

# A LINEARIZED TIMOSHENKO BEAM THEORY IN FINITE DISPLACEMENTS

*By Akio HASEGAWA\**, *Tetsuo IWAKUMA\*\** and *Shigeru KURANISHI\*\*\**

A linearized finite displacement theory of the Timoshenko beam is formulated as the counterpart to so-called beam-column theory of the Bernoulli-Euler beam. The corresponding stiffness equation is derived in a useful form for practical applications. Both theory and stiffness equation result in the same buckling load as the Engesser formula which has been considered reasonable from the theoretical point of view.

## 1. INTRODUCTION

The Timoshenko beam theory takes the effect of shear deformation into account<sup>1)</sup>. This theory has been formulated also in finite displacements<sup>2),3)</sup>, and two basic buckling formulae are obtainable depending on the choice of constitutive equations<sup>3)-5)</sup>. They are known as Engesser's formula and the modified formula. It has been, however, shown recently that the more fundamental approaches result in the Engesser formula rather than the modified one<sup>3),5)</sup>.

In practice, the linearized theory rather than the rigorous nonlinear theory is needed, and the corresponding "geometrical" matrix is most useful for applications. Since the difference between two buckling formulae stems from the different definitions of constitutive laws<sup>3)</sup>, it is necessary to formulate a linearized finite displacement theory carefully and systematically. To this end, the virtual work for the deviated deformation from an arbitrary equilibrium state will be derived first. Then the linearized theory as well as the stiffness equation are automatically obtained from this equation of virtual work with reasonably assumed constitutive relations.

The main objective of this paper is to derive a linearized finite displacement theory and its stiffness matrix of the Timoshenko beam that correspond to the "beam-column" theory of the Bernoulli-Euler beam, and then the consistency of the present results with Engesser's formula is demonstrated.

## 2. A LINEARIZED FINITE DISPLACEMENT THEORY OF THE TIMOSHENKO BEAM

In a rectangular Cartesian coordinates, the equation of virtual work for a body  $V$  with its surface  $S$  can

\* Member of JSCE, Dr. of Eng., Associate Professor, Department of Civil Engineering, University of Tokyo (Bunkyo-ku, Tokyo 113).

\*\* Member of JSCE, Ph.D., Associate Professor, Department of Civil Engineering, Tohoku University (Aoba, Sendai 980).

\*\*\* Member of JSCE, Dr. of Eng., Professor, Department of Civil Engineering, Tohoku University (Aoba, Sendai 980).

be expressed in terms of the 2nd Piola-Kirchhoff stress  $\sigma_{ij}$ , the Green strain  $e_{ij}$ , and the displacement  $u_i$  as

$$\int_V \sigma_{ij} \delta e_{ij} dV - \int_V p_i \delta u_i dV - \int_S T_i \delta u_i dS = 0, \dots\dots\dots (1)$$

where  $p_i$  is the Cartesian component of body force vector,  $T_i$  of traction vector, and repeated indices are summed. The Green strain is defined by

$$e_{ij} \equiv e_{ij}^L + e_{ij}^{N_i}, \quad e_{ij}^L \equiv (u_{i,j} + u_{j,i})/2, \quad e_{ij}^{N_i} \equiv (u_{\kappa,i} u_{\kappa,j})/2, \dots\dots\dots (2)$$

in which comma followed by an index denotes partial differentiation with respect to the corresponding coordinate. Choose an arbitrary state of equilibrium of a body as the reference state in which the body with stresses  $\sigma_{ij}^0$  is in equilibrium with external forces  $p_i^0$  and  $T_i^0$ . Let  $\sigma_{ij}$  denote the perturbed stress field due to the additional external forces  $p_i$  and  $T_i$ . If the corresponding perturbed displacement  $u_i$  due to these forces is measured from the reference state above, then the following equations of virtual work must be satisfied;

$$\int_V \sigma_{ij}^0 \delta e_{ij}^L dV - \int_V p_i^0 \delta u_i dV - \int_S T_i^0 \delta u_i dS = 0, \dots\dots\dots (3)$$

for the reference state, and

$$\int_V (\sigma_{ij}^0 + \sigma_{ij}) \delta (e_{ij}^L + e_{ij}^{N_i}) dV - \int_V (p_i^0 + p_i) \delta u_i dV - \int_S (T_i^0 + T_i) \delta u_i dS = 0, \dots\dots\dots (4)$$

in the neighboring equilibrium state on the reference configuration. Note that the term,  $\sigma_{ij}^0 \delta e_{ij}^{N_i}$ , is not needed in (3) because the small displacement theory holds exactly by selecting momentarily an arbitrary state of equilibrium as the reference state.

Subtracting (3) from (4), we obtain the equation of virtual work for a linearized finite displacement theory as

$$\int_V (\sigma_{ij} \delta e_{ij}^L + \sigma_{ij}^0 \delta e_{ij}^{N_i}) dV - \int_V p_i \delta u_i dV - \int_S T_i \delta u_i dS = 0, \dots\dots\dots (5)$$

where a higher order term,  $\sigma_{ij} \delta e_{ij}^{N_i}$ , has been neglected.

Consider the in-plane Timoshenko beam with its centroidal axis lying along the  $x$ -axis as shown in Fig. 1. Within relatively small rotations, the kinematics of this beam can be given by

$$\begin{aligned} u_1 &= u(x, z) = u(x) + z \lambda(x), & u_2 &= 0, \\ u_3 &= w(x, z) = w(x), & \lambda(x) &= -w'(x) + \gamma(x), \dots\dots\dots (6) \end{aligned}$$

where  $\lambda$  is the rotation of a cross-section, and  $\gamma$  is the uniform shear deformation of a cross-section; a prime denotes differentiation with respect to  $x$ . As is often the case with slender beams, the order of magnitude of  $u(x, z)$  is negligibly small compared with that of  $w(x, z)$ . Therefore the non-zero components of Green's strain are approximately

$$\begin{aligned} e_{xx}^L &= u' + z \lambda', & e_{xx}^{N_i} &\approx (\partial w / \partial x)^2 / 2 = (w')^2 / 2, \\ e_{zx}^L &= (w' + \lambda) / 2. \dots\dots\dots (7) \end{aligned}$$

When we consider a purely compressed equilibrium of a beam as the reference state, the non-zero component of  $\sigma_{ij}^0$  due to the external forces  $P_{xi}^0 = P$  and  $P_{xj}^0 = -P$  at both ends is  $\sigma_{xx}^0 = -P/A$  only, where  $A$  is the cross-sectional area of the beam. Substituting (6) and (7) into (5), we can express the equation of virtual work as

$$\int_0^l [N \delta u' + M \delta \lambda' + V \delta (w' + \lambda) - P \delta \{(w')^2 / 2\}] dx - \int_0^l p_z \delta w dx - [P_{zk} \delta w_k + C_k \delta \lambda_k]_{k=i,j} = 0, \dots\dots\dots (8)$$

where  $p_z$ ,  $P_{zk}$  and  $C_k$  are the deviations of the distributed and end external forces from the reference state and

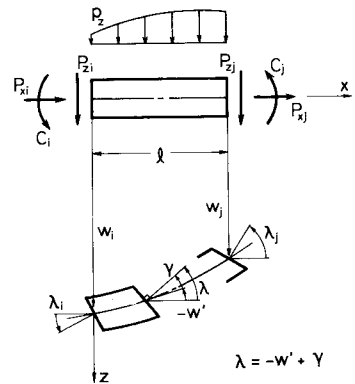


Fig.1 A Timoshenko beam element.

$$N \equiv \int_A \sigma_{xx} dA, M \equiv \int_A \sigma_{xx} z dA, V \equiv \int_A \sigma_{zx} dA. \dots\dots\dots (9)$$

Equation (8) is identical with the generalized formula of the beam-column equation in Ref. 6)

If we assume that the following constitutive relations hold approximately;

$$\sigma_{xx} \simeq E e_{xx}^L = E(u' + z \lambda'), \sigma_{zx} = 2 G e_{zx}^L = G(w' + \lambda), \dots\dots\dots (10)$$

with Young's modulus  $E$  and the shear modulus  $G$ , then the stress resultants in (9) are related to the displacement as

$$N = EA u', M = EI \lambda', V = GKA(w' + \lambda), \dots\dots\dots (11)$$

where  $I$  is the moment of inertia of a cross section, and the shear coefficient  $K$  is newly introduced in such a manner that the constant shear deformation  $\gamma$  across the section becomes equivalent to the non-uniformly distributed shear deformation consistent with the internal work done by the shear stress in equilibrium with the normal stress across the section<sup>7)</sup>.

With this assumption of constitutive relations (11), we obtain the governing equations for bending from (8) as follows: the field equations are

$$V' - P w'' + q = 0, M' - V = 0, M = EI \lambda', V = GKA(w' + \lambda), \dots\dots\dots (12)$$

with the boundary conditions as

$$\left. \begin{aligned} w = w_k \text{ or } n_k(V - P w') = P_{zk}, \\ \lambda = \lambda_k \text{ or } n_k M = C_k, \end{aligned} \right\} \text{ at } k = i, j, \dots\dots\dots (13)$$

where  $n_i = -1$  and  $n_j = 1$ ,

### 3. BUCKLING OF A SIMPLY SUPPORTED BEAM

As an example, consider a simply supported beam subjected to pure compression. The equations (12) and (13) form the eigenvalue problem as

$$(1 - P/GKA)w'''' + P/EI w'' = 0, \text{ in } 0 < x < l, \text{ with } w = 0, \text{ and } w'' = 0, \text{ at } x = 0, l, \dots\dots\dots (14)$$

which leads to the minimum buckling load as

$$P_{cr} l^2 / EI = \pi^2 / [1 + \pi^2 (E/GK)(r/l)^2], \dots\dots\dots (15)$$

where  $r$  is the radius of gyration defined by  $r^2 = I/A$ . Equation (15) turns out to be identical with the Engesser formula<sup>15)</sup>.

### 4. STIFFNESS EQUATION

In order to derive the stiffness equation, we simply follow an FEM procedure. From (8) and (11), the equation of virtual work for bending of the Timoshenko beam can be expressed as

$$\int_0^l [EI \lambda' \delta \lambda' - P w' \delta w' + GKA(w' + \lambda) \delta(w' + \lambda)] dx - [P_{zk} \delta w_k + C_k \delta \lambda_k]_{k=i,j} = 0, \dots\dots\dots (16)$$

where the distributed force is ignored for simplicity. The simplest but sufficient result can be obtained by assuming the 3rd order polynomial for  $w$  and constant for  $\gamma$ <sup>8)</sup>. Noting that the displacement vector of stiffness equation for the Timoshenko beam should not be  $\{w_i - w_i' l \ w_j - w_j' l\}$  but  $\{w_i \ \lambda_i l \ w_j \ \lambda_j l\}$ , we arrive at the following stiffness equation from (16) using (6)<sub>4</sub>:

$$\mathbf{f} = \mathbf{Kd} = EI / (l^3 \Delta) [\mathbf{K}_L^* - (Pl^2/EI) \mathbf{K}_{NL}^*] \mathbf{d}, \dots\dots\dots (17)$$

where  $\mathbf{f}^T \equiv [P_{zi} \ C_i/l \ P_{zj} \ C_j/l]$ ,  $\mathbf{d}^T \equiv [w_i \ \lambda_i l \ w_j \ \lambda_j l]$ , and  $\mathbf{K}_L^*$  and  $\mathbf{K}_{NL}^*$  are symmetric matrices given by

$$\mathbf{K}_L^* = \begin{bmatrix} 12 & & & & \text{Sym.} \\ -6 & 4+12\phi & & & \\ -12 & 6 & 12 & & \\ -6 & 2-12\phi & 6 & 4+12\phi & \end{bmatrix}, \mathbf{K}_{NL}^* = \begin{bmatrix} 6\Delta_1/5 & & & & \text{Sym.} \\ -1/10 & 2/15+\Delta_2 & & & \\ -6\Delta_1/5 & 1/10 & 6\Delta_1/5 & & \\ -1/10 & -1/30-\Delta_2 & 1/10 & 2/15+\Delta_2 & \end{bmatrix} \dots\dots\dots (18)$$

in which  $\phi, \Delta, \Delta_1$ , and  $\Delta_2$  are defined by

$$\phi = (E/GK)(r/l)^2 [1 - (Pl^2/EI)/60], \Delta = 1 + 12\phi, \Delta_1 = 1 + 10\phi, \Delta_2 = \phi. \dots\dots\dots (19)$$

Since  $\phi$  in (19) is a function of  $P$ , the stiffness matrix  $\mathbf{K}$  of (17) is nonlinear with respect to  $P$ , and therefore it is not easy to handle this form of stiffness matrix for the eigenvalue analysis of general structures. But it must be noted that the assumed trial function, the 3rd order polynomial, is not the exact solution, and that  $P$  always appears in the form of  $(Pl^2/EI)$ . Since we are going to use this stiffness equation for rather small finite elements, we can expand this stiffness matrix into the Taylor series with respect to  $(Pl^2/EI)$  and omit its higher order terms to find the approximated stiffness equation as

$$\mathbf{f} = (\mathbf{K}_L + \mathbf{K}_{NL})\mathbf{d} = EI/l^3 \Delta (\mathbf{K}_L^* - (Pl^2/EI)\mathbf{K}_{NL}^*/\Delta)\mathbf{d} \dots\dots\dots (20)$$

where  $\Delta$  and  $\mathbf{K}_L^*$  and  $\mathbf{K}_{NL}^*$  are defined in (18) and (19), but in their definitions,  $\phi$ ,  $\Delta_1$  and  $\Delta_2$  must be replaced by

$$\phi = (E/GK)(r/l)^2, \quad \Delta_1 = 1 + 20\phi + 120\phi^2, \quad \Delta_2 = 2\phi + 12\phi^2 \dots\dots\dots (21)$$

Here  $\mathbf{K}_L$  is the stiffness matrix of the Timoshenko beam for small displacements<sup>6)</sup>, while  $\mathbf{K}_{NL}$  becomes the well-known geometrical stiffness matrix<sup>6)</sup> of the beam-column when shear deformation is neglected ; i. e.  $\phi=0$ .

### 5. EIGENVALUE ANALYSIS OF AN APPROXIMATED STIFFNESS EQUATION

Critical axial loads can be obtained as eigenvalues from the condition of  $\det |\mathbf{K}_L + \mathbf{K}_{NL}| = 0$  of the approximated stiffness equation (20). Numerical computations are carried out for a purely compressed cantilever beam to compare with the existing formulae for the buckling load with shear deformation. In order to exaggerate the differences between the formulae, unrealistic values of the slenderness ratio  $l/r$  are included in computations. Table 1 gives the results for eigenvalues,  $(Pl^2/EI)$ , in  $(\mathbf{K}_L + \mathbf{K}_{NL})$  as well as those from the existing formulae in case of  $E/GK=3.0$ . Explicit expressions of these formulae are enumerated in Appendix I. Incidentally the Euler buckling load of the cantilever beam is  $\pi^2/4 \approx 2.4674$ . In the numerical computations, at most 128 finite elements are used to obtain five-significant-figures accuracy (see Appendix II). As has been expected, Table 1 indicates good agreement between the present results and those from Engesser's formula.

Table 1 Comparison of buckling loads of a cantilever.

Slenderness ratio	10 <sup>6</sup>	10 <sup>3</sup>	20.	10.	5.	10/3
A : present results	2.467 4	2.467 4	2.422 6	2.297 3	1.903 7	1.480 9
B : Engesser's formula	2.467 4	2.467 4	2.422 6	2.297 3	1.903 7	1.480 9
C : modified formula	2.467 4	2.467 4	2.423 4	2.307 6	1.991 4	1.693 3
D : Ziegler's formula	2.467 4	2.467 4	2.437 5	2.354 0	2.091 6	1.809 7
E : B : +shortening	2.467 4	2.467 4	2.436 9	2.344 7	1.986 3	1.509 0
F : C : +shortening	2.467 4	2.467 4	2.437 7	2.356 4	2.110 9	1.850 8
G : 2nd approximation	2.467 4	2.467 4	2.436 9	2.344 9	1.997 3	1.556 7

### 6. SUMMARY

A linearized finite displacement theory of the Timoshenko beam is obtained in the forms both of differential equation (12) with the boundary conditions (13), and of stiffness equation (20), in which the explicit expressions of stiffness matrices are given in (18) with (21). Although the stiffness equation is the approximated one, the eigenvalue analysis of its stiffness matrices yields the same buckling load as that from the theory of the Engesser formula.

### 7. ACKNOWLEDGEMENT

This work has been supported in part by the Grand-in-Aid for Scientific Research from the Japanese Ministry of Education, Science and Culture to Tohoku University.

APPENDIX I — BUCKLING FORMULAE

We here simply list the explicit expressions of the existing formulae. For simplicity, three symbols are introduced as

$$\alpha = E / GK, \quad \beta = r / l, \quad \xi = P_{cr} l^2 / EI, \dots \dots \dots (I-1)$$

where  $\beta$  is the inverse of the slenderness ratio and called thickness parameter. Formulae below apply for a cantilever.

(1) Engesser's formula (B : and E : in Table 1) :

$$\xi = \frac{(\pi/2)^2}{1 + \alpha(\pi\beta/2)^2} \dots \dots \dots (I-2)$$

If the shortening prior to buckling is taken into account, then a possible formula for the critical load is expressed by the equation<sup>3)</sup>

$$(\pi/2)^2 = \frac{\xi(1 - \beta^2 \xi)^2}{1 - (1 + \alpha)\beta^2 \xi} \dots \dots \dots (I-3)$$

(2) Modified formula (C : and F : in Table 1) :

$$\xi = \frac{\sqrt{1 + \alpha(\pi\beta)^2} - 1}{2\alpha\beta^2} \dots \dots \dots (I-4)$$

Similarly to (I-3), (I-4) can be modified to obtain a formula with the effect of shortening as

$$\xi = \frac{1 - \sqrt{1 - (1 - \alpha)(\pi\beta)^2}}{2(1 - \alpha)\beta^2} \dots \dots \dots (I-5)$$

that is, equation (2.67) in Ref. 1).

The difference between Engesser's formula and the modified one stems from the different choices of constitutive laws relating to the direction of axial force  $N^{1)}$ . Note, however, that the definition of the 2nd Piola-Kirchhoff stress automatically leads to the conclusion that the axial force is not acting normal to the cross section but in the direction of the beam axis. Hence the shear component of this axial force is not  $P\lambda$  of the modified formula, but becomes  $-Pw'$  as in (13) consistent with Engesser's formula.

(3) Improved formula by Ziegler (D : in Table 1) :

Ziegler's approach to include the effect of shortening in the Engesser formula results in the following expression<sup>5)</sup> ;

$$\xi = \frac{(\pi/2)^2 [1 + (\pi\beta/2)^2]}{1 + \alpha(\pi\beta/2)^2} \dots \dots \dots (I-6)$$

(4) A "second-order approximation" (G : in Table 1) :

A finite displacement theory of the Timoshenko beam can be approximated by the reasonable assumption on the constitutive relations which include higher-order terms of the axial and shear deformation (see Appendix in Ref. 3). This approximated governing equations lead to the following expression for the buckling load ;

$$(\pi/2)^2 = \frac{\xi}{1 - \alpha\beta^2 \xi} - (\beta\xi)^2 \dots \dots \dots (I-7)$$

Table 2 Convergence of buckling loads.

number of elements slenderness ratio	8	16	32	64	128
10 <sup>6</sup>	2.467 4062	2.467 4014	2.467 4011	2.467 4011	—
10 <sup>3</sup>	2.467 3880	2.467 3832	2.467 3829	2.467 3828	—
20.	2.422 7136	2.422 6052	2.422 5789	2.422 5724	2.422 5707
10.	2.297 8238	2.297 4654	2.297 3764	2.297 3542	2.297 3487
5.	1.904 8089	1.903 9990	1.903 7968	1.903 7463	1.903 7336
10/3	1.481 9991	1.481 1423	1.480 9282	1.480 8747	1.480 8613

## APPENDIX II—CONVERGENCE OF COMPUTED EIGENVALUES

The smaller the slenderness ratio becomes, the more elements are needed to obtain the desired accuracy for eigenvalues. Table 2 shows how eigenvalues are changing in the number with the increasing number of elements when  $E/GK=3.0$ . This table indicates that eight elements may be enough for the practical applications.

## REFERENCES

- 1) Timoshenko, S.P. and Gere, J.M. : Theory of Elastic Stability, 2nd ed., McGraw-Hill Kogakusha Ltd., Tokyo, 1961.
- 2) Reissner, E. : On one-dimensional finite-strain beam theory : the plane problem, ZAMP, Vol.23, pp.795~804, 1972.
- 3) Iwakuma, T. and Kuranishi, S. : How much contribution does the shear deformation have in a beam theory ?, Proc. JSCE, Structural Eng./Earthquake Eng., Vol.1, No.1, pp.103s~113s, 1984.
- 4) Reissner, E. : Some remarks on the problem of column buckling, Ingenieur-Archiv, Vol.52, pp.115~119, 1982.
- 5) Ziegler, H. : Arguments for and against Engesser's buckling formulas, Ingenieur-Archiv, Vol.52, pp.105~113, 1982.
- 6) Nishino, F. : Mechanics of Continua (II), Structural Mechanics, Doboku Kogaku Taikei (Edited by Y. Takahashi), Vol.6, Shokoku-Sha Pub. Co., Tokyo, 1984 (in Japanese).
- 7) Cowper, G.R. : The shear coefficient in Timoshenko's beam theory, J. Appl. Mech., Vol.33, pp.335~340, 1966.
- 8) Iwakuma, T., Ai, M. and Nishino, F. : On derivation of Timoshenko beam stiffness equation, Proc. JSCE, No.312, pp.119~128, 1981.

(Received January 23 1985)