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LAGRANGIAN FINITE ELEMENT METHOD FOR WAVE MOTION USING VELOCITY CORRECTION METHOD

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A new numerical algorithm for the Lagrangian treatment of incompressible fluid flows with free surface has been developed. The novel feature of this new algorithm is the use of the Lagrangian finite element method together with the velocity correction technique. Lagrangian specification permit the favorable treatment of free surfaces, whilst the velocity correction approach greatly simplifies computation in terms of algorithmic structure. Numerical example illustrates the properties of this technique.

1. INTRODUCTION

In this paper, the finite element method for the analysis of transient free surface flow without viscosity in a two-dimensional vertical plane is presented. Recent studies of free surface flow have mostly been based on the assumption that the flow is steady or that change in time is not intensive[1)~10)]. The basic equations can be described following the Eulerian or Lagrangian description or a combination of both. The Eulerian description is concerned with a particular region of the space occupied by the continuum, and is widely employed in fluid flow analysis. Frederiksen and Watts[11)] and Kawahara and Miwa[12)] presented the finite element method based on the Eulerian description. The prime advantage of the Eulerian formulation is that strong distortions can be handled with relative ease. However, it is necessary to discretize the fluid domain at each time step according to the profile of the free surface, and this requires a lot of time to be spent on computation. Hence, Kawahara et al.[13)], examined the finite element method based on the Lagrangian description. The approach is attractive from the numerical point of view because the Lagrangian description focuses on specific particles of the continuum and it offers the most natural approach to transient hydrodynamic problems involving free surfaces or interfaces. Hughes et al.[14)], Liu and Ma[15)] and Donea et al.[16)], examined the combination of the Eulerian and Lagrangian descriptions, providing a tool for the study of fairly complex configuration.

The authors' research group has devoted itself to an explicit type of numerical integration scheme which

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is referred to as the "two step explicit finite element method"[17)~19)]. One of the advantages of this method is that computational core storage can be greatly reduced. Another advantage is that parallel computation is adaptable. However, the storage capacity of modern computers is so large that the problem of core storage does not always have to be considered. Thus, the implicit scheme, which can take a long time increment, is believed to save computation time. In the present Lagrangian finite element method, the domain is assumed to be covered by a mesh of finite elements, the vertices of which move with a fluid. In this process, fluid inside a finite element always remains in that element, and fluid boundaries always move with the element boundaries. In an incompressible Lagrangian calculation the volume of each element must remain constant. To satisfy this constraint a velocity correction procedure, which is originally based on Chorin's[20)] findings with regard to the finite difference method, is employed. Schneider and Raithby[21),22)] and Mizukami and Tsuchiya[23)] adopted a similar approach to the Eulerian formulation of the Navier-Stokes equations in the case of the finite element method. This method is incorporated into the semi-implicit method. In the present paper, the propagation of a solitary wave is analysed as an illustrative example and favorable results of numerical simulation have been obtained.

2. BASIC EQUATIONS

All equations in this paper are expressed in accordance with indicial notation and summation convention. A rectangular cartesian coordinate system is fixed to the inertial space and denoted by $x_i(i=1, 2)$. Let V be a fluid domain which is surrounded by a piece wise smooth boundary S. The equations of the motion and continuity of inviscid and incompressible fluid can be represented in the following forms :

$$\rho Du_i/Dt+p_{,i}+\rho f_i=0 \quad \text{in} \quad V \cdots\cdots (1)$$

$$u_{i,i}=0 \quad \text{in} \quad V \cdots\cdots (2)$$

where $u_i(i=1, 2)$ are the x_i-components of fluid velocity ; p is the pressure ; ρ is the fluid density, f_i is the body force. The material derivative with respect to time t is denoted by D/Dt.

The boundary S consists of two kinds of boundaries, namely the free surface boundary S_1 and the solid wall boundary S_2 of the channel. On the free surface boundary, normal stress should be equal to atmospheric pressure, which is assumed to be zero, and tangential stress should vanish. Thus, these conditions can be expressed as

$$-p\delta_{ij}n_i=0 \quad \text{on} \quad S_1 \cdots\cdots (3)$$

On the solid wall, the free-slip boundary condition is expressed as

$$u_in_i=0 \quad \text{on} \quad S_2 \cdots\cdots (4)$$

where n_i is the direction cosine of the outward normal on the boundary with respect to the x_i-axis. Initial velcoity and pressure are given by the following conditions :

$$u_i=u_i^0 ; p=p^0 \quad \text{at} \quad t=0 \cdots\cdots (5)$$

where u_i^0 and p^0 are functions of the spatial coordinate x_i only and specify the initial behaviour of the fluid.

3. LAGRANGIAN METHOD

In order to implement a numerical solution procedure for the Lagrangian formulation it is necessary to discretize time and material. This is in contrast to a numerical solution of the Eulerian formulation, where one would discretize time and space. The basic concept of the analysis presented in this paper is that the material time derivative can be determined approximately by the functions known at both deformed and undeformed positions during short time increment. The function at the deformed position cannot be obtained before computation. Thus, iterative computation is necessary.

The total time interval is divided into many short time increments, denoted by Δt, and the fluid is discretized into many triangular finite elements. The location of the nodal point of an element is denoted by x_i^n and at n-th time instant $t^n(t^n=n\Delta t)$. The velocities at t^n and t^{n+1} are written as

$$u_i^n=u_i(t^n, x_i^n) \cdots\cdots (6)$$

$$u_i^{n+1}=u_i(t^{n+1}, x_i^{n+1}) \quad (7)$$

The material derivative is the time derivative considering the change of location of a fluid particle. It can be approximated by using equations (6) and (7) as in the following form :

$$Du_i/Dt \doteqdot (u_i^{n+1}-u_i^n)/\Delta t \quad (8)$$

The location of a nodal point after the increment Δt is given by

$$x_i^{n+1}=x_i^n+1/2(u_i^{n+1}+u_i^n)\Delta t \quad (9)$$

Using equation (9), the location can be determined once the velocity has been computed.

The Lagrangian method of this paper can be expressed as follows : at the initial step of iteration. velocity $u_i^{n+1(0)}$, pressure $p^{n+1(0)}$ and position $x_i^{n+1(0)}$ are computed as below.

$$u_i^{n+1(0)}=g_i(u_i^n, p^n, x_i^n) \quad (10)$$

$$p^{n+1(0)}=h(u_i^n, p^n, x_i^n) \quad (11)$$

$$x_i^{n+1(0)}=x_i^n+\Delta t\, u_i^n \quad (12)$$

where $u_i^{n+1(0)}$, for example, means the value of the velocity component at the initial step of iteration in the $(n+1)$-th time interval. g_i and h mean that velocity and pressure can be computed using the velocity correction method (explained in the next section) based on the previously known velocity u_i^n and pressure p^n at position x_i^n and at time t^n. At the m-th iteration cycle (where $m=1, 2, \cdots\cdots$, MAX, MAX is the maximum number of iterations) the values are updated by the following equations :

$$u_i^{n+1(m)}=g_i(u_i^n, p^n, x_i^{n+1(m-1)}) \quad (13)$$

$$p^{n+1(m)}=h(u_i^n, p^n, x_i^{n+1(m-1)}) \quad (14)$$

$$x_i^{n+1(m)}=x_i^n+(\Delta t/2)(u_i^{n+1(m-1)}+u_i^n) \quad (15)$$

Equations (13) and (14) state that velocity and pressure can be computed using the latest position of a fluid particle. Iteration is repeated until the computed results satisfy the following convergence criteria :

$$|x_i^{n+1(m)}-x_i^{n+1(m-1)}|<\varepsilon \quad (16)$$

where ε is a previously defined small value. When the convergence criteria are satisfied during the iteration process, the calculation procedure is repeated for the next time step after assuming the following conditions :

$$u_i^{n+1}=u_i^{n+1(m)} \quad (17)$$

$$p^{n+1}=p^{n+1(m)} \quad (18)$$

$$x_i^{n+1}=x_i^{n+1(m)} \quad (19)$$

Thus velocity and pressure considering the change of position of a fluid particle can be computed. The computation is repeated for all $n=1, 2, \cdots\cdots$, NMAX, where NMAX is the total number of time points.

4. VELOCITY CORRECTION METHOD

To obtain the velocity and pressure at time t^{n+1}, the velocity correction method is successfully used in the present analysis. This section describes how to compute the functions g_i and h in equations (10) and (11) which in turn help to compute u_i and p in equations (13) and (14).

Using equation (8), the equation of motion (1) can be discretized into the following form :

$$\tilde{u}_i^{n+1}=u_i^n-\Delta t\{(1/\rho)p_{,i}^n+f_i^n\} \quad (20)$$

where $\tilde{u}_i^{n+1}$ is the approximate velocity field not satisfying the incompressibility condition. The exact velocity which satisfies the incompressibility constraint is denoted by u_i^{n+1} and pressure by p^{n+1} at time t^{n+1}, and it is assumed that they satisfy the following equations :

$$u_i^{n+1}=u_i^n-\Delta t\{(1/\rho)p_{,i}^{n+1}+f_i^n\} \quad (21)$$

$$u_{i,i}^{n+1}=0 \quad (22)$$

Taking rotation on both sides of equations (20) and (21) the following relation can be obtained :

$$\mathrm{Curl}\, u_i^{n+1}=\mathrm{Curl}\, \tilde{u}_i^{n+1} \quad (23)$$

From the above relation it is clear that

$$u_i^{n+1}=\tilde{u}_i^{n+1}+\phi_{,i} \quad (24)$$

where ϕ is some scalar potential. Taking divergence on both sides of equation (24) together with the incompressibility constraint (22), the following equation for ϕ can be derived.

$$\phi_{,ii}=-\tilde{u}^{n+1}_{i,i} \quad (25)$$

To solve the above equation the following boundary conditions are applied:

$$\phi=0 \quad \text{on} \quad S_1 \quad (26)$$

$$\phi_{,i}n_i=0 \quad \text{on} \quad S_2 \quad (27)$$

Equation (25) can be solved by the finite element method. Thus the potential at all nodal points in the flow field can be obtained. Substitution of the resulting potential in equation (24) yields the corrected velocities at all nodal points. Pressure p^{n+1} can be calculated as

$$p^{n+1}_i=p^n_i-\phi_{,i}/\Delta t \quad (28)$$

The above equation can be derived by subtracting equation (20) from equation (21) and using equation (24). Based on the procedures expressed by equations (20), (25), (24) and (28), velocity u^{n+1}_i and pressure p^{n+1} can be obtained.

5. FINITE ELEMENT METHOD

Equations (20) with (5); (24), (25) with (26) and (27); and (28) can be discretized by finite element analysis. Then, recalling equations (6) and (7), one can evaluate all functions at time t^{n+1} or t^n. Superscript ^{n} denotes the function evaluated at time t^n and position x^n_i. A weak formulation of the problem defined by equations (20), (24) and (25) is obtained by multiplying the differential equations by suitable weighting functions and integrating over the domain V. By multiplying the equations (20) and (24) by the weighting function v_i, and equation (25) by the weighting function q and integrating them over the domain V, the following weighted residual equations of the original problem can be obtained:

$$\int_{Vn+1}(v^{n+1}_i\tilde{u}^{n+1}_i)dV=\int_{Vn}(v^n_iu^n_i)dV+(\Delta t/\rho)\int_{Vn}(v^n_i, p^n)dV-(\Delta t/\rho)\int_{Sn}(v_ipn_i)dS-\Delta t\int_{Vn}(v^n_if^n_i)dV \quad (29)$$

$$\int_{Vn+1}(v^{n+1}_iu^{n+1}_i)dV=\int_{Vn+1}(v^{n+1}_i\tilde{u}^{n+1}_i)dV+\int_{Vn+1}(v^{n+1}_i\phi_{,i})dV \quad (30)$$

$$\int_{Vn+1}(q^{n+1}_{,i}\phi_{,i})dV=\int_{Vn+1}(q^{n+1}\tilde{u}^{n+1}_{i,i})dV+\int_{Sn+1}(q^{n+1}\phi_{,i}n_i)dS \quad (31)$$

Assume that the flow field to be analysed is divided into a number of small domains called finite elements. The velocity and pressure in each finite element are interpolated by using linear shape functions and are expressed in the following forms:

$$u^n_i=\Phi^n_\alpha u^n_{\alpha i} \quad (32)$$

$$p^n=\Phi^n_\alpha p^n_\alpha \quad (33)$$

where Φ^n_α is the shape function of the α-th node at time t^n and location x^n_i. $u^n_{\alpha i}$ denotes the nodal velocity at the α-th node in the i-th direction at time t^n and location x^n_i, and p^n_α is the pressure at the α-th node at time t^n. The weighting functions are interpolated in a manner similar to equations (32) and (33) as

$$v^n_i=\Phi^n_\alpha v^n_{\alpha i} \quad (34)$$

$$q^n=\Phi^n_\alpha q^n_\alpha \quad (35)$$

where $v^n_{\alpha i}$ and q^n_α denote the values at the α-th node at time t^n. Introducing equations (32) through (35) into equations (29) through (31) and using the arbitrariness of the weighting functions, the finite element equations are derived as follows:

$$\overline{M}^{n+1}_{\alpha\beta}\tilde{u}^{n+1}_{\beta i}=\overline{M}^n_{\alpha\beta}u^n_{\beta i}-\Delta t(N^n_\alpha f^n_{\alpha i}-H^n_{\alpha i\beta}p^n_\beta+\hat{\Omega}^n_{\alpha i}) \quad (36)$$

$$\overline{M}^{n+1}_{\alpha\beta}u^{n+1}_{\beta i}=\overline{M}^{n+1}_{\alpha\beta}\tilde{u}^{n+1}_{\beta i}+H^{n+1}_{\alpha i\beta}\phi_\beta \quad (37)$$

$$A^{n+1}_{\alpha\beta}\phi_\beta=H^{n+1}_{\alpha i\beta}\tilde{u}^{n+1}_{\beta i}+\hat{\sum}^{n+1}_\alpha \quad (38)$$

where

$$M^{n+1}_{\alpha\beta}=\int_{Vn+1}(\Phi^{n+1}_\alpha\Phi^{n+1}_\beta)dV \;;\; M^n_{\alpha\beta}=\int_{Vn}(\Phi^n_\alpha\Phi^n_\beta)dV \;;\; H^n_{\alpha i\beta}=\int_{Vn}(\Phi^n_{\alpha,i}\Phi^n_\beta)dV$$

$$H^{n+1}_{\alpha i\beta}=\int_{V^{n+1}}(\Phi^{n+1}_{\alpha,i}\Phi^{n+1}_{\beta})dV \;;\; A^{n+1}_{\alpha\beta}=\int_{V^{n+1}}(\Phi^{n+1}_{\alpha,i}\Phi^{n+1}_{\beta,i})dV \;;\; \hat{\Omega}^{n}_{\alpha i}=\int_{S^{n}}(\Phi^{n}_{\alpha}p^{n}n_i)dS$$

$$N^{n}_{\alpha}=\int_{V^{n}}(\Phi^{n}_{\alpha})dV \;;\; \hat{\Sigma}^{n+1}_{\alpha}=\int_{S^{n+1}}(\Phi^{n+1}_{\alpha}\phi_{,i}n_i)dS$$

In equations (36) and (37), $\overline{M}^{n}_{\alpha\beta}$ means the lumped matrix obtained from the consistent mass matrix $M^{n}_{\alpha\beta}$.

6. ANALYSIS OF SOLITARY WAVE PROPAGATION

A solitary wave propagation and run up on a vertical wall have been analysed for the purpose of illustrating the validity of the present finite element method. It is assumed that the bottom of the channel is horizontal and that no obstacle exists. Hence the solitary wave will travel on the surface with no change of form. Laitone's[24] approximations for free surface profile, velocity and pressure are used for the initial conditions. The dimension of the channel is shown in Fig. 1. The still water depth d is 10 m, the wave

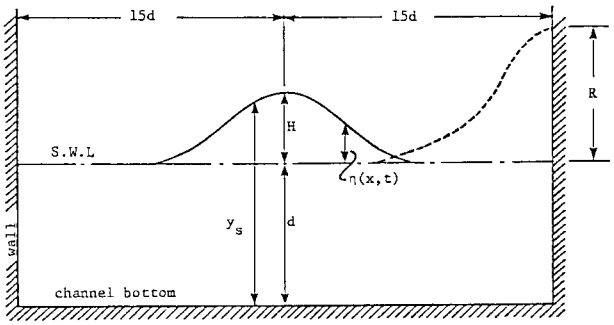

Fig. 1 Definition sketch of a solitary wave.

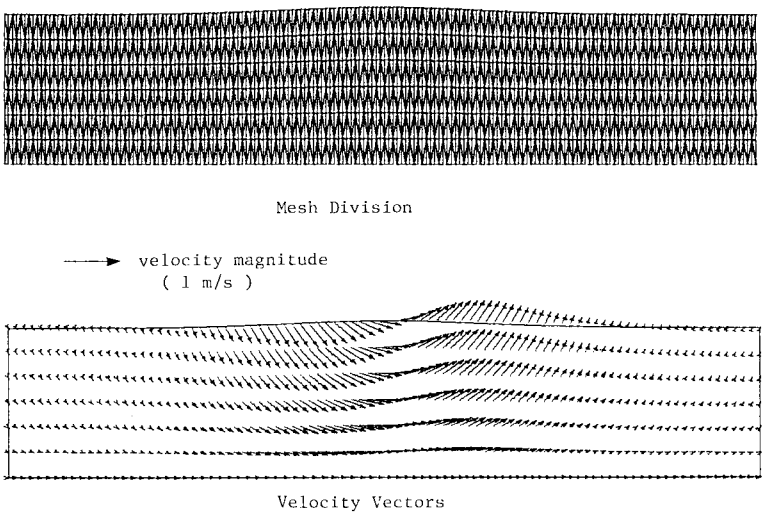

Fig. 2 Initial conditions.

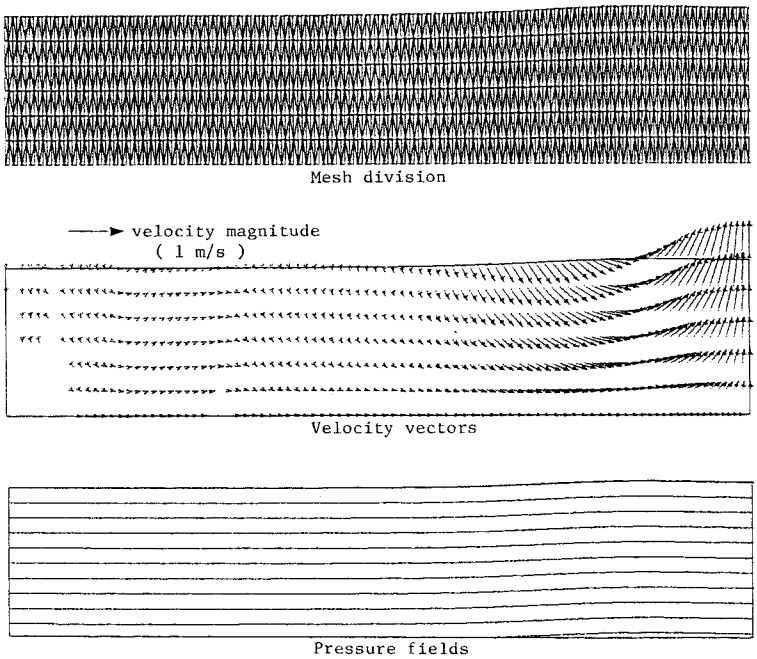

Fig. 3 Computed results at time $t=10$ s.

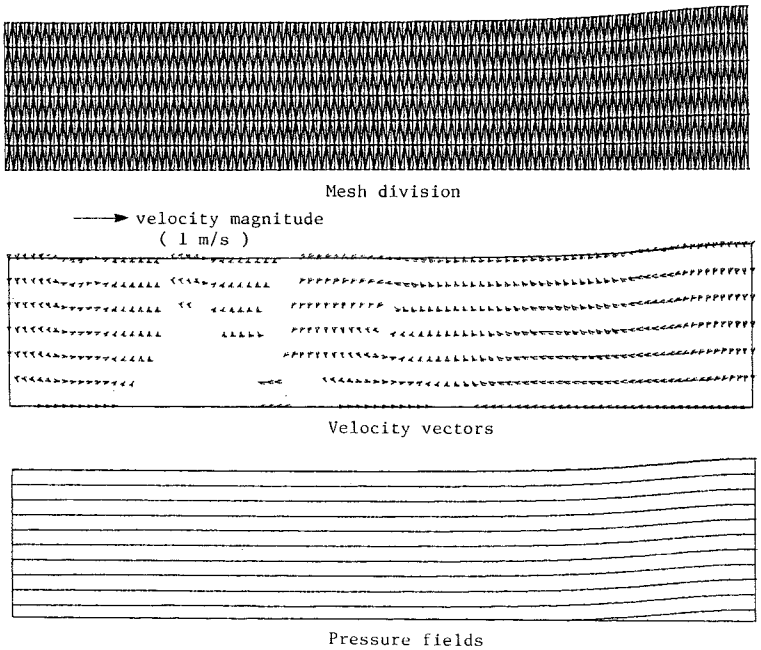

Fig. 4 Computed results at time $t=15$ s.

height H is 0.5 m and the horizontal length of the channel is $30\,d=300$ m.

For computation purposes, density is assumed to be constant, and the time increment $\Delta t=0.05$ s and parameter $\varepsilon=0.00001$ m are used. The finite element idealization at the initial stage is shown in Fig. 2. The total number of nodal points and finite elements are 1 267 and 2 160, respectively. The initial velocity and pressure computed by Laitone's approximation are also shown in Fig. 2. Starting from the initial

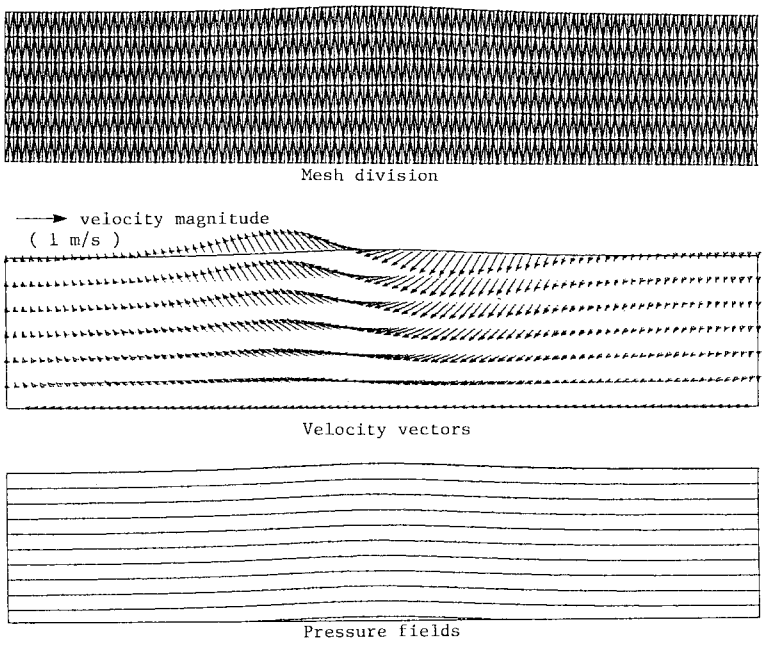

Fig. 5 Computed results at time $t=30$ s.

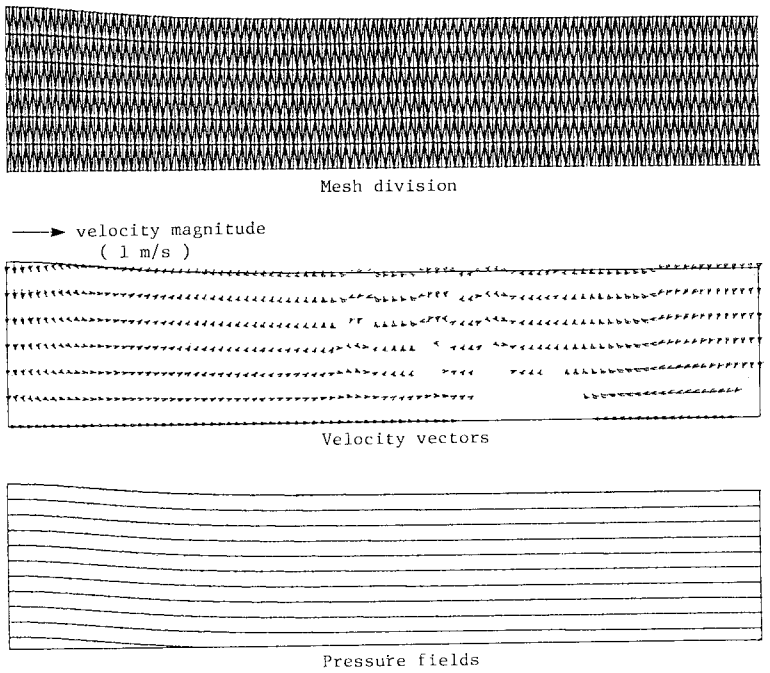

Fig. 6 Computed results at time $t=45$ s.

condition, the behaviour of the solitary wave can be computed. The maximum iteration cycle in each time step was 3 to 4 for normal computation. Computed wave profile, velocity and pressure at elapsed times 10 s, 15 s, 30 s and 45 s are shown in Figs. 3, 4, 5 and 6, respectively. The time when the wave crest arrives at the right hand vertical wall from the center of the channel is 14.8 s, which is close to Laiton's result viz., 14.79 s. The run up height on the vertical wall obtained by Laitone's approximation is $R=1.01$ m.

The computed value in the present analysis is $R=1.029$ m and the error is less than 2 %. From Figs. 4 to 6 it can be seen that the computed results are not affected by any artificial damping effect.

It should be noted that computing time was approximately 1 hour per 1 000 time steps based on single precision using the FACOM M 170 F computer of Chuo University.

7. CONCLUSION

The Lagrangian finite element method has been presented for calculating the transient dynamics of an incompressible fluid having a free surface, and has been applied to the analysis of the propagation of a solitary wave. This new method involves an extremely simple algorithmic structure, which is achieved by adopting the velocity correction method to the time integration of the unsteady Navier-Stokes equations. Lagrangian specification permit the favorable treatment of free surfaces, and maintain proper material interface without the numerical instability associated with convective terms. In the present paper an inviscid fluid was treated, but it would be easy to extend the present analysis to viscous fluids.

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