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ON CONVERGENCE OF GEOMETRICALLY NONLINEAR DISCRETIZATION AT LIMIT ELEMENT DIVISION

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By focusing on to what extent strain energies are approximated, this paper intends to derive the conditions for a discrete model to converge at the limit element division to the original finite-displacement elasticity. A criterion for that convergence is presented in a general form. Further, especial expansions are made to find out which discretized relations might be linearized at the limit. As a result, a natural classification of discretizations for finite-displacement small-strain problems is proposed.

Keywords : discretization, geometrically nonlinear, convergence

1. INTRODUCTION

Regardless of small or large displacements, being of a variational problem is one of the most fundamental properties of an elastic deformation. Therefore, a set of governing equations formulated are examined on their completeness by seeing whether they are mathematically of a variational problem. For instance, there are various theories for the beam bending such as the primary linear theory, the beam-column theory and recent formulations for truely large displacements[e.g. 1),10),13)]. While these theories are suject to their own restrictions on displacements and/or strains regarding the actual phenomena, it is to be noted that even beyond those restrictions, the governing equations of the above theories remain to be variational problems, respectively. Thus, whether or not a set of governing equations are of a variational problem depends upon analytic relations between relevant functions and/or functionals, but is not affected by how large values they take, in general. Besides the actual restrictions, the preservation of being a variational problem is a basic necessity for the consistency among governing equations. In the recent analyses with using digital computers, various discretizations of geometrically-nonlinear-elasticity problems were made. While simplifying rigorous relations is sometimes effective to economically obtain useful results, in some existing such discretizations, however, we can find the absurdities to yield variational problems. Obviously, remaining of a variational problem is necessary for the completeness also in a set of discretized relations, since it is so in the original elasticity.

A time has past since we recognized it useful in discretizing the geometrically nonlinear problems to separate the degrees of total freedom of an element into the parameters of its rigid position and those of its

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deformation. There exist now many researches and numerical calculations made on that approach[e.g. 1),3)~6),8),9),11),12),14),16)]. Recently, a few studies on those discretizations as for the beams were presented to find out what simplifications can be made to obtain the exact solutions at infinitesimal divisions[5),6)]. Soon later, a discussion about the conclusion of those studies was given[2)]. In this paper, further considerations are made to attain, in a general form, a criterion for that convergence. According to an especial consideration made further on which discrete relations might be linearized at the limit, a natural classification of the discretizations for finite-displacement small-strain problems is proposed.

2. COMPATIBILITIES IN DISCRETIZATION

Assume a set of differential governing equations for a finite-displacement elastic continuum are prescribed, where the body's spatial configuration is described in terms of a finite number of functions of some or all of the Lagrangean coordinates, $\{\chi^{\delta}(\zeta^{\lambda}),\ \delta=1,\cdots,\Delta,\ \lambda=1,\cdots,\Lambda\}$ (fundamental unknowns). Denoting the remaining convected coordinates by $\{s^{\sigma},\ \sigma=1,\cdots,\Sigma\}$ where $\Lambda+\Sigma=3$, a set of $\{\xi^1,\ \xi^2,\ \xi^3\}=\{(\zeta^{\lambda}),\ (s^{\sigma})\}$ are complete as 3-D (dimensional) Lagrangean coordinates : for a 3-D field problem, $\{\xi^{\alpha}\}=\{(\zeta^1,\ \zeta^2,\ \zeta^3)\}$; and for a problem of 2-D or 1-D kinematic field, $\{\xi^{\alpha}\}=\{(\zeta^1,\ \zeta^2),\ (s^1)\}$ or $=\{(\zeta^1),\ (s^1,\ s^2)\}$, respectively. Let the domain of $\{\zeta^{\lambda}\}$ in the body be denoted by L, and the domain of $\{s^{\sigma}\}$ at each $\{\zeta^{\lambda}\}$, by $C(\zeta^{\lambda})$.

In those equations, we can make the following summarizations : A set of coordinates $\{\zeta^{\lambda}\}$ which will be called a *point* in kinematic field $\{\chi^{\delta}(\zeta^{\lambda})\}$ refer to a material point ($\Lambda=3$) or a set of such points on $C(\zeta^{\lambda})$ ($\Lambda<3$). In the kinematic field, we can consider such a set of parameters $\boldsymbol{X}(\zeta^{\lambda})$, which will be called a *spatial position* of point $\{\zeta^{\lambda}\}$, that their values at $\{\zeta^{\lambda}\}$ determine spatial positions of all the material points on $C(\zeta^{\lambda})$. On the other hand, the spatial configuration of domain $C(\zeta^{\lambda})$ is to be determined by the analyticities of functions $\{\chi^{\delta}(\zeta^{\lambda})\}$ at $\{\zeta^{\lambda}\}$. Thus, in general, $\boldsymbol{X}(\zeta^{\lambda})$ is related to $\{\chi^{\delta}(\zeta^{\lambda})\}$ and their derivatives in the form :

$$\boldsymbol{X}(\zeta^{\lambda})=\boldsymbol{D}_X\{\chi^{\delta}(\zeta^{\lambda})\} \quad \cdots\cdots (1)$$

where $\boldsymbol{D}_X$ is a set of nonlinear differential operators on $\{\zeta^{\lambda}\}$-field. It is to be mentioned that the parameters of $\boldsymbol{X}$ might take any values at a point, but they can not distribute independently over domain L. For, they are restricted derivable by (1) from differentiable $\{\chi^{\delta}(\zeta^{\lambda})\}$. Similarly, we can consider a set of *strain parameters* $\boldsymbol{\phi}(\zeta^{\lambda})$ whose values at $\{\zeta^{\lambda}\}$ describe the strain distributions on $C(\zeta^{\lambda})$. These parameters are also related to $\{\chi^{\delta}(\zeta^{\lambda})\}$ in the form :

$$\boldsymbol{\phi}(\zeta^{\lambda})=\boldsymbol{D}_{\phi}\{\chi^{\delta}(\zeta^{\lambda})\} \quad \cdots\cdots (2)$$

where $\boldsymbol{D}_{\phi}$ is a set of differential operators, usually containing once higher derivatives than $\boldsymbol{D}_X$. If $\boldsymbol{\phi}=$ *const.* in a domain of $\{\zeta^{\lambda}\}$, we call it a *constant-stain* state of the domain in the kinematic field. For given $\boldsymbol{\phi}$ at $\{\zeta^{\lambda}\}$, the conventional strains, i. e. the Green's strain components $\boldsymbol{e}(\xi^{\alpha})$, at material points of $C(\zeta^{\lambda})$ are assigned by their $\{s^{\sigma}\}$-coordinates :

$$\boldsymbol{e}(\xi^{\alpha})=\boldsymbol{\Gamma}_e(\boldsymbol{\phi}\,;\,s^{\sigma}) \quad \cdots\cdots (3)$$

With the above preliminaries, we consider the following discretization of the original continuum : Among various ways of element subdivisions, we choose such one that for any M of an infinite sequence $\{M\}$ of ordered natural numbers, a division into M elements is uniquely given, and that each element domain vanishes at limit $M\to\infty$:

$$\lim_{M\to\infty}\|\Delta\zeta^{\lambda}\|_{(e)}=0 \quad \cdots\cdots (4)$$

where $\|\Delta\zeta^{\lambda}\|_{(e)}$ denotes the norm of maximum differences of $\{\zeta^{\lambda}\}$ within element (e). Let the size of an element be denoted by

$$L_{(e)}\left(=\int_{(e)}dL\right)=\int\cdots\int_{(e)}d\zeta^1\cdots d\zeta^{\Lambda} \quad \cdots\cdots (5)$$

and $L=\sum_{e=1}^{M}L_{(e)}$: the total size (we use L to denote both a domain of $\{\zeta^{\lambda}\}$ and its size). Let the joints associated to M-division be numbered as $j=1,\cdots,N$.

The spatial positions of the joints are called *joint positions* and denoted by $\{X_j\}=\{X_j,\ j=1,\cdots,N\}$. Let a set of those X_j which lie on element (e), or a set of parameters in a regular transformation from those, be denoted by $\{X\}_{(e)}$ and called an *element position*. While, to yield a discrete model, each element is to be geometrically interpolated in terms of its $\{X\}_{(e)}$, we further separate the parameters of $\{X\}_{(e)}$ into the two sets of parameters, $\boldsymbol{v}_{(e)}$ and $\boldsymbol{\epsilon}_{(e)}$, to define its *rigid position* and *deformation*, respectively :

$$\left.\begin{aligned} \boldsymbol{v}_{(e)}&=\boldsymbol{\Gamma}_v(\{X\}_{(e)}) \\ \boldsymbol{\epsilon}_{(e)}&=\boldsymbol{\Gamma}_\epsilon(\{X\}_{(e)}) \end{aligned}\right\} \qquad (6\cdot\text{a, b})$$

For instance, after constraining the element's rigid displacements by applying a certain statically-determinate support at its joints, we can take the remaining degrees of freedom in $\{X\}_{(e)}$ as $\boldsymbol{\epsilon}_{(e)}$, and the spatial position of the support itself as $\boldsymbol{v}_{(e)}$[1]. In case of this separation, first, a deformed configuration of (e) is assigned for $\boldsymbol{\epsilon}_{(e)}$ through a certain interpolation, and, next, by translating and rotating it as a rigid according to $\boldsymbol{v}_{(e)}$, we obtain a final spatial configuration for $\{X\}_{(e)}$. Let the configuration be denoted by $\{\hat{\chi}^\sigma(\zeta^\lambda)\}_{(e)}$, $\{\zeta^\lambda\}\in L_{(e)}$ in the same way to $\{\chi^\sigma(\zeta^\lambda)\}$, where superimposed ^ means a quantity after discretization.

For any $\{\chi^\sigma(\zeta^\lambda)\}$, we estimate with the use of (1) the associated joint positions, and, by the interpolation, we reconstruct another configuration, $\{\hat{\chi}^\sigma(\zeta^\lambda)\}_{(e)}$, $\{\zeta^\lambda\}\in L_{(e)}$, $e=1,\cdots,M$. Here, we assume the following compatibility in the discretization : when the spatial positions of the joints are estimated still more for the interpolated configuration, they precisely coincide at limit $M\to\infty$ with the joint positions used to produce the $\{\hat{\chi}^\sigma(\zeta^\lambda)\}_{(e)}$.

3. CONVERGENCE CONDITIONS

The strain energy of a continuum subject to the kinematic field is a functional of $\{\chi^\sigma(\zeta^\lambda)\}$, and the functional reflects any elastic properties. Then, by whether or not the sequence of M-division models attains to reproduce the exact strain energies stored in any configurations of the continuum, we can see the convergence of the discretization at limit $M\to\infty$ to the original elasticity. This has been a conventional way of the proof for linear elasticity problems[e.g. 7), 15)].

By denoting the conventional strain-energy-density function related to $\boldsymbol{e}(\xi^\alpha)$ by $\Phi(\boldsymbol{e})$, the associated stress tensor is represented as

$$\boldsymbol{\sigma}(\boldsymbol{e})=(\partial\Phi/\partial\boldsymbol{e})^T \qquad (7)$$

In the kinematic field, we define *stress resultants* $\boldsymbol{M}(\zeta^\lambda)$ conjugate to $\boldsymbol{\psi}(\zeta^\lambda)$ such that inner product $(\boldsymbol{M}\cdot\delta\boldsymbol{\psi})\,dL$ gives the real work done in domain $C(\zeta^\lambda)\times dL$ by the stresses during infinitesimal $\delta\boldsymbol{\psi}$ from $\boldsymbol{\psi}$. By relation (3) and equation $\left(\int_C \boldsymbol{\sigma}\cdot\delta\boldsymbol{e}dC\right)dL=(\boldsymbol{M}\cdot\delta\boldsymbol{\psi})\,dL$, the constitutive relations between $\boldsymbol{M}$ and $\boldsymbol{\psi}$ are represented as

$$\boldsymbol{M}(\boldsymbol{\psi})=\int_{C(\zeta^\lambda)}[\partial\boldsymbol{\Gamma}_e/\partial\boldsymbol{\psi}]^T\boldsymbol{\sigma}dC \qquad (8)$$

Strain-energy-density function $A(\boldsymbol{\psi})$ defined by $\delta A=\boldsymbol{M}\cdot\delta\boldsymbol{\psi}$ on $\{\zeta^\lambda\}$-field is represented in terms of $\Phi(\boldsymbol{e})$ as

$$A(\boldsymbol{\psi})=\int_{C(\zeta^\lambda)}\Phi(\boldsymbol{\Gamma}_e(\boldsymbol{\psi}\,;\,s^\sigma))\,dC \qquad (9)$$

Here, as an allowable configuration subject to the kinematic field, we assume any $\{\chi^\sigma(\zeta^\lambda)\}$ for which $A(\boldsymbol{\psi})$ is (piecewise) continuous in L.

In the discrete model, we define *deformation force* $\boldsymbol{f}_{(e)}$ of element, conjugate to deformation $\boldsymbol{\epsilon}_{(e)}$, such that for infinitesimal $\delta\boldsymbol{\epsilon}_{(e)}$ from $\boldsymbol{\epsilon}_{(e)}$, inner product $\boldsymbol{f}_{(e)}\cdot\delta\boldsymbol{\epsilon}_{(e)}$ gives the real internal work during $\delta\boldsymbol{\epsilon}_{(e)}$ done by the forces acting at the element's joints. Similarly, *element force* $\{F\}_{(e)}$ is defined such that for $\delta\{X\}_{(e)}$ from $\{X\}_{(e)}$, $\{F\}_{(e)}\cdot\delta\{X\}_{(e)}$ represents the real external work done by the forces at the joints.

In an interpolated element with the separation of $\{X\}_{(e)}$ into $\boldsymbol{v}_{(e)}$ and $\boldsymbol{\epsilon}_{(e)}$, the distribution of strain parameters $\hat{\boldsymbol{\psi}}(\zeta^\lambda)$ is determined by deformation $\boldsymbol{\epsilon}_{(e)}$, and we obtain its strain energy in terms of $\boldsymbol{\epsilon}_{(e)}$:

$$\hat{U}_{(e)}(\boldsymbol{\epsilon})=\int_{L(e)} A(\hat{\boldsymbol{\psi}}(\boldsymbol{\epsilon}\,;\,\zeta^{\lambda}))\,dL \quad (10)$$

Or, apart from the geometrical interpolation, by directly assuming a (convex) scalar function $\hat{U}_{(e)}(\boldsymbol{\epsilon})$, we can suppose an associated elastic element. From the above definition of $\boldsymbol{f}_{(e)}$, we can relate $\boldsymbol{f}_{(e)}$ to $\hat{U}_{(e)}(\boldsymbol{\epsilon})$ as

$$\boldsymbol{f}_{(e)}(\boldsymbol{\epsilon})=(\partial\hat{U}/\partial\boldsymbol{\epsilon})^{T}_{(e)} \quad (11)$$

By considering the statics as a rigid on the deformed configuration (or by taking account of the reactions in the statically-determinate support) (: $[Q'_{F(e)}]$), and by rotating the force components according to $\boldsymbol{v}_{(e)}$ (: $[T_{F(e)}]$), we can relate the element force at arbitrary $\{\boldsymbol{X}\}_{(e)}$ to $\boldsymbol{f}_{(e)}$ in the form[1] :

$$\left.\begin{aligned}\{\boldsymbol{F}\}_{(e)}&=[Q_{F(e)}(\{\boldsymbol{X}\})]\,\boldsymbol{f}_{(e)}\\ [Q_{F(e)}]&=[T_{F(e)}(\boldsymbol{v})][Q'_{F(e)}(\boldsymbol{\epsilon})]\end{aligned}\right\} \quad (12\cdot\text{a, b})$$

It is to be noted that matrix $[Q_{F(e)}]$ is not indpendent of the $\epsilon_{(e)}-\{\boldsymbol{X}\}_{(e)}$ relation : by denoting the derivatives of (6·b) as

$$\delta\epsilon_{(e)}=[Q_{X(e)}(\{\boldsymbol{X}\})]\,\delta\,\{\boldsymbol{X}\}_{(e)} \quad (13)$$

from the equation of virtual work, $\{\boldsymbol{F}\}^{T}_{(e)}\,\delta\,\{\boldsymbol{X}\}_{(e)}=\boldsymbol{f}^{T}_{(e)}\,\delta\epsilon_{(e)}$, we can see the contragredience :

$$[Q_{F(e)}(\{\boldsymbol{X}\})]=[Q_{X(e)}(\{\boldsymbol{X}\})]^{T} \quad (14)$$

When for simplicity, each $\{\boldsymbol{X}\}_{(e)}$ be a set of the relevant joint positions lying on (e), element positions $\{\boldsymbol{X}_{(E)}\}=\{\{\boldsymbol{X}\}_{(e)},\ e=1,\cdots,M\}$ are then obtained by picking out their relevant joint positions from $\{\boldsymbol{X}_{J}\}$, which is expressed in the form :

$$\{\boldsymbol{X}_{(E)}\}=[S_{X}]\,\{\boldsymbol{X}_{J}\} \quad (15)$$

where $[S_X]$ is geometrical continuity matrix of elements of zero and unity. Here, *joint force* $\boldsymbol{F}_j$ is defined, in the similar way to $\boldsymbol{f}_{(e)}$ and $\{\boldsymbol{F}\}_{(e)}$, such that for infinitesimal $\delta\boldsymbol{X}_j$ from arbitrary $\boldsymbol{X}_j$, inner product $\boldsymbol{F}_j\cdot\delta\boldsymbol{X}_j$ represents the real work done by the forces on j during the $\delta\boldsymbol{X}_j$. For element forces $\{\boldsymbol{F}_{(E)}\}=\{\{\boldsymbol{F}\}_{(e)},\ e=1,\cdots,M\}$ known, we can obtain each of joint forces $\{\boldsymbol{F}_J\}=\{\boldsymbol{F}_j,\ j=1,\cdots,N\}$ by collecting their relevant elements of $\{\boldsymbol{F}_{(E)}\}$:

$$\{\boldsymbol{F}_J\}=[S_F]\,\{\boldsymbol{F}_{(E)}\} \quad (16)$$

where from the equation of virtual work, $\{\boldsymbol{F}_J\}^{T}\,\delta\,\{\boldsymbol{X}_J\}=\{\boldsymbol{F}_{(E)}\}^{T}\,\delta\,\{\boldsymbol{X}_{(E)}\}$, the force continuity matrix $[S_F]$ is related to $[S_X]$ as

$$[S_F]=[S_X]^{T} \quad (17)$$

Denoting the external forces at joints by $\{\overline{\boldsymbol{P}}_J\}$ in the same way to $\{\boldsymbol{F}_J\}$, we obtain the equilibrium equations as

$$(\{\boldsymbol{F}_J(\{\boldsymbol{X}_J\})\}=)\,[S_F]\,\{\boldsymbol{F}_{(E)}\}=\{\overline{\boldsymbol{P}}_J\} \quad (18)$$

In the above relations, the existence of elements' strain-energy functions $\hat{U}_{(e)}(\boldsymbol{\epsilon})$ together with contragredience relations (14) and (17) holds the discrete model complete as an assembly of elastic elements, or being of a variational problem.

By dividing the entire domain of the continuous body according to the M-division, let the exact strain energy in each subdomain for any $\{\chi^{\sigma}(\zeta^{\lambda})\}$ be denoted by

$$U_{(e)}=\int_{L(e)} A(\boldsymbol{\psi})\,dL \quad (19)$$

When $A(\boldsymbol{\psi})$ is continuous, by condition (4), quotient $U_{(e)}/L_{(e)}$ converges to the strain-energy density over the limit domain :

$$\lim_{M\to\infty} U_{(e)}/L_{(e)}=A_{(e)} : \textit{finite} \quad (20)$$

On the other hand, with the use of (1), we evaluate the joint positions associated to $\{\chi^{\sigma}(\zeta^{\lambda})\}$. And, by the use of (6·b), (10) and (15), we obtain the strain energies $\hat{U}_{(e)}$ of the discrete elements. Now, we assume that the approximation is attained in each element to the extent :

$$\lim_{L(e)\to 0}(\hat{U}_{(e)}-U_{(e)})/L_{(e)}=0 \quad (21)$$

Then, the difference of the two total strain energies is developed as follows :

$$|\lim_{M\to\infty}\sum_{e=1}^{M}\hat{U}_{(e)}-U(\{\chi^{\sigma}\})|\leq\lim_{M\to\infty}\sum_{e=1}^{M}|\hat{U}_{(e)}-U_{(e)}|=L\lim_{M\to\infty}\sum_{e=1}^{M}\theta_{(e)}^{M}\cdot|\hat{U}_{(e)}-U_{(e)}|/L_{(e)} \quad (22)$$

where $\theta_{(e)}^{M}=L_{(e)}/L$. Now, condition (21) together with relation $\sum_{e=1}^{M}\theta_{(e)}^{M}=1$ leads to the convergence :

$$\lim_{M\to\infty}\sum_{e=1}^{M}\hat{U}_{(e)}=U(\{\chi^{\sigma}\}) \quad (23)$$

The above expansions are a little more generalized as stated in Appendix I to deal with finite jumps in $A(\psi)$-distribution.

4. REQUIREMENTS OF CONSTANT STRAINS AND FOR LINEARIZATION

By the assumption of $A(\psi)$ being continuous for allowable $\{\chi^{\sigma}(\zeta^{\lambda})\}$, strain parameters $\psi(\zeta^{\lambda})$ are also continuous in the body. At limit $M\to\infty$, obviously, $\psi(\zeta^{\lambda})$ become relatively constant, say $\psi_{(e)}$, within each subdomain. Assume that the sequence of M-division models with the joint positions associated to $\{\chi^{\sigma}(\zeta^{\lambda})\}$ attains to reproduce the same constant strains in each element :

$$\lim_{M\to\infty}\hat{\psi}(\zeta^{\lambda})=\psi_{(e)}:\ const.\quad(\{\zeta^{\lambda}\}\in L_{(e)}) \quad (24)$$

In this case, provided that the strain energy of each element is exactly estimated for the geometrical interpolation, it is easy to show that criterion (21) is satisfied : since $\hat{\psi}\to\psi_{(e)}$, then

$$\lim_{M\to\infty}(\hat{U}_{(e)}-U_{(e)})/L_{(e)}=\lim_{M\to\infty}\int_{L(e)}[A(\hat{\psi})-A(\psi_{(e)})]\,dL/L_{(e)}=0 \quad (25)$$

Further, it is sufficient for the above requirement that by choosing adequate values of $\{X\}_{(e)}$ (or $\epsilon_{(e)}$), each element is capable at limit $L_{(e)}\to 0$ to produce any pertinent constant strains. For, a set of constant strain parameters uniquely determine the deformation state of domain $L_{(e)}$, regardless of the domain being of the continuous body or of the discrete model. As the joint positions are evaluated for $\{\chi^{\sigma}(\zeta^{\lambda})\}$ through (1) (satisfying any compatibilities at the limit), the discrete elements with the above ability yield the same constant-strain states $\psi_{(e)}$, $e=1\cdots,M$, at $M\to\infty$, by necessity.

Next, we consider that a strain-energy function $U_{(e)}(\epsilon)$ sufficient for (21) is decomposed into several terms :

$$\hat{U}_{(e)}(\epsilon)=\hat{U}_{(e)1}(\epsilon)+\hat{U}_{(e)2}(\epsilon)+\cdots+\hat{U}_{(e)K}(\epsilon) \quad (26)$$

To attain the same convergence, then, we can neglect any such term $\hat{U}_{(e)k}$ that

$$\lim_{L(e)\to 0}\hat{U}_{(e)k}/L_{(e)}=0 \quad (27)$$

We make it by the Taylor expansion as

$$\hat{U}_{(e)}(\epsilon)=\boldsymbol{f}_{(e)}^{0T}\epsilon_{(e)}+\frac{1}{2}\epsilon_{(e)}^{T}[k^{0}]_{(e)}\epsilon_{(e)}+\Delta\hat{U}_{(e)}^{N}(\epsilon^{3}) \quad (28)$$

where $\boldsymbol{f}_{(e)}^{0}$=initial deformation force at $\epsilon_{(e)}=\boldsymbol{0}$; $[k^{0}]_{(e)}$=initial tangent stiffness between $\boldsymbol{f}_{(e)}$ and $\epsilon_{(e)}$; and $\Delta\hat{U}_{(e)}^{N}$=higher term than ϵ^{2}, containing any nonlinear effects. Under the former requirement of constant strains, we can say that if

$$\lim_{L(e)\to 0}\Delta\hat{U}_{(e)}^{N}/L_{(e)}=0\quad\text{for}\quad\epsilon_{(e)}\quad\text{producing any constant strains} \quad (29)$$

convergence (21) is held by using the associated linear interpolation :

$$\hat{U}_{(e)}^{L}(\epsilon)=\boldsymbol{f}_{(e)}^{0T}\epsilon_{(e)}+\frac{1}{2}\epsilon_{(e)}^{T}[k^{0}]_{(e)}\epsilon_{(e)} \quad (30)$$

5. APPLICATION TO PLANE-FRAMES

We follow the discretization of plane-frames developed in Ref. 1), and, here, emphasis is put on how the $\boldsymbol{f}_{(e)}-\epsilon_{(e)}$ relation can ultimately be simplified with preserving the convergence to the exact solutions.

(1) $\epsilon_{(e)}-\{X\}_{(e)}$ and $\{F\}_{(e)}-\boldsymbol{f}_{(e)}$ Relations

Consider a plane-beam element (e) in two dimensions, with $\{x,\ y\}$ as the background Cartesian coordinates and θ as the deflection angle measured from the x-direction. We choose parameters of $\{X\}_{(e)}$ as

$\{X\}_{(e)}=\{(x,\ y,\ \theta)_A,\ (x,\ y,\ \theta)_B\}_{(e)}$ ………… (31)

By applying the simple support at the two ends, A and B, to constrain its rigid displacements, we have the following parameters for $\epsilon_{(e)}$ and $v_{(e)}$, respectively (: Fig. 1) :

$$\left.\begin{array}{l}\epsilon_{(e)}=\{\epsilon,\ \varphi_A,\ \varphi_B\}_{(e)}\\ v_{(e)}=\{x_A,\ y_A,\ \tau\}_{(e)}\end{array}\right\}\cdots\cdots\cdots\cdots\ (32\cdot\text{a, b})$$

Those are related to $\{X\}_{(e)}$ of (31) as

$$\epsilon=\sqrt{\bar{x}^2+\bar{y}^2}\,/\,l-1$$
$$\varphi_A=\theta_A-\arctan(\bar{y}/\bar{x})$$
$$\varphi_B=\theta_B-\arctan(\bar{y}/\bar{x})$$
$$\tau=\arctan(\bar{y}/\bar{x})\cdots\cdots\cdots\cdots\ (33\cdot\text{a}\sim\text{d})$$

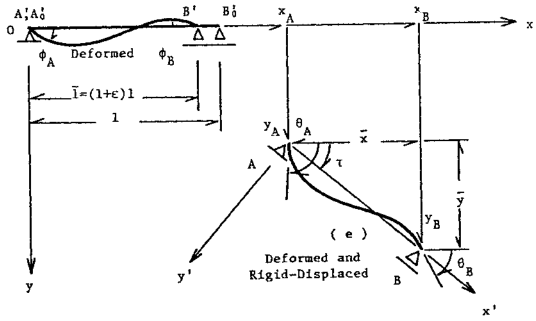

Fig. 1 Parameters of Deformation and Rigid Position.

where $\{\bar{x},\ \bar{y}\}=\{x_B-x_A,\ y_B-y_A\}$; and l=original (arc) length of a beam element.

By decomposing the forces acting at the ends into components in the way stated in Sec. 3, we obtain the following $\{F\}_{(e)}$ conjugate to $\{X\}_{(e)}$ of (31) :

$\{F\}_{(e)}=\{(F_x,\ F_y,\ M)_A,\ (F_x,\ F_y,\ M)_B\}_{(e)}$ ………………………………………… (34)

where $\{F_x,\ F_y\}=\{x,\ y\}$-components of a force;and M=moment in θ-direction. And, we obtain the deformation force conjugate to $\epsilon_{(e)}$ of (32·a) as

$f_{(e)}=\{Hl,\ M_A,\ M_B\}_{(e)}$ ………………………………………… (35)

where H=tensile chord force between the two ends. The associated $[Q_{F(e)}]$ of (12·a) is obtained as

$$[Q_{F(e)}(\{X\})]=\begin{bmatrix}-\cos\tau/l, & -\sin\tau/\bar{l}, & -\sin\tau/\bar{l}\\ -\sin\tau/l, & \cos\tau/\bar{l}, & \cos\tau/\bar{l}\\ 0, & 1, & 0\\ \cos\tau/l, & \sin\tau/\bar{l}, & \sin\tau/\bar{l}\\ \sin\tau/l, & -\cos\tau/\bar{l}, & -\cos\tau/\bar{l}\\ 0, & 0, & 1\end{bmatrix}_{(e)}\cdots\cdots\cdots\ (36)$$

where $\bar{l}\,(=\sqrt{\bar{x}^2+\bar{y}^2})=(1+\epsilon)\,l$. Now, it is not difficult to confirm contragredience (14) in (33·a~c) and (36).

(2) *A Sufficient* $f_{(e)}-\epsilon_{(e)}$ *Relation*

When a plane beam with uniform cross-section undergoes large deformations under the Bernoulli-Euler hypothesis, they cause only the normal strains in the logitudinal direction. Describing those strains by the unit elongation, not by the relevant component of the Green's strain tensor, we assume that the strains make the associated normal stresses arise proportionally even if they are finite. Under those assumptions, the following governing equations are obtained[1])

$$\phi-\{\chi^0\}\ \textit{Relations}:\ \left.\begin{array}{l}e_G(\zeta)=\sqrt{x_G'^2+y_G'^2}-1\\ k(\zeta)=\{-y_G'x_G''+x_G'y_G''\}/\{x_G'^2+y_G'^2\}\end{array}\right\}\cdots\cdots\cdots\ (37\cdot\text{a, b})$$

$$\textit{Equilibrium Equations}:\ \frac{d}{d\zeta}\left(\frac{1}{g_G}\begin{bmatrix}x_G', & -y_G'\\ y_G', & x_G'\end{bmatrix}\begin{Bmatrix}N\\ (M'-\overline{m})/g_G\end{Bmatrix}+\begin{Bmatrix}\overline{p}_x\\ \overline{p}_y\end{Bmatrix}\right)=\begin{Bmatrix}0\\ 0\end{Bmatrix}\cdots\cdots\ (38)$$

Constitutive Relations : $N=EAe_G,\quad M=-EI\,(k-k^0)$ ………………………………… (39·a, b)

where ζ=Lagrangean coordinate along original length of neutral line (prime means the differentiation with respect to ζ) ; E=Young's modulus ; A and I=area and moment of inertia of cross-section ; $k^0(\zeta)$=initial curvature ; $\{\overline{p}_x(\zeta),\ \overline{p}_y(\zeta)\}$ and $\overline{m}(\zeta)$=distributed external force components and moment per unit of ζ ; $\{x_G(\zeta),\ y_G(\zeta)\}$=one-parameter equilibrium curve ; $e_G(\zeta)$ and $k(\zeta)$=unit elongation and curvature of a neutral line ($g_G=1+\epsilon_G$) ; and $N(\zeta)$ and $M(\zeta)$=axial force and bending moment.

Suppose element (e) in a plane frame which undergoes large deformations. We can see that at limit l

$\to 0$, $\{\varphi_A, \varphi_B\}\to\{0, 0\}$, and the initial curvature becomes relatively constant. To focus on its deformations, we transfer (e) as a rigid to the basis position and make it subject to the simple support (: Fig. 2). Then, $y_G(\zeta)\ll x_G(\zeta)$ for small enough l. By introducing $\{x'_G, y'_G\}\to\{g_G, 0\}$ with no external forces, we can reduce (37·a, b) and (38) as follows :

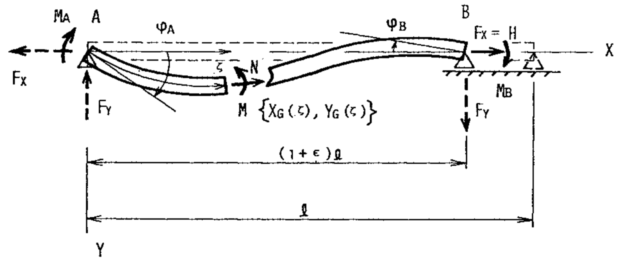

Fig. 2 Deformation of Beam Element

$$e_G=x'_G-1,\quad k=y''_G/x'_G \quad\cdots\cdots (40\cdot a, b)$$

$$\frac{dN}{d\zeta}=0,\quad \frac{d}{d\zeta}\left(\frac{M'}{g_G}\right)=0 \quad\cdots\cdots (41\cdot a, b)$$

And, the associated boundary conditions are reduced to

$$\left.\begin{aligned} &x_G(0)=0,\quad x_G(l)=(1+\epsilon)\, l\\ &y_G(0)=0,\quad y_G(l)=0\\ &\left.\frac{dy_G}{dx_G}\right|_{\zeta=0}=\varphi_A,\quad \left.\frac{dy_G}{dx_G}\right|_{\zeta=l}=\varphi_B \end{aligned}\right\} \quad\cdots\cdots (42\cdot a\sim f)$$

We can solve the above differential equations together with (39·a, b) as follows :

$$\left.\begin{aligned} &x_G(\xi)=l\,(1+\epsilon)\,\xi\\ &y_G(\xi)=l\,(1+\epsilon)\{(\xi^3-2\xi^2+\xi)\,\varphi_A+(\xi^3-\xi^2)\,\varphi_B\} \end{aligned}\right\} \quad\cdots\cdots (43\cdot a, b)$$

$$\left.\begin{aligned} &N(\xi)=EA\epsilon \ :\ const.\\ &M(\xi)=EI\,\{(-6\xi+4)\,\varphi_A/l+(-6\xi+2)\,\varphi_B/l+k^0\} \end{aligned}\right\} \quad\cdots\cdots (44\cdot a, b)$$

where $\xi=\zeta/l$: normalized coordinate. The associated $f_{(e)}-\epsilon_{(e)}$ relation and strain-energy function are then obtained as

$$\begin{Bmatrix} Hl\\ M_A\\ M_B \end{Bmatrix}=\begin{bmatrix} EAl & & \\ & 4EI/l, & 2EI/l\\ & 2EI/l, & 4EI/l \end{bmatrix}\begin{Bmatrix} \epsilon\\ \varphi_A\\ \varphi_B \end{Bmatrix}+\begin{Bmatrix} \\ EIk^0\\ -EIk^0 \end{Bmatrix} \quad\cdots\cdots (45)$$

$$\hat{U}_{(e)}=EIk^0(\varphi_A-\varphi_B)+EAl/2\cdot\epsilon^2+2EI/l\cdot(\varphi_A^2+\varphi_B^2+\varphi_A\varphi_B) \quad\cdots\cdots (46)$$

It is to be noted that excepting only $y_G(\xi)$ in which ϵ might be finite, the above $f_{(e)}-\epsilon_{(e)}$ relation and $\hat{U}_{(e)}$ are exactly same to those of the linear theory for small displacements. The limits relevant to $\{\varphi_A, \varphi_B\}$ at $l\to 0$ are related to the continuous quantities as follows :

$$\left.\begin{aligned} &\lim_{l\to 0}\{\varphi_A+\varphi_B,\ \varphi_A-\varphi_B\}/l=\{0,\ M/EI-k^0\}\\ &\lim_{l\to 0}(\varphi_A+\varphi_B)/l^2=-(1+e_G)\,Q/6EI \end{aligned}\right\} \quad\cdots\cdots (47\cdot a, b)$$

where $Q(\zeta)=(1+e_G)^{-1}\,dM/d\zeta$: shear force in cross-section. By the use of these limits, we can estimate limit (20) for the present $\hat{U}_{(e)}$ of (46) as

$$\lim_{l\to 0}\frac{\hat{U}_{(e)}}{L_{(e)}}=-EIk^0k+\frac{EA}{2}\,e_G^2+\frac{EI}{2}\,k^2 \quad\cdots\cdots (48)$$

As for the continuous beam, in accordance with the foregoing paper[1], the spatial vector element corresponding to material $d\zeta$ with distance η from the neutral line is given by $d\boldsymbol{l}=(1+e_G-\eta k)\,d\zeta\,\boldsymbol{i}_\zeta$, and the longitudinal force acting on area element dA of the cross section is represented as $d\boldsymbol{F}=\sigma_{\zeta\zeta}dA\,\boldsymbol{i}_\zeta=Ee_{\zeta\zeta}dA\,\boldsymbol{i}_\zeta$, where $e_{\zeta\zeta}$=unit elongation defined by $(|d\boldsymbol{l}|-|d\boldsymbol{l}^0|)/d\zeta$; $\sigma_{\zeta\zeta}$=stress component conjugate to $e_{\zeta\zeta}$; and $\boldsymbol{i}_\zeta$=unit vector into ζ-direction. Then, by integrating $\delta(dU/d\zeta)=\int_A d\boldsymbol{F}\cdot\delta(d\boldsymbol{l}/d\zeta)=\int_A E\,(e_G-\eta\,(k-k^0))(\delta e_G-\eta\delta k)\,dA=EA\,e_G\delta e_G+EI\,(k-k^0)\,\delta k$, we can see that the exact $\lim_{l\to 0} U_{(e)}/L_{(e)}=dU/d\zeta$ is equal to the result of (48). Now, we confirm that convergence (21) holds for the interpolation defined by (45) and (46), and that the higher terms ignored on the simplification are really negligible in the sense of

(27).

6. DISCUSSION ON FURTHER SIMPLIFICATION IN SMALL-STRAIN PROBLEMS

In the discretization into small enough elements of a finite-displacement small-strain problem, we can see that each element is largely translated and rotated, but that its deformations are small. Then, the matrix $[Q'_{F(e)}(\epsilon)]$ of (12·b) which holds the element force in equilibrium as a rigid on the deformed configurations does not change so much as the matrix $[T_{F(e)}(v)]$ does for large rigid displacements. It seems, therefore, reasonable to use the initial $[Q'^*_{F(e)}]=[Q'_{F(e)}(0)]$: *const.* on any stage, instead of $[Q'_{F(e)}(\epsilon)]$: *variable*. Let the alternative element force be denoted by

$$\{F\}^*_{(e)}=[Q^*_{F(e)}(v)]\,f_{(e)}=[T_{F(e)}(v)][Q'^*_{F(e)}]\,f_{(e)} \quad\cdots\cdots (49)$$

However, when the tangent stiffness is derived by differentiating the above $\{F\}^*_{(e)}$ with respect to $\{X\}_{(e)}$, the expansion yields at least asymmetry of its geometrical stiffness matrix, with implying no path-independent work done by $\{F\}^*_{(e)}$ on paths of $\{X\}_{(e)}$. In some existing studies, we can see such discretizations as employ the above $\{F\}^*_{(e)}$. In the following, we discuss the admissibility of that approximation.

We focus on the difference between the works done by $\{F\}_{(e)}$ and by $\{F\}^*_{(e)}$ during a common change of $\{X\}_{(e)}$ within small strains. For precise discussion, consider the plane-beam element dealt with in Sec. 5. For the exact $[Q_{F(e)}]$ of (36), we can obtain the associated $[Q^*_{F(e)}]$ by replacing $\bar{l}\,(=(1+\epsilon)\,l)$ by l in the matrix. Consider a typical element position $\overline{\{X\}}_{(e)}$ in a largely deformed plane-frame, $\{\bar{x}_G(\zeta),\ \bar{y}_G(\zeta)\}$, within small strains, with $\bar{\epsilon}_{(e)}$ and $\bar{v}_{(e)}$ being its deformation and rigid position. Let the element be displaced from the basis position, $\epsilon_{(e)}=0$ and $v_{(e)}=0$, to the $\overline{\{X\}}_{(e)}$ in the following way : first, be deformed to $\bar{\epsilon}_{(e)}$ with preserving $v_{(e)}\equiv 0$, which we denote as $\epsilon_{(e)}(\rho)$, $0\leq\rho\leq 1$ ($\epsilon_{(e)}(0)=0$ and $\epsilon_{(e)}(1)=\bar{\epsilon}_{(e)}$) ; and next be displaced as a rigid to $\bar{v}_{(e)}$ with $|\epsilon_{(e)}\equiv\bar{\epsilon}_{(e)}$, which be denoted as $v_{(e)}(\rho')$, $0\leq\rho'\leq 1$ ($v_{(e)}(0)=0$ and $v_{(e)}(1)=\bar{v}_{(e)}$), see Fig. 1. On the first path, the element position is represented in terms of the independent $\epsilon_{(e)}(\rho)$ as

$$\{(x,\ y,\ \theta)_A,\ (x,\ y,\ \theta)_B\}_{(e)}=\{0,\ 0,\ \varphi_A(\rho),\ l\,(1+\epsilon\,(\rho)),\ 0,\ \varphi_B(\rho)\} \quad\cdots\cdots (50)$$

Trivially, the work $\hat{V}_{(e)}$ done by $\{F\}_{(e)}$ is equal to the stored strain energy $\hat{U}_{(e)}$. By integrating $\delta V^*_{(e)}=\{F\}^{*T}_{(e)}\,\delta\,\{X\}_{(e)}=f^T_{(e)}[Q^*_{F(e)}]^T\,\delta\,\{X\}_{(e)}$ with the use of the $[Q^*_{F(e)}]$, we can see that the work $V^*_{(e)}$ done by $\{F\}^*_{(e)}$ is also equal to $\hat{U}_{(e)}$, exactly. On the second path of rigid displacement, the element position is expressed as

$$\{(x,\ y,\ \theta)_A,\ (x,\ y,\ \theta)_B\}_{(e)}=\{x_A(\rho'),\ y_A(\rho'),\ \bar{\varphi}_A+\tau\,(\rho'),\ x_A(\rho')+(1+\bar{\epsilon})\,l\cos\tau\,(\rho'),$$
$$y_A(\rho')+(1+\bar{\epsilon})\,l\sin\tau\,(\rho'),\ \bar{\varphi}_B+\tau\,(\rho')\} \quad\cdots\cdots (51)$$

The exact $\{F\}_{(e)}$ obviously does no work on (51). While, since $\{F\}^*_{(e)}$ is slightly out of equilibrium due to ignoring the effect of deformation, it does a work on the rigid displacement. By integrating $\delta V^*_{(e)}=f^T_{(e)}[Q^*_{F(e)}]^T\,\delta\{X\}_{(e)}$ on (51), we estimate it at $-\bar{\epsilon}\,\bar{\tau}\,(\bar{M}_A+\bar{M}_B)$, where $\bar{M}_A$ and $\bar{M}_B$ are associated to $\bar{\epsilon}_{(e)}$ by a sufficient $f_{(e)}-\epsilon_{(e)}$ relation. Then, the total work done by $\{F\}^*_{(e)}$ is written as

$$V^*_{(e)}=\hat{U}_{(e)}-\bar{\epsilon}\,\bar{\tau}\,(\bar{M}_A+\bar{M}_B) \quad\cdots\cdots (52)$$

Here, it is to be mentioned that the work done by $\{F\}^*_{(e)}$ depends upon the paths of $\{X\}_{(e)}$; for, $\{F\,(\{X\})\}^{*T}_{(e)}\,\delta\,\{X\}_{(e)}$ is not in the exact differential form. For instance, if the element is, first, displaced as a rigid to $\bar{v}_{(e)}$, and succeedingly deformed to $\bar{\epsilon}_{(e)}$ with $v_{(e)}\equiv\bar{v}_{(e)}$, then exactly $V^*_{(e)}=\hat{V}_{(e)}=\hat{U}_{(e)}$ after the two paths.

We introduce (46) into (52) as a strain-energy function sufficient for the convergence. By taking limit of $V^*_{(e)}/l$ at $l\rightarrow 0$ with the use of (47·a, b), we can estimate in the sense of (21) the negligibility of the additional work :

$$\lim_{l\rightarrow 0}\frac{V^*_{(e)}}{l}=-\frac{EI}{2}\,k^{02}+\frac{1}{2}\,\bar{N}\bar{e}_G-\frac{1}{2}\,\bar{M}\,(\bar{k}-k^0)+\bar{Q}\bar{e}_G\bar{\theta}\,(1+\bar{e}_G) \quad\cdots\cdots (53)$$

where a quantity with superimposed bar means associated to $\{\bar{x}_G(\zeta),\ \bar{y}_G(\zeta)\}$. Since the present shear force $\bar{Q}\,(\zeta)$ and axial force $\bar{N}\,(\zeta)$ are of the same order, with $\theta\,(\zeta)$ being finite, the last term related to using $\{F\}^*_{(e)}$

instead of $\{F\}_{(e)}$ is comparable in magnitude at least to the second term of axial elongation. As long as the second is to be taken into account, the additional work is thus also innegligible.

As for a general discrete element, roughly, we can say as follows : *The work done by the slightly-out-of-equilibrium* $\{F\}^{*}_{(e)}$ *during a large rigid displacement is innegligible compared with such a strain energy as stored by the exact* $\{F\}_{(e)}$ *for an equally small deformation.*

7. CONCLUDING REMARKS

As long as the compatibilities at the joints, stated in Sec. 2, are held in the discretization, it is sufficient for the present convergence that the strain-energy density of each (dominant) element attains at limit $M\to\infty$ to the corresponding value of the associated continuum in any configuration, or at $L_{(e)}\to 0$ to the exact value for any constant-strain state of the continuum's corresponding portion (while those of the vanishing elements may converge to any finite values). Alternatively, provided that the estimation of strain energies is correct for given configurations, it is sufficient that the geometrical interpolation of each (dominant) element is capable by choosing adequate values of its element position (or deformation) to produce at the limit any pertinent constant strains (while the vanishing elements may yield any finite strains). On this criterion, we can deduce which terms can be eliminated in a strain-energy function to attain the same convergence ; e. g. , by estimating the nonlinear-term's contribution to the strain-energy density with the use of (29) within the frame of constant strains, we can see whether or not the deformation force-deformation relation might be linearized.

From the discussion made in Sec. 6, we can say that if the inequilibrium due to ignoring an element's deformation be estimated by the work, that amounts comparable to its strain energy even in small-strain problems ; in the way of the deformation being smaller, the two energies are of the same order. The $\{F\}_{(e)}$—$f_{(e)}$ relation of a discrete element corresponds in their role to the differential equations of equilibrium for the continuous body. Those differential equations describe, literally, the equilibrium conditions on a deformed differential volume element, an absolute infinitesimal element. And, we know that we can not make them linear by choosing any small subdomains and any local coordinates. It is the same way with a discrete element : the effect of the element deformation is to be taken into the $\{F\}_{(e)}-f_{(e)}$ relation. The disregard leads to the contradiction as an assembly of elastic elements, with getting out of a variational problem. In addition, the discrete $f_{(e)}-\epsilon_{(e)}$ relation corresponds to the continuous constitutive equations. And, we know the validity of linear constitutive equations even for large displacements, if restricted within small strains. We can expect the similarity for the $f_{(e)}-\epsilon_{(e)}$ relation.

By the existence of elements' strain-energy functions together with the relations of contragredience, (14) and (17), the discrete model is assured complete as an elastic assembly. Then, in accordance with ordered numbers of element divisions, $\{M\}$, there are an associated sequence of the complete assemblies. It is another and our final proposition to find out such actual strain-energy functions as make the sequence convergent to the original elasticity. Now, if there are such strain-energy functions obtained that reflect at the limit the exact constitutive relations only within small deformations, as a natural classification, we can call them discretizations for large-displacement small-strain problems.

APPENDIX I. A GENERALIZATION OF CRITERION (21)

Consider elements of M-division are distinguished into two groups, $e=1, \cdots, M'$ and $e=M'+1, \cdots, M$, where number M' is dependent on M, in such a manner that the sum of the latter elements' domains vanishes at the limit :

$$\lim_{M\to\infty}\sum_{e=1}^{M'} L_{(e)}=L\cdot\lim_{M\to\infty}\sum_{e=1}^{M'}\theta^{M}_{(e)}=L \quad\cdots\cdots (A\cdot 1)$$

$$\lim_{M\to\infty}\sum_{e=M'+1}^{M} L_{(e)}=L\cdot\lim_{M\to\infty}\sum_{e=M'+1}^{M}\theta^{M}_{(e)}=0 \quad\cdots\cdots (A\cdot 2)$$

Let the former and latter elements be called *dominant* and *vanishing*, respectively. We apply criterion (21) only to the dominant elements :

$$\lim_{L_{(e)}\to 0} (\hat{U}_{(e)} - U_{(e)})/L_{(e)} = 0 \quad \text{for} \quad e=1, \cdots, M' \cdots\cdots (A\cdot 3)$$

While, densities $\hat{U}_{(e)}/L_{(e)}$ of the vanishing elements are allowed to converge to any finite values. In this case, the difference of the two total strain energies is expanded as follows :

$$|\lim_{M\to\infty} \sum_{e=1}^{M} \hat{U}_{(e)} - U(\{\chi^{\sigma}\})| \leq \lim_{M\to\infty} \sum_{e=1}^{M} |\hat{U}_{(e)} - U_{(e)}| = L\cdot \lim_{M\to\infty} (\sum_{e=1}^{M'} + \sum_{e=M'+1}^{M}) \theta_{(e)}^{M} \cdot |\hat{U}_{(e)} - U_{(e)}| / L_{(e)} \cdots\cdots (A\cdot 4)$$

Now, i) by the use of (A·1) and (A·3), the first sum converges to zero ; and ii) since densities $(\hat{U}_{(e)} - U_{(e)})/L_{(e)}$ of the vanishing elements converge to finite values, the total of their domains vanishing, the second sum also converges to zero. Those lead to convergence (23).

When the distribution of strain-energy density $A(\psi)$ has finite jumps in an entire continuum, those jumps can be covered by the vanishing elements. Thus, we can take any $\{\chi^{\sigma}(\zeta^{\lambda})\}$ producing piecewise continuous $A(\psi)$ as an allowable configuration of the body.

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