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FUNDAMENTAL SOLUTION FOR TRANSIENT INCOMPRESSIBLE VISCOUS FLOW AND ITS APPLICATION TO THE TWO DIMENSIONAL PROBLEM

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The three dimensional and two dimensional fundamental solutions for the transient incompressible viscous flow are determined by solving the singular differential equations, which are obtained in the induction process of the integral equation, by means of the Fourier transform. The fundamental solutions determined in this paper have been proved to be practical in applying to a two dimensional problem by the boundary element method.

1. INTRODUCTION

In accordance with the rapid development of scientific technology, the analysis of transient incompressible viscous flow has become an important subject to be urgently established these days. Various investigations[1)~3)], like F. D. M. or F. E. M. etc, have been made for this subject, and recently other studies using B. E. M.[4)~6)] are also coming out, all of these assume a new aspect of the fluid mechanics.

Navier-Stokes equation is a well known governing equation of transient viscous flow, but it's difficult to obtain solutions directly, even though it may satisfy the condition of incompressibility. Because the simultaneous differential equations include four more unknowns (pressure and three components of velocity), the general method of analysis usually depends on calculations making use of vorticity or stream function, or penalty function etc. However, even if we make vorticity or stream function an unknown, it's hard to grasp the physical meaning and moreover not a few restrictions exist when the velocity on the boundary is calculated. The use of stream function also makes the application to the three dimensional (3-D) problem difficult. Although usage of penalty function can define physical quantities, velocity and pressure, as unknowns, it's difficult to determine pressure.

The purpose of paper is to determine velocity and pressure directly, where we derived two integral equations based upon N-S equation and the formula once differentiated N-S equation with respect to coordinates and further applied summation convention, both of which we regard as governing equations on velocity and pressure.

Considering the above, to obtain the integral equation on velocity, first of all we combine weighted residuals which are the results of multipling the N-S equation by the fundamental solution of velocity, and

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the equation of incompressibility by the fundamental solution of pressure, as a weighting function. After partially integrating them according to Gauss' divergence theorem, we can obtain integral equation on velocity. This manipulation defines Singular Differential Equation (SDE) which satisfies the fundamental solutions of velocity and pressure. Fundamental solutions are obtained by solving this formula. Here, the 3-D and two dimensional (2-D) fundamental solutions were shown through manipulation of Fourier transform. These fundamental solutions enabled us to confirm that transient viscous flow is theoretically the same as transient heat transfer phenomenon[7] very well. On the other hand, as governing equation satisfies Poisson's equation, the integral equation on pressure is lead by weighted residuals obtained by multiplying the fundamental solution of Laplace equation to governing equation on pressure as weighting function.

An iterative calculation is required for the numerical analysis as the integral equation on velocity includes non-linear terms. Here, we apply the boundary type method of iterative calculation in which unknown quantities appear only on the boundary. Through this, the velocity on the boundary, and in the domain and surface traction come out, and then we can decide the pressure accordingly, by substituting them for the integral equation on pressure.

Attention should be paid to singular integrations in case of this analysis. As is mentioned later, we can't calculate the pressure on the boundary analytically. After the analysis it was clarified that the B. E. M. (direct method) enables us to express the basic movement of the transient incompressible viscous flow.

2. INTRODUCTION OF INTEGRAL EQUATIONS

(1) Governing differential equation

N-S equation is given as follows[8]

$$\rho\frac{Du_i}{Dt}=\rho\left(\frac{\partial u_i}{\partial t}+u_j u_{i,j}\right)=\rho X_i-p_{,i}+\mu u_{i,jj}+\frac{u_{j,ji}}{3} \qquad (1)$$

where ρ is density of a fluid, u_i is velocity component of fluid particles, X_i is body force component, μ is viscous coefficient and p is pressure. Commas represent infinitesimal operators and accompanying letters are governed by the summation convention.

Incompressibility of a fluid satisfies eq. (2)[8]. Modification of eq. (1) using eq. (2) gives the governing equation of velocity, eq(3).

$$u_{i,i}=0 \qquad (2)$$

$$\rho\frac{\partial u_i}{\partial t}-\mu(u_{i,jj}+u_{j,ji})+p_{,i}+\rho u_j u_{i,j}-\rho X_i=0 \qquad (3)$$

The other governing equation, of the pressure, can be given as follows, by differentiating eq. (3) with respect to direction and adopting the summation convention

$$\rho\frac{\partial u_{i,i}}{\partial t}-\mu(u_{i,jji}+u_{j,jii})+p_{,ii}+(\rho u_j u_{i,j})_{,i}-\rho X_{i,i}=0 \qquad (4)$$

Substituting eq. (2) for eq. (4), eq. (4) shows the pressure which satisfies the Poisson's equation[9].

(2) Definition of SDE and introduction of integral equations with respect to velocity

Considering the time and 3-D space shown in Fig. 1, u^*_{Ki} is to be the fundamental solution for velocity in the direction k on point B generated by the unit force working toward direction i on point A, and p^*_K is to be the normal component of fundamental solution for pressure working in the direction of k on point B generated by the unit pressure working on point A. And suffix 's' and 'o' on the domain Ω and the boundary Γ in Fig. 1 represent source points and observe points respectively, however, they are virtually $\Omega_s=\Omega_0$ and $\Gamma_s=\Gamma_0$.

After multiplying u^*_{Ki} by eq. (3) as weighting function, and equally p^*_K by eq. (2), we can get weighted residuals as follows by differential and/or integral calculation of Ω_s and t_s in Fig. 1. Herein u^*_{Ki} has reciprocity according to the property of Green function which is also clarified by eq. (26) mentioned in the latter part[10].

$$0=\int_{\Omega_s}\int_{t_s}^{t}\left[u^*_{Ki}\left\{\rho\frac{\partial u_i}{\partial t_s}-\mu(u_{i,jj}+u_{j,ji})+p_{,i}+\rho u_j u_{i,j}-\rho X_i\right\}+(-p^*_K)u_{i,i}\right]dt_s d\Omega_s \quad (5)$$

Further, assuming that the fundamental solution for velocity satisfies the incompressibility, we get

$$u^*_{Ki,i}=0 \quad (6)$$

Then, integrating eq. (5) in the direction by means of Gauss' divergence theorem, and substituting eq. (2) and (6) for it, we get the following formula (in which upper and lower limits of integrations are omitted).

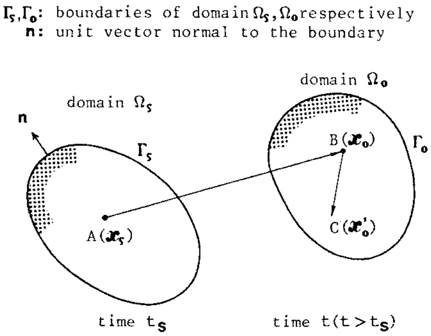

Fig. 1 Schematic Diagram.

$$-\iint\left(-\rho\frac{\partial u^*_{Ki}}{\partial t_s}-\mu u^*_{Ki,jj}+p^*_{K,i}\right)u_i dt_s d\Omega_s-\iint\{-\tilde{\delta}_{ij}p^*_K+\mu(u^*_{Ki,j}+u^*_{Kj,i})\}n_j u_i dt_s d\Gamma_s$$
$$=-\iint u^*_{Ki}\{-\tilde{\delta}_{ij}p+\mu(u_{i,j}+u_{j,i})\}n_j dt_s d\Gamma_s+\iint \rho u^*_{Ki}u_j n_j u_i dt_s d\Gamma_s+\int \rho u^*_{Ki}u_i|^t_{t_s}d\Omega_s$$
$$-\iint(\rho u^*_{Ki,j}u_j u_i+\rho u^*_{Ki}X_i)dt_s d\Omega_s \quad (7)$$

where Γ_s is boundary of domain Ω_s, n_j is j directed component of unit vector on Γ_s, $\tilde{\delta}_{ij}$ is Kronecker's delta. Eq. (7) can define the following SDE which the fundamental solutions satisfy.

$$-\rho\frac{\partial u^*_{Ki}}{\partial t_s}-\mu u^*_{Ki,jj}+p^*_{K,i}+\tilde{\delta}_{Ki}\delta(x_0-x_s)\delta(t-t_s)=0 \quad (8)$$

where both $\delta(x_0-x_s)$ and $\delta(t-t_s)$ represent Dirac's delta function.

Eq. (8) corresponds to the SDE of transient Stokes flow. If the fundamental solutions satisfy eq. (8) eq. (7) is consequently, rewritten as the integral equation for observe point B.

$$C_{Ki}u_i(x_0,t)+\iint T^*_{Ki}u_i dt_s d\Gamma_s=\iint u^*_{Ki}T_i dt_s d\Gamma_s+\int \rho u^*_{Ki}u_i|^t_{t_s}d\Omega_s+\iint \rho u^*_{Ki}u_j n_j u_i dt_s d\Gamma_s$$
$$-\iint(\rho u^*_{Ki,j}u_j u_i+\rho u^*_{Ki}X_i)dt_s d\Omega_s \quad (9)$$

where T^*_{Ki}, T_i and C_{Ki} represent the following equations.

$$\left.\begin{aligned}T^*_{Ki}&=-\{-\tilde{\delta}_{ij}p^*_K+\mu(u^*_{Ki,j}+u^*_{Kj,i})\}n_j\\ T_i&=-\{-\tilde{\delta}_{ij}p+\mu(u_{i,j}+u_{j,i})\}n_j\end{aligned}\right\} \quad (10)$$

$$C_{Ki}=\begin{cases}\tilde{\delta}_{Ki}\in\Omega_0\\ \tilde{C}_{Ki}\in\Gamma_0\\ 0\ \notin\Omega_0\end{cases} \quad (11)$$

Further, the stress working on the fluid is defined by eq. (12)[8].

$$\sigma_{ij}=-\tilde{\delta}_{ij}p+\mu(u_{i,j}+u_{j,i}) \quad (12)$$

Therefore, eq. (10) shows contraction affecting boundaries.

(3) Integral equation on pressure

Considering domain Ω_0 at $t(=t_s)$ gives the following equation indicating weighted residuals which was obtained by the application of the fundamental solution H^* to eq. (4) as a weighting function. (Assuming that the differentiation and integration is only for point B and the integral domain is Ω_0)

$$0=\int H^*\left\{\rho\frac{\partial u_{i,i}}{\partial t_s}-\mu(u_{i,jji}+u_{j,jii})+p_{,ii}+\rho(u_j u_{i,j})_{,i}-\rho X_{i,i}\right\}d\Omega_0 \quad (13)$$

Integrating eq. (13) in the direction i and j, using the Gauss' divergence theorem, and noting velocity and pressure on the boundary satisfy eq. (3), we have integral equation on pressure for point C as follows.

$$p(x'_0)=-\int H^*_{,i}T_i d\Gamma_0-2\int \mu H^*_{,ij}n_j u_i d\Gamma_0+\int \rho H^*_{,ij}u_j u_i d\Omega_0-\int \rho H^*_{,i}u_j n_j u_i d\Gamma_0$$

$$-\int \rho H^*_{,i}\left(\frac{\partial u_i}{\partial t_s}-X_i\right)d\Omega_0 \cdots\cdots (14)$$

where,

$$H^*=\frac{1}{4\pi R} \qquad \text{(3-D)}^{15)} \cdots\cdots (15)$$

$$H^*=\frac{1}{2\pi}\ln\frac{1}{R} \quad \text{(2-D)}^{15)} \cdots\cdots (16)$$

3. FUNDAMENTAL SOLUTIONS

(1) Fundamental solutions for 3-D

By the Fourier transform the Dirac's delta function can be defined to time and space as follows[30].

$$\delta(\boldsymbol{x})\delta(\tau)=\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\boldsymbol{K}\boldsymbol{x}}\, e^{-iK_t\tau}\, d\boldsymbol{K}dK_t \cdots\cdots (17)$$

where i represents imaginary unit, $\boldsymbol{K}$ represents vector of parameter K, $\boldsymbol{x}=\boldsymbol{x}_0-\boldsymbol{x}_s$ and $\tau=t-t_s$.

Also considering A_{Ki} and B_K as constants with respect to coordinates and time, u^*_{Ki} and p^*_K can be defined by the Fourier transform as follows.

$$u^*_{Ki}=\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_{Ki}\, e^{-i\boldsymbol{K}\boldsymbol{x}}\, e^{-iK_t\tau}\, d\boldsymbol{K}dK_t \cdots\cdots (18)$$

$$p^*_K=\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_K\, e^{-i\boldsymbol{K}\boldsymbol{x}}\, e^{-iK_t\tau}\, d\boldsymbol{K}dK_t \cdots\cdots (19)$$

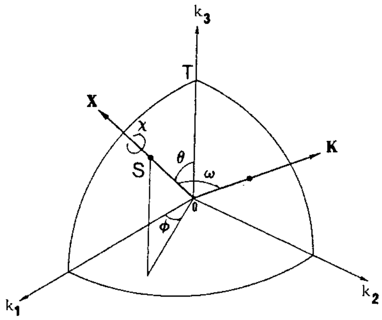

Fig. 2 Spherical Coordinates System.

Substituting eq. (17) through (19) for eq. (6) and (8) respectively, we get

$$-i\rho K_t A_{Ki}+\mu K^2 A_{Ki}+iK_i B_K+\tilde{\delta}_{Ki}=0 \cdots\cdots (20)$$

$$K_i\cdot A_{Ki}=0 \quad (K^2=K_j\cdot K_j) \cdots\cdots (21)$$

Then, we have A_{Ki} and B_K by reducing above equations.

$$A_{Ki}=\frac{\tilde{\delta}_{Ki}K^2-K_iK_K}{(i\rho K_t-\mu K^2)K^2} \cdots\cdots (22)$$

$$B_K=-\frac{K_K}{iK^2} \cdots\cdots (23)$$

Integration with respect to $\boldsymbol{K}$ and K_t after substituting eq. (22) and (23) for eq. (18) and (19), can give u^*_{Ki} and p^*_K. To be more specific, K_t can be integrated by means of the residue theorem and/or definition of delta function, and $\boldsymbol{K}$ can be integrated by converging origins of coordinates of $\boldsymbol{K}$ space and $\boldsymbol{X}$ space, which rotates the projected point S of X on the unit sphere to the zenith T, and by converting integral transforms[11]. Then $d\boldsymbol{K}$ is given as below.

$$d\boldsymbol{K}=K^2\sin\omega dK d\omega d\chi \qquad K[0,\infty],\ \omega[0,\pi],\ \chi[0,2\pi] \cdots\cdots (24)$$

And the fundamental solutions can be described as follows. (see Appendix 1)

$$u^*_{Ki}=\frac{-1}{8\pi^{3/2}\rho}\left[\left(\tilde{\delta}_{Ki}-\frac{x_Kx_i}{R^2}\right)u^*+\left(\tilde{\delta}_{Ki}-\frac{3x_Kx_i}{R^2}\right)\left\{\frac{2\nu\tau}{R^2}u^*-\frac{4}{R^3}\mathrm{Erf}\left(\frac{R}{2\sqrt{\nu\tau}}\right)\right\}\right] \cdots\cdots (25)$$

$$p^*_K=\frac{x_K}{4\pi R^3}\delta(\tau) \cdots\cdots (26)$$

where,

$$u^*=\frac{1}{(\nu\tau)^{3/2}}e^{-\frac{R^2}{4\nu\tau}} \cdots\cdots (27)$$

$$\mathrm{Erf}(x)=\int_0^x e^{-y^2}dy=\frac{\sqrt{\pi}}{2}-\int_x^{\infty} e^{-y^2}dy=\frac{\sqrt{\pi}}{2}-\mathrm{Erfc}(x) \cdots\cdots (28)$$

$$R^2 = x_i \cdot x_i \quad \cdots\cdots (29)$$

ν represents coefficient of kinematic viscosity $(=\mu/\rho)$, and $\mathrm{Erfc}(x)$ represents error function.

(2) Fundamental solutions for 2-D

We can derive the 2-D fundamental solutions from the 3-D ones[13]. Supposing that the unit force working on point A has continuously moved along the axis X_2 from $-\infty$ to ∞ as shown in Fig. 3, the 2-D fundamental solutions ${}^2u^*_{Ki}$ and ${}^2p^*_K$ are therefore given by integrating eq. (26) and eq. (27) respectively[19]. (see Appendix 2)

$$\begin{aligned} {}^2u^*_{Ki} &= \int_{-\infty}^{\infty} u^*_{Ki} dx_j \quad (k \neq j,\ i \neq j) \\ &= \frac{-1}{4\pi\rho}\left\{\left(\tilde{\delta}_{Ki} - \frac{x_K x_i}{R^2}\right)\frac{1}{\nu\tau} e^{-\frac{R^2}{4\nu\tau}} - \frac{2}{R^2}\left(\tilde{\delta}_{Ki} - \frac{2x_K x_i}{R^2}\right)\left(1 - e^{-\frac{R^2}{4\nu\tau}}\right)\right\} \quad \cdots\cdots (30) \end{aligned}$$

$$\begin{aligned} {}^2p^*_K &= \int_{-\infty}^{\infty} p^*_K dx_j \quad (k \neq j) \\ &= \frac{x_K}{2\pi R^2}\delta(\tau) \quad \cdots\cdots (31) \end{aligned}$$

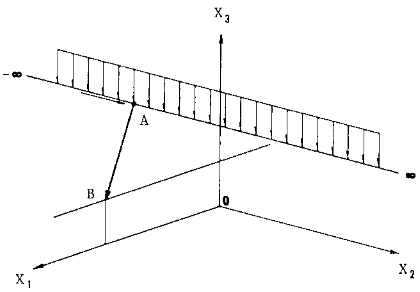

Fig. 3 Continuously Acting Unit Force Parallel to the X_2 Axis.

Incidentally, the fundamental solution of transient Oseen flow is shown by intuitive investigation in ref. (14), but it can be also induced by the same method noted in this paper.

Assume that the velocity $u_i (=\bar{u}_i + \hat{u}_i)$ is the total of constant components $(\bar{u}_i)$ and variable ones $(\hat{u}_i)$. Substituting it for eq. (3) and taking the same procedure as we induced eq. (8), we get

$$-\rho\frac{\partial u^*_{Ki}}{\partial t_s} - \mu u^*_{Ki,jj} - \rho\bar{u}_j u^*_{Ki,j} + p^*_{K,i} + \tilde{\delta}_{Ki}\delta(x)\delta(\tau) = 0 \quad \cdots\cdots (32)$$

And using the same method to reduce the 3-D fundamental solutions, we get

$$A_{Ki} = \frac{\tilde{\delta}_{Ki}K^2 - K_K K_i}{(i\rho K_t - \mu K^2 + i\rho K_j \bar{u}_j)K^2} \quad \cdots\cdots (33)$$

$$B_K = -\frac{K_K}{iK^2} \quad \cdots\cdots (34)$$

Above results (eq. (33) and (34)) clarify, after all, that the fundamental solutions of transient Oseen flow can be obtained by replacing x_i with $x_i - \bar{u}_i\tau$ on that of transient Stokes flow in eq. (25). Obtained p^*_K is identical to that of transient Stokes flow despite u^*_{Ki} is different each other.

Same manipulation can be used for the 2-D case.

(3) Fundamental solutions for steady state flow

As is mentioned later, we use fundamental solutions for steady state flow in numerical analysis to compare steady state flow with transient one. According to ref. (18), they are given as follows

(3-D)

$$u^*_{Ki} = \frac{-1}{8\pi\mu R}\left(\tilde{\delta}_{Ki} + \frac{x_K x_i}{R^2}\right) \quad \cdots\cdots (35)$$

$$p^*_K = \frac{x_K}{4\pi R^3} \quad \cdots\cdots (36)$$

(2-D)

$$u^*_{Ki} = \frac{-1}{4\pi\mu}\left(\tilde{\delta}_{Ki}\ln\frac{1}{R} + \frac{x_K x_i}{R^2}\right) \quad \cdots\cdots (37)$$

$$p^*_K = \frac{x_K}{2\pi R^2} \quad \cdots\cdots (38)$$

4. APPLICATION TO 2-D PROBLEM

(1) Time quadrature of fundamental solutions

By minimizing the time interval (Δt) and assuming a constant velocity during this interval, the fundamental solutions can be integrated with respect to time. The fundamental solutions after time quadrature can be indicated with ~ (tidal) as follows

$$\tilde{u}^*_{Ki}=\int_{t_s}^{t(=t_s+\Delta t)} {}^2u^*_{Ki}\,dt_s=\frac{1}{4\pi\rho}\left\{\frac{\tilde{\delta}_{Ki}}{2\nu}E_i(-X_0)+\frac{2\Delta t}{R^2}\left(\tilde{\delta}_{Ki}-\frac{2x_Kx_i}{R^2}\right)(1-e^{-X_0})\right\} \quad (39)$$

$$\tilde{p}^*_K=\int_{t_s}^{t_s+\Delta t} {}^2p^*_K\,dt_s=\frac{x_K}{2\pi R^2} \quad (40)$$

where,

$$X_0=\frac{R^2}{4\nu\Delta t} \quad (41)$$

$$-E_i(-X_0)=\int_{X_0}^{\infty}\frac{e^{-y}}{y}\,dy \qquad \text{(exponential integral function)}^{17)} \quad (42)$$

$\tilde{T}^*_{Ki}$ is also obtained by differentiating eq. (39) and substituting it for eq. (10).

(2) Evaluation of singularity

a) Concerning with integral equation on velocity

$\tilde{C}_{Ki}$ in eq. (11) can be evaluated by removing the limit of $r(\varepsilon\rightarrow 0)$ considering the node as a point within the domain after attaching the infinitesimal small circle ($r=\varepsilon$) to the node of the boundary, shown in Fig. 4. Consequently, $\tilde{C}_{Ki}$ is obtained as the following matrix for the 2-D. Then the integration of the 2nd term in the left latus of eq. (9) is obtained as Cauchy's principal value[19].

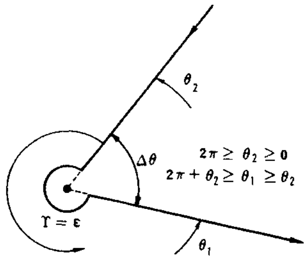

Fig. 4 Definition of Angles for Computation of $\tilde{C}_{Ki}$.

$$\tilde{C}_{Ki}=\begin{bmatrix}\dfrac{\Delta\theta}{2\pi}-\dfrac{\sin2\theta_1-\sin2\theta_2}{4\pi} & \dfrac{\cos2\theta_1-\cos2\theta_2}{4\pi}\\[2ex] \dfrac{\cos2\theta_1-\cos2\theta_2}{4\pi} & \dfrac{\Delta\theta}{2\pi}+\dfrac{\sin2\theta_1-\sin2\theta_2}{4\pi}\end{bmatrix} \quad (43)$$

b) Concerning with integral equation on pressure

In case we're going to calculate the pressure on the boundary by eq. (14), concerned with the 2-D problem, $O(R^{-1})$ (right latus, the 1st and 4th term), and $O(R^{-2})$ (right latus, the 2nd term) appear on the boundary integual term, and $O(R^{-1})$ (right latus, the 5 th term), and $O(R^{-2})$ (right latus, the 3rd term), does on the domain integral term.

$O(R^{-1})$ on the domain integral term does not bear such a singularity as to be noted, and $O(R^{-2})$ on the domain integral term can be analytically evaluated through the same method as a). However, $O(R^{-2})$ on the boundary integral term can't be solved analytically as it becomes divergent integral.

Herein, we will describe the boundary integral which has the singularity of $O(R^{-2})$.

Generally, the principal value of the integral is defined by

$$p\int_a^b\frac{1}{x^m}\,dx=\lim_{\varepsilon\to 0}\left[\int_a^{-\varepsilon}\frac{1}{x^m}\,dx+\int_{\varepsilon}^{b}\frac{1}{x^m}\,dx\right] \qquad (a<0<b,\ \varepsilon>0) \quad (44)$$

and in case m is an odd number, ε cancels each other, but in case m is an even number, ε remains after the integration[20]. Therefore, in the latter case, the principal value of the integral needs to be evaluated using the Hadamard's finite part of divergence integral ($pf\int$)[21].

$$pf\int_a^b\frac{1}{x^m}\,dx=\frac{m-1}{a^{m-1}}-\frac{m-1}{b^{m-1}} \quad (45)$$

Consequently, we have to approximately calculate such a singularity, because we give a minimal value to a remaining ε.

The integration of the 2nd term in the right latus of eq. (14) can be made as follows. If the singularity is to be found at point (1) shown in Fig. 5, the principal value of the boundary integral can be obtained as follows, where it should be noted that velocity component u_i is assumed to make linear changes between point (1) and (2), that suffix figure in a parenthesis represents node number, and $d\Gamma_0=dR$.

$$H^*_{,1j}\, n_j = -\frac{\sin\theta}{2\pi R^2} \quad \cdots\cdots (46)$$

$$H^*_{,2j}\, n_j = -\frac{\cos\theta}{2\pi R^2} \quad \cdots\cdots (47)$$

$$u_i = \left(1-\frac{R}{R_2}\right) u_{i(1)} + \frac{R}{R_2}\, u_{i(2)} \quad \cdots\cdots (48)$$

$$\int \mu H^*_{,ij}\, n_j u_i d\Gamma_0 = \frac{-\sin\theta}{2\pi}\left[pf\int_0^{R_2}\mu\left\{\left(\frac{1}{R^2}-\frac{1}{R_2 R}\right)u_{1(1)}+\frac{1}{R_2 R}\, u_{1(2)}\right\}dR\right] + \frac{\cos\theta}{2\pi}\left[pf\int_0^{R_2}\mu\left\{\left(\frac{1}{R^2}-\frac{1}{R_2 R}\right)u_{2(1)}+\frac{1}{R_2 R}\, u_{2(2)}\right\}dR\right] \quad \cdots\cdots (49)$$

The principal value of the integral at singular node should be evaluated in the manner mentioned in a), which attaches the infinitesimal small circle with radius ε to the node, but remaining ε is given a minimal value. Therefore, the integration of the 2 nd term in the right latus of eq. (14) is consequently, made approximately.

(3) Flow chart of numerical computation

B. E. M. using internal cell to discretize the boundaries and domains can numerically obtain velocity and pressure at the time of $t(=t_s+\Delta t)$. As is shown in Fig. 6, we adopt the boundary type formulation, where unknowns appear only on the boundary, and (i), (ii) and (iii) represent following calculations respectively.

(i) $C_{Ki}u_i^n+\int \tilde{T}^*_{Ki}u_i^n d\Gamma_s,\ \int \tilde{u}^*_{Ki}T_i^n d\Gamma_s$

(ii) $\int \rho u^*_{Ki}u_i d\Omega_s^{(t=t_s)},\ \int \rho\tilde{u}^*_{Ki}X_i d\Omega_s$

(iii) $\int \rho\tilde{u}^*_{Ki}u_j^{n-1}n_j u_i^{n-1}d\Gamma_s,\ \int \rho\tilde{u}^*_{Ki,j}u_j^{n-1}u_i^{n-1}d\Omega_s$

This way of calculation is known as simple iteration method[22] which has such advantages that influence matrices made in prior step can be left as it is unless the boundary conditions and Δt are changed. Incidentally, the detail of the dicretization method are omitted here[16].

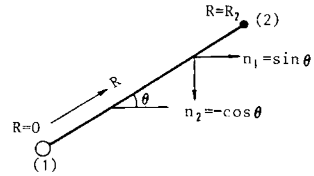

Fig. 5 Geometrical Mapping of a Boundary Element for a Singular Boundary Integral.

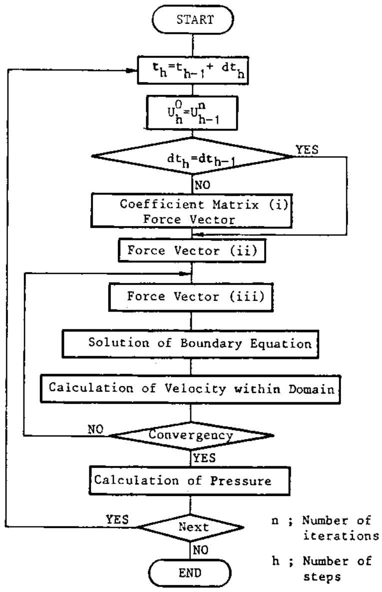

Fig. 6 Flow Diagram for Computation.

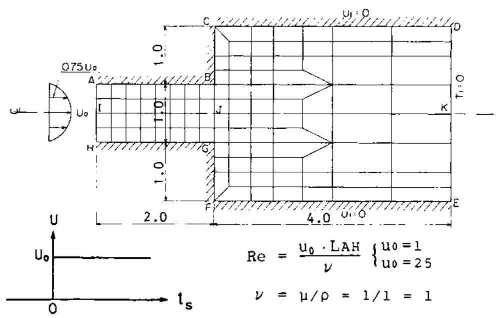

Fig. 7 Boundary Element Mesh and Boundary Conditions.

(4) The result of numerical analysis

Fig. 7 shows the model and the boundary conditions for numerical analysis. The current given as the boundary condition is to be fixed in terms of time. We've applied $\Delta t=0.1$ for the increment of time and carried out our analysis to the degree of $t_s=0\sim 2$. Although we've fixed convergence check (standard) to be 10^{-2}, it is observed that the convergence frequency is $2\sim 3$ times in case of $Re=1$, and $7\sim 10$ times in case of $Re=25$. Body force was disregarded in this analysis. We've approximate the acceleration, the 5th term on the right latus of eq. (14) according to the following formula.

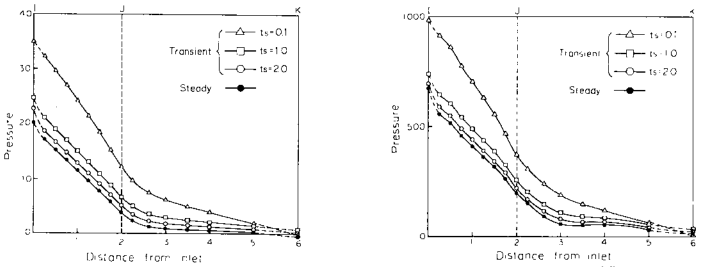

Fig. 8 Pressure Curve along the Center Line.

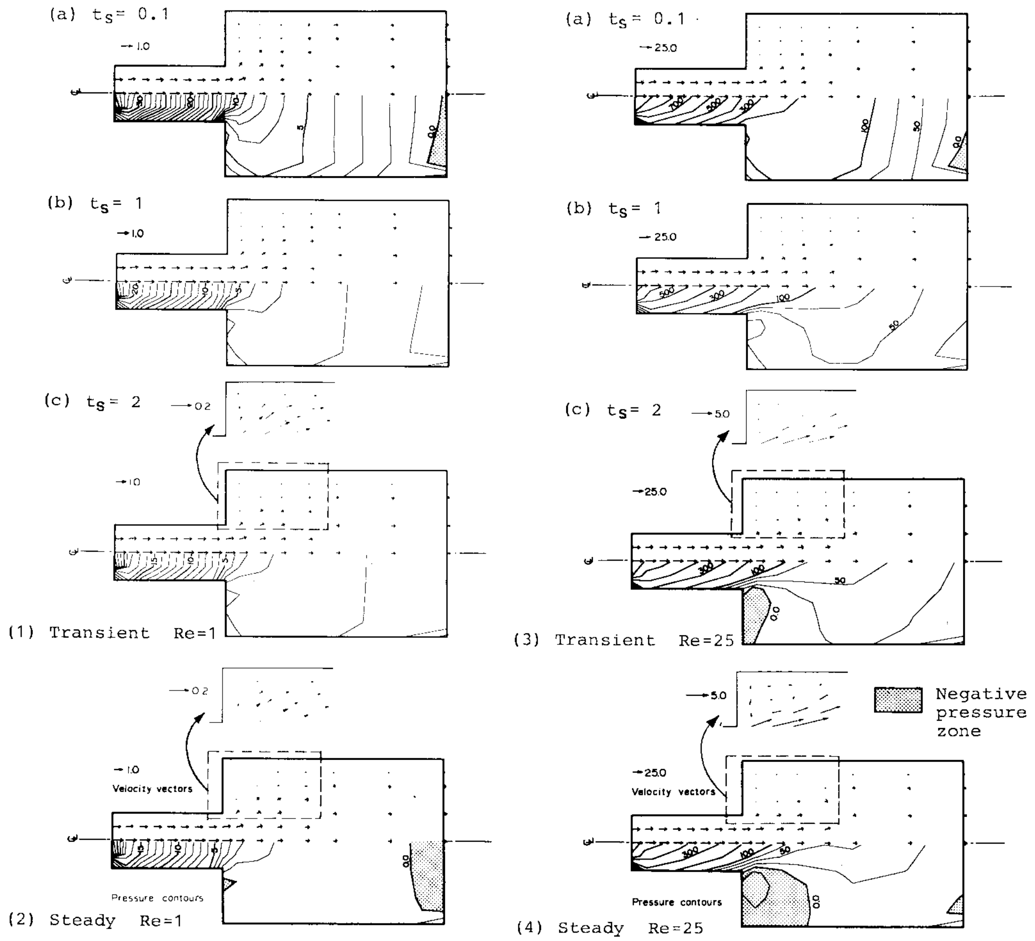

Fig. 9 Velocity Vectors and Dimensionless Pressure Contours.

$$\frac{\partial u_i}{\partial t_s}=\frac{u_i^h-u_i^{h-1}}{\Delta t} \quad \cdots\cdots (50)$$

Fig. 9 shows current and pressure distribution on each time of t_s=0.1, 1, and 2. Incidentally, results using the steady state fundamental solutions are also presented for comparison. In case of Re=25, the domain forming vortex and the domain generating negative pressure coincide very well except for the area near the outlet.

Fig. 8 shows pressure inclination along the model center line (I-J-K) in Fig. 7 for the both case of Re=1 and $Re=25$. As the pressure on the boundary was calculated approximately, $\varepsilon=10^{-4}$, noted in 4. (2), b), we indicated the pressure inclination around the inlet/outlet with dashed line in Fig. 8 where it is shown that the approximate solution on the boundary does not loose the qualitative tendency of pressure. It is well described in Fig. 8 and Fig. 9 that both $Re=1$ and $Re=25$ between I-J have nearly the same current distribution, and of which condition is laminar because of the pressure falling rectilinearlly[23]. According to the above results, it is expected that the transient solution would coincide with the steady solution by means of $t_s \to \infty$ as the current distribution and the pressure distribution become close to steady with the increase of time ($t_s=0\sim2$). And this fact also indicates that sufficient results can be expected even if we apply approximate acceleration of velocity to the transient analysis as we did in eq. (50). After comparing flow quantity passing through its section on the inlet boundary with the one on the outlet boundary, we've also confirmed that the continuous condition was formed with an aberration no more than 1 %.

5. CONCLUDING REMARKS

The contents of this paper are summarized as follows :

(1) Regarding transient incompressible viscous flow problem, we've induced the integral equation which enabled us to calculate the velocity of fluid directly by applying the fundamental solutions of velocity and pressure. And, in order to induce the integral equation on pressure, we made use of Laplace solution.

(2) We have investigated the concrete form of fundamental solutions for velocity and pressure of Stokes flow in the three dimension by solving the singular differential equation, which was obtained through mathematical manipulation with the use of Fourier transform. We've also got the fundamental solutions in the two dimension.

(3) According to (1) and (2), we've confirmed the appropriateness (of our paper) by application of B. E. M. (direct method), where to the two dimensional sudden expansion nodel, the velocity or contraction on the boundary are defined as unknowns, and comparing it with the result previously calculated by the fundamental solutions of the steady problem.

Results of the analysis on this paper showed a tendency that convergence frequency increases with the increase of Reynolds number. However, we suppose it might become impossible, from the financial viewpoint, to deal with problems of high Reynolds number by means such as B. E. M. using internal cell which uses simple iteration.

According to the recent investigations[24], however, it has become possible to analyze them to the degree of high Reynolds number by means of Hybrid method only with a couple of convergences. We do expect further investigation on this subject.

APPENDIX 1[25),26)] ($a>0$)

$$K=\begin{bmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{Bmatrix} K\sin\omega\cos\chi \\ K\sin\omega\sin\chi \\ K\cos\omega \end{Bmatrix}$$

$$\int_0^\infty e^{-a^2x^2}\cos bx\,dx=\frac{\sqrt{\pi}}{2a}\,e^{-\frac{b^2}{4a^2}}$$

$$\int_0^\infty \frac{\sin bx}{x}\,e^{-a^2x^2}dx=\frac{\pi}{2}-\sqrt{\pi}\,\mathrm{Erfc}\left(\frac{b}{2a}\right)$$

APPENDIX 2[27)~29)] ($a>0,\ b>0$)

$$\int_0^\infty e^{-a^2x^2}dx=\frac{\sqrt{\pi}}{2a}=\frac{1}{a}\,\mathrm{Erfc}(0)$$

$$\int_0^\infty \frac{e^{-a^2x^2}}{x^2+b^2}\,dx=\sqrt{\pi}\,\frac{1}{b}\,e^{a^2b^2}\mathrm{Erfc}(ab)$$

$$\int_0^\infty \frac{b^2 e^{-a^2x^2}}{(x^2+b^2)^2}\,dx=\frac{1}{2}\int_0^\infty \frac{e^{-a^2x^2}}{x^2+b^2}\,dx+a^2\int_0^\infty \frac{x^2 e^{-a^2x^2}}{x^2+b^2}\,dx$$

$$\int_{t_s}^{t} \frac{1}{(\nu\tau)^{3/2}}\, e^{-\frac{R^2}{4\nu\tau}} dt_s=\frac{4}{\nu R}\,\mathrm{Erfc}\left(\frac{R}{2\sqrt{\nu\tau}}\right) \quad (\tau=t-t_s)$$

$$\int \frac{1}{(x^2+b^2)^{3/2}}\,dx=\frac{x}{b^2(x^2+b^2)^{1/2}}$$

$$\int \frac{1}{(x^2+b^2)^{5/2}}\,dx=\frac{1}{b^4}\left[\frac{x}{(x^2+b^2)^{1/2}}-\frac{x^3}{3(x^2+b^2)^{3/2}}\right]$$

REFERENCES

1) Kawai, T. ed. : Finite Element Flow Analysis, Univ. of Tokyo press, 1973.

2) Okajima, A. : Strouhal Numbers of Rectangular Cylinders, J. F. M., Vol. 123, pp. 379~398, 1982.

3)* Yoshida, Y. et al. : A Solution Procedure for the Finite Element Equations of Transient Incompressible Viscous Flows, Proc. of JSCE. Soc. Civil Eng., 351/ I I -2, pp. 59~68, 1984.

4) Wu, J. C. et al. : Problems of Time-dependent Navier-Stokes Flow, Developments in Boundary Element Methods-3, Elsevier Applied Science Publishers, pp. 137~170, 1984.

5) Brebbia, C. A. et al. : Boundary Element Techniques, Springer-Verlag, pp. 386~399, 1984.

6)* Tosaka, N. and Kakuda, K. : Boundary Element Analysis of the Unsteady Viscous Flows, Proc. of 1st Jap. Natio. Sympo. on B. E. M., pp. 241~246, 1984.

7)* Uematsu, T. : Fluid Mechanics, Kyoritsu Pub. p. 92, 1968.

8)* Landau, L. D. and Lifshitz E. M. (Takeuchi, H trans.) : Fluid Mechanics 1, Tokyo Tosho, p. 53, 1982.

9)* Honma, M. and Aki K. : Mononobe's Hydraulics, Iwanami Shoten, p. 458, 1978.

10)* Imamura, T. : Physics and Green Function, Iwanami Shoten, p. 22, 1981.

11)* Inui, T. : Differential Equations and Its Applications, Corona Co., p. 136, 1967.

12)* Moriguchi, S. et al. : Mathematical Formulae Collection (Ⅲ), Iwanami Shoten, p. 21~22, 1977.

13) preceding 10, p. 38

14) Miyake, Y. et al. : Study of Three-Dimensional Unsteady Oseen Flow, J. F. M., Vol. 86, Part 4, pp. 609~622, 1978.

15) preceding 10, pp. 26~27

16) Brebbia, C. A. and Walker, S. : Boundary Element Techniques, Newnes-Butterworths, 1980.

17) preceding 12, p. 21

18) Youngren, G. K. and Acrivos, A. : Stokes Flow Past a Particle of Arbitary Shape : a Numerical Solution, J. F. M., Vol. 69, Part 2, pp. 377~403, 1975.

19)* Tanaka, M. et al. : Boundary Element Method-Its Foundation and Application, Maruzen, p. 19, 1982.

20)* Imai, I. : Applied Hyperfunction (Ⅰ), Science Co., pp. 41~42, 1982.

21)* Moriguchi, S. et al. : Mathematical Formulae Collection (Ⅰ), Iwanami Shoten, pp. 51~52, 1977.

22)* Onishi, K. : Transient Convective Diffusion Problem, Mathematical Sciences, Science Co., p. 40, 1984. 8.

23)* Honma, M. : Hydraulics, Maruzen, p. 79, 1967.

24)* Tosaka, N. and Kakuda, K. : Steady Analysis of Incomperssible Viscous Fluid by Boundary Integral Method, Proc. of 2nd Jap. Natio. Sympo. on B. E. M., pp. 161 ~166, 1985.

25) preceding 21, p. 233

26) Gradshteyn, I. S. and Ryzhik, I. M. : Table of Integralas, Series, and Products, Academic Press, p. 495, 1983.

27) preceding 21, p. 232

28) preceding 26, p. 338

29) preceding 12, p. 111

30) preceding 10, p. 234

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