of a long span bridge depends crucially upon the difference of seismic wave forms observed on piers at both ends of the bridge. And such time functions as earthquake generated ground acceleration and wind velocity have been analyzed based on the random vibration theory[1)-4)] dealing not only with a single time series but also with a multiple time series.

In the field of earthquake engineering, most of the strong-motion accelerograms have been obtained at a number of separate observing sites. And for earthquake-resistant design of general structures, dynamic analyses have been made by the use of such records. However, for structures with large foundations, it has recently been proposed that changes in characteristics of the seismic waves along the large base of the foundation should be taken into consideration. These changes are caused by the propagation of seismic waves, by the spatial variations in the elastic parameters of soils, etc. In order to investigate such changes, accelerograms of strong-motion events are recorded on arrays in several regions[5)-7)].

As an example, accelerograms written during the 1980 Central Chiba Prefecture earthquake and provided by the Tokyo Metropolitan Government[5)] are reproduced in Fig. 1. The upper trace is the horizontal component of ground acceleration measured by the seismograph at the observing site on the right bank of the Naka river. The lower trace is the same component on the opposite (left) bank. A comparison

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of the two sets of curves of Fig. 1 makes it evident that these two are similar to each other. However, it should be noticed that a time delay is observed between the arrival of the wave peak or trough at one site and the arrival of the corresponding one at the other site. Also, it should be noticed that these wave forms are slightly different from each other, namely, they are distorted.

From such observations of earthquake motions, not only the maximum amplitude and the power spectrum but also the direction of propagation and the speed of the wave have been obtained[8]. In addition, spectra of spatially variant ground motions and coherence functions have been examined[9)-13)].

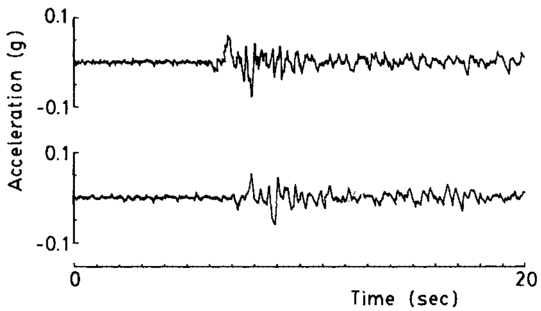

Fig. 1 Accelerograms written during the 1980 Central Chiba Prefecture earthquake and provided by the Tokyo Metropolitan Government (from Reference 5).

On the other hand, it has been pointed out that the deformation of the ground, in general, have a marked effect especially on the earthquake response of underground structures[10]. Therefore, it is important to evaluate such effects accurately. However, a general method for investigating the seismic effects on such structures is based on the assumption that the wave forms are not modified during their propagation. In practice, however, the wave is distorted as mentioned previously. Indeed, a recent investigation by the authors has examined in some detail such problems ; expressing the degree of distortion of the wave form in terms of a coherence function, the authors have evaluated the effects of such distortion on the ground strain[11].

As noted above, coherence functions have been used in a number of fields, such as earthquake, wind[12], and ocean[13] engineering fields. However, the properties or errors in estimates of such functions have not been examined sufficiently.

The purpose of this paper is to examine the effects of a few factors on the coherence function estimate. In Section 2, a general method for computing the coherence function estimate is reviewed briefly. In Sections 3 and 4, the following two important properties of the estimates, illustrated in Fig. 2, are described in detail :

(1) Even for the two time functions whose forms are equal, the obtained coherence estimate is not equal to unity when the time delay between them is not zero.

(2) Even when the two processes are statistically independent, the coherence function estimate is sometimes considerably different from zero.

The former result has been pointed out in part already by Kimura[13] using the records of ocean waves. As to the latter result, the errors introduced by using the time series of finite length have been examined, and the empirical results have been obtained[4].

In this paper, the effects of spectral windows used in the smoothing procedure on the coherence estimate are analyzed for several specific examples. Then, by the use of the obtained results, a modified method of calculating coherence function estimates is proposed in Section 5, and, as an example, it is applied to the actual data shown in Fig. 1.

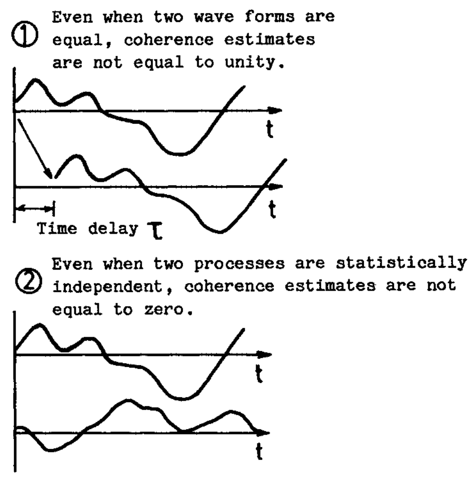

Fig. 2 Two important properties of coherence function estimates when smoothed by frequency averaging.

2. CONVENTIONAL METHOD OF CALCULATING COHERENCE FUNCTION ESTIMATES

(1) Definition

As one of the approximate methods for expressing the degree of change in the wave form with its propagation, a coherence function, $\mathrm{coh}^2(\omega)$, defined by Eq. 1 has often been used. (Precisely speaking, this function does not always correspond to the degree of difference between two wave forms ; if one time series is accurately expressed as a linear function of the other, then the coherence function equals unity, although these two wave forms are not equal.)

$$\mathrm{coh}^2(\omega)=\frac{|S_{xy}(\omega)|^2}{S_x(\omega)\,S_y(\omega)} \quad (1)$$

in which ω is the angular frequency ; $S_x(\omega)$ and $S_y(\omega)$ are the power spectral density functions of the time functions, $x(t)$ and $y(t)$, respectively ; and $S_{xy}(\omega)$ is the cross-spectral density function, namely,

$$S_x(\omega)=\frac{2\pi}{T}E[X^*(\omega)\,X(\omega)] \quad (2)$$

$$S_y(\omega)=\frac{2\pi}{T}E[Y^*(\omega)\,Y(\omega)] \quad (3)$$

$$S_{xy}(\omega)=\frac{2\pi}{T}E[X^*(\omega)\,Y(\omega)] \quad (4)$$

in which $E[\ \ ]$ indicates the ensemble average ; $*$ is the complex conjugate operation ; and $X(\omega)$ and $Y(\omega)$ are finite Fourier transforms of $x(t)$ and $y(t)$, respectively, over the record length T, namely,

$$X(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-T/2}^{T/2}x(t)\,e^{-i\omega t}\,dt \quad (5)$$

$$Y(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-T/2}^{T/2}y(t)\,e^{-i\omega t}\,dt \quad (6)$$

Mathematical work for the coherence function indicates that this function equals unity for all frequencies when $x(t)$ and $y(t)$ are fully coherent, and that this equals zero for all ω when they are statistically independent[2)~4)].

In what follows it will be assumed that direct Fourier transform procedures (and especially, fast Fourier transform [FFT] procedures) are used to determine the spectra, although several other spectral analysis procedures have been proposed[1)~4)].

In order to demonstrate the difficulties in getting an accurate determination of the coherence function from a pair of time functions, Bendat and Piersol[4)] have demonstrated the following result : If the power and cross-spectral density estimates are smoothed neither by ensemble nor frequency averaging, the obtained coherence estimate is always equal to unity, independent of the actual coherence function value. And this result has been explained as follows : If the spectral density estimates are not smoothed, they are expressed as modified Eqs. 2 through 4 in which the symbols, $E[\ \ ]$, are omitted. Substitution in Eq. 1 from the modified Eqs. 2 through 4 yields[4)]

$$\mathrm{coh}^2(\omega)=\frac{S_{xy}^*(\omega)\,S_{xy}(\omega)}{S_x(\omega)\,S_y(\omega)}=\frac{\left[\frac{2\pi}{T}X^*Y\right]^*\left[\frac{2\pi}{T}X^*Y\right]}{\left[\frac{2\pi}{T}X^*X\right]\left[\frac{2\pi}{T}Y^*Y\right]}=1 \quad (7)$$

Actually in most previous studies on the estimation of coherence functions, the average has been thought of as either across the ensemble or along frequency. And the average across the ensemble has been considered to be equivalent to the average along frequency. However, the errors introduced by such approximations have not been examined sufficiently.

(2) Smoothing Operations

Although a number of spectral windows have been proposed previously[1)~4)], in this paper, the "hanning" window expressed as

$$\hat{S}(\omega_k)=0.25S(\omega_{k-1})+0.5S(\omega_k)+0.25S(\omega_{k+1}) \quad (8)$$

is used as a fundamental spectral window. And by applying this fundamental window repeatedly, spectral windows with various widths of frequency bands are considered. Then the obtained windows are characterized by the parameter n reflecting the number of repetition of smoothing operations. The weights P_k used in the smoothing operation should satisfy the relations

$$\sum_{k=-n}^{n} P_k=1 \quad (9)$$

$$P_k=P_{-k} \quad (10)$$

The equivalent width $b(n)$ can be defined as[1)~4)]

$$b(n)=\frac{1}{\sum_{k=-n}^{n} P_k^2}\Delta\omega \quad (11)$$

in which $\Delta\omega=\omega_{k+1}-\omega_k$, and it equals $2\pi/T$ (T : the length of the record) when the FFT procedure is used. The weights P_k and the equivalent width $b(n)$ are illustrated in Fig. 3 for values of n ranging from 1 to 5. It should be noticed that the value of $b(n)$ increases with increasing values of n.

The coherence function estimate expressed in terms of spectral density estimates thus obtained is

$$\hat{\mathrm{coh}}^2(\omega_k)=\frac{|\hat{S}_{xy}(\omega_k)|^2}{\hat{S}_x(\omega_k)\hat{S}_y(\omega_k)} \quad (12)$$

in which ^ indicates the estimate derived by the frequency smoothing procedure.

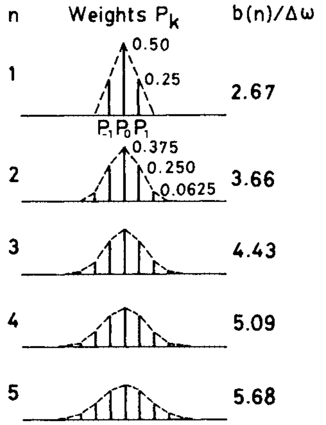

Fig. 3 Weights P_k and equivalent widths $b(n)$.

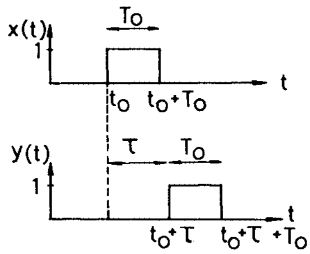

Fig. 4 Rectangular time functions with a time delay.

3. EFFECTS OF TIME DELAYS ON COHERENCE FUNCTION ESTIMATES

As one of the possible factors which might affect the coherence function estimates, the time delay between the two wave forms is treated in this section. For simplicity, however, the wave form is assumed, throughout this section, not to be modified during its propagation. In such cases, it is considered to be most convenient and practically desirable that a value of about one is obtained as a coherence estimate.

(1) Case for Rectangular Time Function

As one of the simple examples in which the wave form is not modified during its propagation, the case of the deterministic time functions, $x(t)$ and $y(t)$, illustrated in Fig. 4 and given as

$$x(t)=\begin{cases}1 & (t_0<t<t_0+T_0)\\ 0 & (t<t_0,\ t_0+T_0<t)\end{cases} \quad (13)$$

$$y(t)=\begin{cases}1 & (t_0+\tau<t<t_0+\tau+T_0)\\ 0 & (t<t_0+\tau,\ t_0+\tau+T_0<t)\end{cases} \quad (14)$$

is considered. Each consists of a single wave of a rectangle of a unit height and a width of T_0. And the difference in arrival times between the two places is assumed to be a fixed value τ. The following equations are derived from Eqs. 13 and 14 by Fourier transformation :

$$X(\omega)=\frac{1}{\sqrt{2\pi}}\int_{t_0}^{t_0+T_0} e^{-i\omega t}dt=\frac{1}{i\omega\sqrt{2\pi}}[e^{-i\omega t_0}-e^{-i\omega(t_0+T_0)}] \quad (15)$$

$$Y(\omega)=\frac{1}{\sqrt{2\pi}}\int_{t_0+\tau}^{t_0+\tau+T_0} e^{-i\omega t}dt=\frac{1}{i\omega\sqrt{2\pi}}[e^{-i\omega(t_0+\tau)}-e^{-i\omega(t_0+\tau+T_0)}] \quad (16)$$

Substitution from Eqs. 15 and 16 into Eqs. 2 through 4 yields

$$S_x(\omega)=S_y(\omega)=\frac{2}{T\omega^2}[1-\cos(\omega T_0)] \quad (17)$$

$$S_{xy}(\omega)=\frac{2}{T\omega^2}[1-\cos(\omega T_0)]e^{-i\omega\tau} \quad (18)$$

However, when, instead of by ensemble averaging, such estimates are smoothed by frequency averaging, they are given as

$$\begin{aligned}\hat{S}_x(\omega_k)&=\hat{S}_y(\omega_k)\\&=0.25\times\frac{2}{T\omega_{k-1}^2}[1-\cos(\omega_{k-1}T_0)]+0.5\times\frac{2}{T\omega_k^2}[1-\cos(\omega_kT_0)]\\&\quad+0.25\times\frac{2}{T\omega_{k+1}^2}[1-\cos(\omega_{k+1}T_0)] \quad (19)\end{aligned}$$

$$\begin{aligned}|\hat{S}_{xy}(\omega_k)|^2&=Re^2\{\hat{S}_{xy}(\omega_k)\}+Im^2\{\hat{S}_{xy}(\omega_k)\}\\&=0.25^2\left(\frac{2}{T\omega_{k-1}^2}\right)^2[1-\cos(\omega_{k-1}T_0)]^2+0.5^2\left(\frac{2}{T\omega_k^2}\right)^2[1-\cos(\omega_kT_0)]^2\\&\quad+0.25^2\left(\frac{2}{T\omega_{k+1}^2}\right)^2[1-\cos(\omega_{k+1}T_0)]^2\\&\quad+0.25\left(\frac{2}{T\omega_{k-1}\omega_k}\right)^2[1-\cos(\omega_{k-1}T_0)][1-\cos(\omega_kT_0)]\cos(\Delta\omega\tau)\\&\quad+0.25\left(\frac{2}{T\omega_k\omega_{k+1}}\right)^2[1-\cos(\omega_kT_0)][1-\cos(\omega_{k+1}T_0)]\cos(\Delta\omega\tau)\\&\quad+0.125\left(\frac{2}{T\omega_{k-1}\omega_{k+1}}\right)^2[1-\cos(\omega_{k-1}T_0)][1-\cos(\omega_{k+1}T_0)]\cos(2\Delta\omega\tau) \quad (20)\end{aligned}$$

in which $\Delta\omega$ is the width of an elementary angular frequency band, and when the FFT procedure is used, it is expressed as a function of the record length T, namely,

$$\Delta\omega=\omega_{k+1}-\omega_k=\frac{2\pi}{T} \quad (21)$$

Substitution in Eq. 12 from Eqs. 19 and 20 then leads to the coherence function estimate. It should be noticed that the last three terms of the right-hand side of Eq. 20 are functions of $\Delta\omega\tau$, which equals $2\pi(\tau/T)$ when the FFT procedure is used. For the case $\Delta\omega\tau=0$, $\cos(\Delta\omega\tau)=1$, and then the coherence function estimate equals unity. In this case, the coherence function expresses accurately the degree of distortion of the wave form. However, for the case $\Delta\omega\tau\neq0$, $\cos(\Delta\omega\tau)\neq1$, and then the coherence function estimate is not equal to unity but is less than unity, although the wave form is not modified. It may thus be concluded that the value of $\Delta\omega\tau$ is a factor which affects the errors in coherence function estimates. Of course, such results do not contradict the previous studies on stationary random processes. Indeed, it is clear from Eq. 21 that the value of $\Delta\omega$ decreases with increasing values of the record length T. And for infinite values of the record length T, which are always assumed when a stationary process is considered, $\Delta\omega=0$, and then the coherence function estimate increases to unity. However, in the practical analysis of seismic waves, the ratio of the time delay to the record length, τ/T, is not always equal to zero. This is a reason why such effects on the coherence function estimate should be examined in the following analysis.

Coherence function estimates obtained from Eqs. 12, 19 and 20 are presented in **Fig. 5** for specific values of $\Delta\omega\tau=0.5$, 1.0 and 2.0, and for different values of T_0. Inspection of **Fig. 5** shows that the coherence function estimates are kept constant for different angular frequencies and for different values of T_0, although its variability increases with increasing values of T_0. And it should be noticed that the coherence function estimate decreases with increasing values of $\Delta\omega\tau$; for values of $\Delta\omega\tau=0.5$, 1.0 and 2.0, the coherence function estimates are approximately 0.9, 0.8 and 0.3, respectively. The obtained results indicate that when the time delay is not zero, the coherence estimate is not equal to unity, but less than unity, even when the two wave forms are equal. And the coherence estimate is not significantly affected by

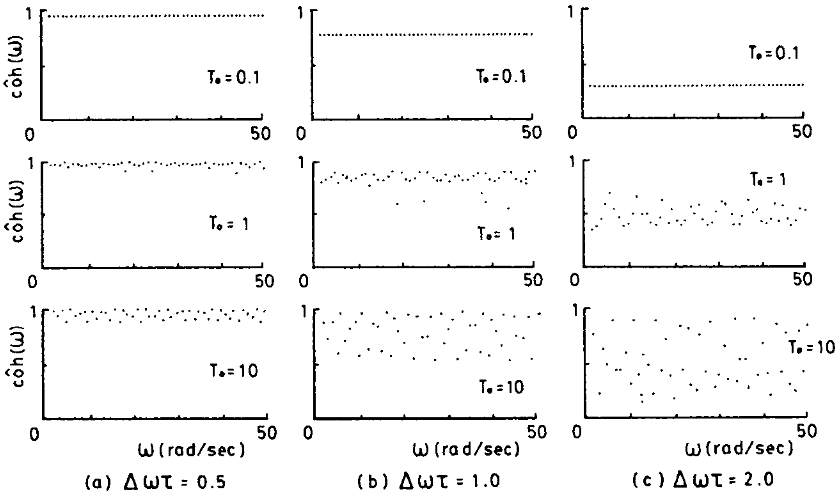

Fig. 5 Coherence estimates for rectangular time functions with time delays.

the difference in angular frequencies ω, but it is a function of $\Delta\omega\tau$, the product of the width of the elementary angular frequency band $\Delta\omega$ and the time delay τ.

(2) Case for Time Functions Having Power Spectrum $S(\omega)$

Consider the case in which a time function has a power spectrum $S(\omega)$ and the wave propagates without any modification of the form. The power and cross-spectra of such functions can be expressed as

$$S_x(\omega)=S_y(\omega)=S(\omega) \quad \cdots\cdots (22)$$

$$S_{xy}(\omega)=S(\omega)\,e^{-i\omega\tau} \quad \cdots\cdots (23)$$

When the estimates are smoothed by frequency averaging instead of by the ensemble averaging, the power spectral density estimates, and the real and the imaginary parts of the cross-spectral density estimates are expressed as

$$\hat{S}_x(\omega)=\hat{S}_y(\omega)=\sum_{k=-n}^{n} P_k S(\omega+k\Delta\omega) \quad \cdots\cdots (24)$$

$$Re\{\hat{S}_{xy}(\omega)\}=\sum_{k=-n}^{n} P_k S(\omega+k\Delta\omega)\cos\{(\omega+k\Delta\omega)\tau\} \quad \cdots\cdots (25)$$

$$Im\{\hat{S}_{xy}(\omega)\}=-\sum_{k=-n}^{n} P_k S(\omega+k\Delta\omega)\sin\{(\omega+k\Delta\omega)\tau\} \quad \cdots\cdots (26)$$

in which P_k represent the weights as shown in Fig. 3. From Eqs. 12, 24, 25 and 26, the coherence function estimate is then obtained as

$$\hat{\text{coh}}(\omega)=\frac{\sqrt{\sum_{k=-n}^{n}\sum_{l=-n}^{n} P_k P_l S(\omega+k\Delta\omega)\,S(\omega+l\Delta\omega)\cos\{(k-l)\Delta\omega\tau\}}}{\sum_{k=-n}^{n} P_k S(\omega+k\Delta\omega)} \quad \cdots\cdots (27)$$

The obtained value is less than unity when the time delay τ is not zero. This result is similar to that presented previously for the case of the rectangular time function. It should be noticed from Eq. 27 that the coherence estimate is affected not only by the product $\Delta\omega\tau$, but also by the shape of the power spectrum function $S(\omega)$. However, the latter effects will not be treated in detail in this paper.

(3) Case for Time Functions Having Smooth Power Spectrum

To be more specific, consider a time function whose power spectrum function is smooth and gently changing even before the estimate is smoothed either by ensemble or frequency averaging. This corresponds to the case in which white noise

$$S(\omega) = a \quad \cdots\cdots (28)$$

is considered within each range of the width of the spectral window. In particular, for the limiting case where $n=1$, substitution in Eq. 27 from Eq. 28 yields

$$\hat{\mathrm{coh}}(\omega) = \sqrt{0.375 + 0.5\cos(\Delta\omega\tau) + 0.125\cos(2\Delta\omega\tau)} \quad \cdots\cdots (29)$$

As has been pointed out earlier in this discussion, the coherence function estimates in Eqs. 27 and 29 are functions of $\Delta\omega\tau$; for the case $\Delta\omega\tau=0$, $\hat{\mathrm{coh}}(\omega)=1$, and for the case $\Delta\omega\tau \neq 0$, $\hat{\mathrm{coh}}(\omega)<1$.

Using Eq. 27, the effects of the value of $\Delta\omega\tau$ on coherence function estimates are studied as follows. First, coherence estimates are plotted in Fig. 6 as functions of angular frequency for specific values of $\Delta\omega = 1.0$ rad/sec and $\tau=0.5$ sec and for different values of n. It is clear from Fig. 6 that the coherence estimate is not significantly affected by the angular frequency. Then, such estimates are plotted in Fig. 7 as functions of $\Delta\omega\tau$ for different values of n. For any values of n, for the case $\Delta\omega\tau=0$, $\hat{\mathrm{coh}}(\omega)=1$. However, the value of $\hat{\mathrm{coh}}(\omega)$ decreases with increasing values of $\Delta\omega\tau$. And as the value of n is increased, a marked decrease in the value of $\hat{\mathrm{coh}}(\omega)$ is shown; for example, when $\Delta\omega\tau=1.0$, the coherence estimates for the cases of $n=1$, 2, 4, and 8 are $\hat{\mathrm{coh}}(\omega)=0.77$, 0.59, 0.34 and 0.12, respectively. Considering the fact that the equivalent width of the spectral window, $b(n)$, increases with increasing values of n and $\Delta\omega$ (see Fig. 3 and Eq. 11), it may thus be concluded that Fig. 7 indicates that the coherence estimate decreases with increasing values of the equivalent width $b(n)$ and the time delay τ.

(4) Effects of Widths of Spectral Windows

In this section, in order to study analytically on the effects of widths of spectral windows, a continuous spectral window is used. Now consider the case in which waves propagate without being modified; the power spectrum of the seismic wave recorded at each place is assumed to be a fixed function $S(\omega)$, and the time delay between the two records is assumed to be a fixed value τ. In such a case, the cross-spectral density function $S_{xy}(\omega)$ is given by Eq. 23. For simplicity, consider the case in which such a time function is represented as a white noise process.

$$S_x(\omega) = S_y(\omega) = S(\omega) = a \quad \cdots\cdots (30)$$

As a spectral window $Q(\omega)$, consider a rectangle of width B.

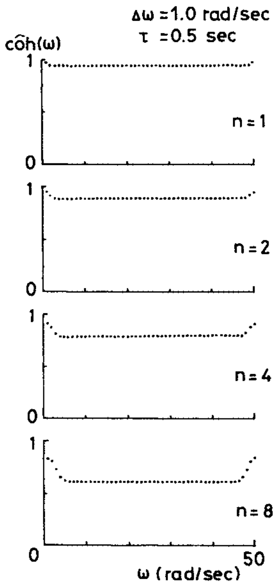

Fig. 6 Coherence function estimates for white noise processes with a time delay.

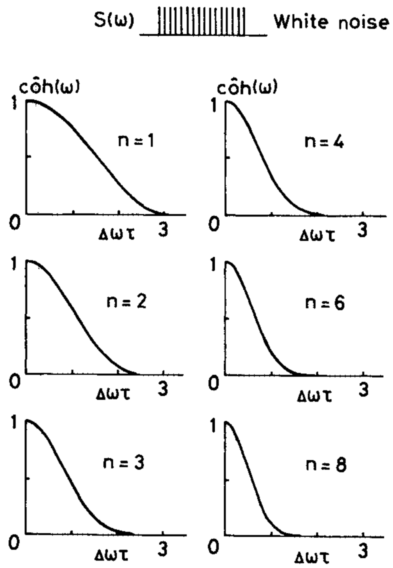

Fig. 7 Effects of time delays on coherence estimates.

$$Q(\omega)=\begin{cases} 1/B & (|\omega|\leqq B/2) \\ 0 & (|\omega|>B/2) \end{cases} \quad \cdots\cdots (31)$$

By frequency averaging, the estimates for power and cross-spectra are given by

$$\hat{S}_x(\omega)=\hat{S}_y(\omega)=a \quad \cdots\cdots (32)$$

$$\hat{S}_{xy}(\omega)=\int_{-B/2}^{B/2}\frac{1}{B}\,a\exp\{-i(\omega+\omega')\tau\}\,d\omega'=a\frac{\sin(B\tau/2)}{B\tau/2}\exp(-i\omega\tau) \quad \cdots\cdots (33)$$

Then, the coherence function estimate is obtained as

$$\hat{\mathrm{coh}}(\omega)=\left|\frac{\sin(B\tau/2)}{B\tau/2}\right| \quad \cdots\cdots (34)$$

It should be noticed that for small values of $B\tau$, $\hat{\mathrm{coh}}(\omega)\doteqdot 1$, however, that for higher values of $B\tau$, smaller values of the coherence estimate are observed, although the wave is not modified.

It may thus be concluded that when the time delay is not zero, the coherence estimate is not equal to unity even when the wave forms are not modified. This value is affected by the value of $\Delta\omega\tau$ and by the number of smoothing operations, n. In other words, this value is affected by the time delay τ and by the width of the spectral window, B. The coherence estimate decreases with increasing values of the two parameters just described in two ways.

4. EFFECTS OF NUMBER OF SMOOTHING OPERATIONS ON COHERENCE ESTIMATES FOR STATISTICALLY INDEPENDENT PROCESSES

When the coherence function is considered to be used as a measure of the degree of distortion of the wave form, it is most convenient and practically desirable to estimate this function under the following conditions : For the case of processes of equal wave form, the coherence estimate should be equal to unity ; and for statistically independent processes, this value should become vanishingly small. In practice, however, the obtained coherence estimate is not about zero, even when the two processes are statistically independent from each other. In this paper, such statistically independent processes are defined as those having fixed power spectrum and having independent phase angles at each frequency. However, phases of different frequencies are not assumed to be statistically independent. In what follows, for such processes, coherence function estimates will be studied in relation to the spectral windows.

(1) Case for Power Spectra of $S_x(\omega)$ and $S_y(\omega)$

From Eqs. 2 through 6 the cross-spectral density function of two statistically independent processes is given as

$$S_{xy}(\omega)=E\left[\sqrt{S_x(\omega)S_y(\omega)}\exp[i\{\psi(\omega)-\phi(\omega)\}]\right] \quad \cdots\cdots (35)$$

in which $\psi(\omega)$ and $\phi(\omega)$ represent the phase angles of the time functions, $x(t)$ and $y(t)$, respectively. As mentioned previously, independent phases are assumed for each value of ω. When the cross-spectral density estimate is determined from Eq. 35 by the ensemble averages, it is obviously zero.

However, when it is determined by frequency averaging instead of by ensemble averaging, the power and cross-spectral density estimates and the coherence function estimate are given by

$$\hat{S}_x(\omega)=\sum_{k=-n}^{n}P_kS_x(\omega+k\Delta\omega) \quad \cdots\cdots (36)$$

$$\hat{S}_y(\omega)=\sum_{k=-n}^{n}P_kS_y(\omega+k\Delta\omega) \quad \cdots\cdots (37)$$

$$Re[\hat{S}_{xy}(\omega)]=\sum_{k=-n}^{n}P_k\sqrt{S_x(\omega+k\Delta\omega)S_y(\omega+k\Delta\omega)}\cos\{\psi(\omega+k\Delta\omega)-\phi(\omega+k\Delta\omega)\} \quad \cdots\cdots (38)$$

$$Im[\hat{S}_{xy}(\omega)]=\sum_{k=-n}^{n}P_k\sqrt{S_x(\omega+k\Delta\omega)S_y(\omega+k\Delta\omega)}\sin\{\psi(\omega+k\Delta\omega)-\phi(\omega+k\Delta\omega)\} \quad \cdots\cdots (39)$$

$$\hat{\text{coh}}^2(\omega)=\frac{\sum_{k=-n}^{n}\sum_{l=-n}^{n}[P_kP_l\sqrt{S_x(\omega+k\Delta\omega)S_y(\omega+k\Delta\omega)S_x(\omega+l\Delta\omega)S_y(\omega+l\Delta\omega)}}{\left\{\sum_{k=-n}^{n}P_kS_x(\omega+k\Delta\omega)\right\}}\cdot$$

$$\cdot\frac{\cos\{\psi(\omega+k\Delta\omega)-\psi(\omega+l\Delta\omega)-\phi(\omega+k\Delta\omega)+\phi(\omega+l\Delta\omega)\}]}{\left\{\sum_{k=-n}^{n}P_kS_y(\omega+k\Delta\omega)\right\}} \qquad (40)$$

It should be noticed that the coherence function estimate in Eq. 40 is still expressed in terms of the phase angles, $\psi(\omega)$ and $\phi(\omega)$. And then the estimate varies from sample to sample. In this sense, averages across the ensemble are not equivalent to averages along frequency. And smoothing over frequency cannot be substituted for smoothing over an ensemble. Moreover, the assumption of independent phases has not been used yet. Then, the ensemble average of the coherence estimate expressed in Eq. 40 is calculated (see Eq. 42). And it is considered to be necessary that such average values should be equal to zero in the case of statistically independent processes. A detailed discussion will be presented in what follows. When $\psi(\omega)$ and $\phi(\omega)$ are statistically independent,

$$E[\cos\{\psi(\omega+k\Delta\omega)-\psi(\omega+l\Delta\omega)-\phi(\omega+k\Delta\omega)+\phi(\omega+l\Delta\omega)\}]=\begin{cases}0 & \text{for } k\neq l\\ 1 & \text{for } k=l\end{cases} \qquad (41)$$

in which $E[\ ]$ represents the ensemble average. Substituting Eq. 41 into Eq. 40 gives

$$E[\hat{\text{coh}}^2(\omega)]=\frac{\sum_{k=-n}^{n}P_k^2S_x(\omega+k\Delta\omega)\,S_y(\omega+k\Delta\omega)}{\left\{\sum_{k=-n}^{n}P_kS_x(\omega+k\Delta\omega)\right\}\left\{\sum_{k=-n}^{n}P_kS_y(\omega+k\Delta\omega)\right\}} \qquad (42)$$

It should be noticed that the average value of coherence estimates is not equal to zero but is positive, even for statistically independent processes. This value is the bias of the estimate, and is given as a function of the shape of the spectral window and also of the power spectra of the processes.

(2) Case for White Noise Processes

In the case of white noise processes, substitution in Eq. 42 from Eq. 9 yields

$$E[\hat{\text{coh}}^2(\omega)]=\sum_{k=-n}^{n}P_k^2 \qquad (43)$$

For white noise processes, the average value of coherence estimates depends only on the shape of the spectral window. Such values are shown by the solid lines in Fig. 8 as a function of the number n. When $n=1$, $E[\hat{\text{coh}}^2(\omega)]=0.375$, and when $n=2$, $E[\hat{\text{coh}}^2(\omega)]=0.273$. Although such values decrease with increasing values of n, they are not negligible for small values of n.

(3) Lower Bound for Average Value of Coherence Estimates

In particular, for the case of two records having the equal power spectral density function, the value of $E[\hat{\text{coh}}^2(\omega)]$ is examined as follows. First, when the product, $P_kS(\omega+k\Delta\omega)$, in Eq. 42 is expressed as R_k, Eq. 42 may be reduced to the form

$$E[\hat{\text{coh}}^2(\omega)]=\frac{\sum_{k=-n}^{n}R_k^2}{\left\{\sum_{k=-n}^{n}R_k\right\}^2} \qquad (44)$$

The value of $E[\hat{\text{coh}}^2(\omega)]$ increases with an increase in the sum of the squares of R_k over all k (see numerator in Eq. 44) for a fixed value of the sum of R_k over all k (see denominator in Eq. 44). Namely, it increases with increasing values of the coefficients of variation of R_k until ultimately it approaches a

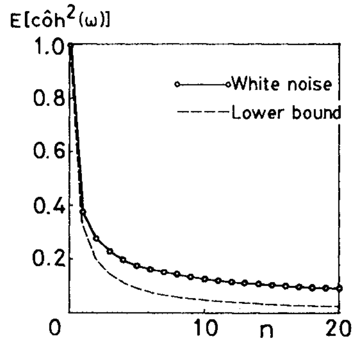

Fig. 8 Effects of numbers of smoothing operations on average values of coherence estimates for statistically independent processes.

maximum value of unity. Meanwhile, when R_k is the same for all k, it reaches a minimum value

$$E[\hat{\text{coh}}^2(\omega)]=1/(2n+1) \qquad (45)$$

This minimum value is a function only of n, and is shown by the dashed lines in Fig. 8.

5. MODIFIED METHOD OF CALCULATING COHERENCE ESTIMATES AND NUMERICAL EXAMPLES

In the preceding sections, the two problems described briefly in Section 1 (1) (2) have been examined. And it has been shown that such problems may arise when the conventional computational method is used. Therefore, in order to describe the degree of distortion of the wave form by the coherence function, and for ease of interpretation of the obtained results, the conventional method is desired to be modified. And in order to compensate for the imperfections in the conventional method, the following procedures are proposed :

(1) Before the conventional method for calculating coherence functions is applied, the cross-correlogram between the two records should be calculated. And by noting the observed peak in the cross-correlogram, the records should be shifted along the time axis so that the time delay becomes equal to zero.

(2) Considering that the average value of coherence estimates for statistically independent processes should be equal to zero, the obtained estimate should be modified as follows :

$$\underline{\text{coh}}^2(\omega)=\frac{\hat{\text{coh}}^2(\omega)-E[\hat{\text{coh}}^2(\omega)]}{1-E[\hat{\text{coh}}^2(\omega)]} \qquad (46)$$

in which $\hat{\text{coh}}^2(\omega)$ is the coherence estimate by the use of the conventional equation (see Eq. 12), and $E[\hat{\text{coh}}^2(\omega)]$ represents the average value of coherence estimates for statistically independent processes (see Eq. 42 or 43).

As a numerical example, coherence function estimates are obtained for the two records shown in Fig. 1. And the effects of the two kinds of modifications are investigated. The sample estimates of $\hat{\text{coh}}^2(\omega)$ determined by the conventional procedure are shown in Fig. 9. Fig. 10 shows the results for the case where only the first modification is made, and Fig. 11 for the case where both modifications are made. For each of these three cases, the results are presented for $n=1$ and 8. A comparison of Figs. 9, 10 and 11 indicates the effects of each of these two modifications. In Fig. 9, it should be noticed that the coherence estimates are considerably different from unity for lower frequencies, in particular, for $n=8$. And no apparent relation between the estimates and frequency is derived. In Fig. 10, when only the first modification is made, the estimates become larger than those in Fig. 9. And for lower frequencies, the estimates are

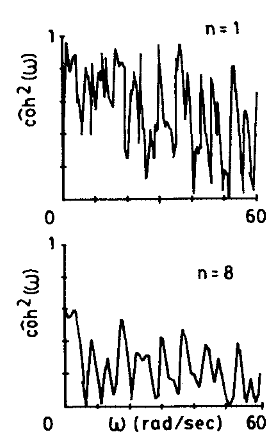

Fig. 9 Coherence estimates by the conventional method.

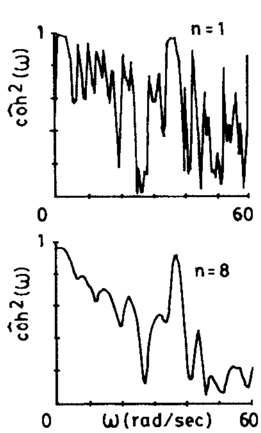

Fig. 10 Coherence estimates when only the first modification is made.

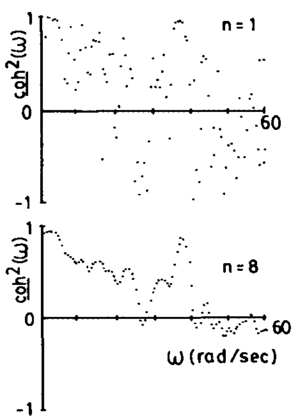

Fig. 11 Coherence estimates when both modifications are made.

approximately equal to unity. And the estimates exhibit a marked decrease with frequency. Such results are in agreement with those presented by Ishii[9]. However, the estimate for $n=1$ is considerably different from that for $n=8$, especially for high frequencies ; for example, at frequencies near $\omega=50$ rad/sec, the coherence estimates are about 0.3 for $n=1$, while for $n=8$ they are about 0.1. Such discrepancy seems to be due to the errors introduced by bias in the coherence estimates. On the contrary, the similarity between the coherence estimates for $n=1$ and those for $n=8$ can be pointed out in Fig. 11, although the variance of the estimate is larger for $n=1$ than that for $n=8$. It should also be noticed that the two records are totally independent or incoherent at frequencies near 50 rad/sec. Considering that the estimate determined by the conventional procedure is always positive, it may be concluded that a less biased estimate seems to have been obtained by the proposed modified method.

6. CONCLUSIONS

(1) When the time delay between the two records is not zero, the coherence function is underestimated. In particular, when the time delay is not zero, the coherence estimate is not equal to unity, even when these two records are exactly equal to each other. This value is affected by the value $\Delta\omega\tau$ and by the number of smoothing operations n. In other words, this value is affected by the time delay τ and by the width of the spectral window, B. The coherence estimate decreases with increasing values of the two parameters just described in two ways.

(2) Even when the two processes are statistically independent, the coherence function estimate is not equal to zero but is always positive. The bias of the estimate is a function of the number of smoothing operations.

(3) The conventional method for calculating coherence estimates should be modified as follows : (a) Before the conventional method is applied, the records should be shifted along the time axis so that the time delay becomes equal to zero ; and (b) Using the average value of coherence estimates for statistically independent processes, the coherence function estimate should be improved.

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