[PROC. OF JSCE No. 351/II-2 (Hydraulic and Sanitary Eng.) November 1984]

ON THE AUTOCORRELATION FUNCTION OF A BRP PROCESS

By Takeharu ETOH and Yasuhiro HAYASHI***

The autocorrelation function of a binary relaxed Poisson process is derived. The process is defined as the following series of transformations of a nonhomogeneous Poisson process : 1) the process of energy bursts (a nonhomogeneous Poisson process), 2) relaxation process of the point process, and 3) its binary transformation.

For a stationary BRP process, the autocorrelation function is reduced to a doubly exponential function. For a periodic BRP process with a dominant primal periodic component, it is also reduced to a simple expression in terms of the first kind modified Bessel's function of the 0-th order.

The present theory is applicable to describe and analyze intermittent time series such as precipitation series, intermittent turbulence in flows, etc.

1. INTRODUCTION

The autocorrelation function of a binary relaxed Poisson process is derived. The process is defined by the author as the following series of transformations of a nonhomogeneous Poisson process.

1) The process of energy bursts (Fig. 1 (a)) : a point process of the events which trigger energy release ; the process is assumed to be a nonhomogeneous Poisson process.

2) Relaxation process (Fig. 1 (b), (c)) : the process in which the energy is gradually released through a relaxation mechanism ; the duration of the relaxation process is assumed to be exponentially distributed, and independent of the properties of the point process.

3) Binary transformation (Fig. 1 (d)) : if the time is included in a relaxation process induced by one or some events, $x(t)=1$, otherwise $x(t)=0$.

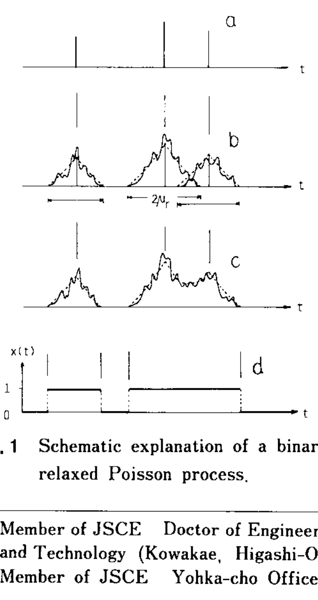

Fig. 1 Schematic explanation of a binary relaxed Poisson process.

Namely, a BRP process $x(t)$ takes a value 1 or 0, alternately. The process was originally presented to analyze a dry-and-wet process. Thus, the states represented by $x(t)=0$ and $x(t)=1$ are called a dry state and a wet state, respectively.

The process is basically described in terms of two parameters,

* Member of JSCE Doctor of Engineering, Professor of Kinki University, Department of Civil Engineering, Faculty of Science and Technology (Kowakae, Higashi-Osaka City, Osaka 577 JAPAN)
** Member of JSCE Yohka-cho Office (Yohka Cho, Yabu Gun, Hyogo 667 JAPAN)

i. e., the rate of occurrence of events $\lambda(t)$, and the scale parameter of the exponential density of the duration of a single relaxation process $\beta_r(t)$.

First, we shall assume that $\lambda(t)$ changes with time t, and a scale parameter is fixed at a constant value β_r. Then, the rate of occurrence is assumed to be a periodic function and remains periodic until a very last stage of the analysis, at which the autocorrelation function of a BRP process is finally reduced to its simplest and most basic form by putting $\lambda(t)$ a constant value.

From a physical point of view, generating mechanisms of the point process and the subsequent relaxation process are different. Therefore, it may be appropriate to analyze the characteristics of each mechanism individually. To analyze the macroscale characteristics of an intermittent series, it may be reasonable to transform the series into a binary process as described above to eliminate the disturbance generated in the relaxation process. On the other hand, microscale characteristics may be reasonablly analyzed through observing only the variation within the relaxation processes, i. e., by means of a conditional sampling technique.

2. THEORY

(1) Autocorrelation function of a D-W process

We shall derive an expression of the autocorrelation function in terms of the probabilities of a dry state and of the co-occurrence of dry states at two points at a distance τ.

We shall employ the following notations,

$$P_D(t)=\text{Prob}\{x(t)=0\}, \quad P_W(t)=\text{Prob}\{x(t)=1\} \quad \cdots (1), (2)$$

$$P_{DD}(t, \tau)=\text{Prob}\left\{x\left(t-\frac{\tau}{2}\right)=0 \cap x\left(t+\frac{\tau}{2}\right)=0\right\} \quad \cdots (3)$$

$P_{DW}(t, \tau)$, $P_{WD}(t, \tau)$, and $P_{WW}(t, \tau)$ are similarly defined. Let $\mu(t)$, $\sigma^2(t)$, $\text{Cov}(t, \tau)$, and $r(t, \tau)$ denote mean, variance, covariance and autocorrelation coefficient of a binary process. Then,

$$\mu(t)=0\cdot\text{Prob}\{x(t)=0\}+1\cdot\text{Prob}\{x(t)=1\}=1-P_D(t) \quad \cdots (4)$$

Similarly,

$$\sigma^2(t)=\{1-P_D(t)\}P_D(t) \quad \cdots (5)$$

$$\text{Cov}(t, \tau)=P_{DD}(t, \tau)-P_D\left(t-\frac{1}{2}\tau\right)\cdot P_D\left(t+\frac{1}{2}\tau\right) \quad \cdots (6)$$

$$r(t, \tau)=\text{Cov}(t, \tau)/\sqrt{\sigma\left(t-\frac{1}{2}\tau\right)\cdot\sigma\left(t+\frac{1}{2}\tau\right)} \quad \cdots (7)$$

It should be noticed that all the statistical parameters μ, σ^2 and r are evaluated only by using the dry probabilitities, P_D and P_{DD}. For a stationary time series,

$$r(\tau)=\{P_{DD}(\tau)/P_D^2-1\}/(1/P_D-1) \quad \cdots (8)$$

We shall derive the expressions of the parameters for a periodic binary process with the primal period T.

$$\bar{\mu}=\frac{1}{T}\int_0^T \mu(t)\,dt=1-\bar{P}_D \quad \cdots (9)$$

$$\bar{\sigma}^2=\frac{1}{T}\int_0^T \{(0-\bar{\mu})^2 P_D(t)+(1-\bar{\mu})^2 P_W(t)\}\,dt$$

$$=\{1-\bar{P}_D\}\bar{P}_D \quad \cdots (10)$$

Similarly,

$$\tilde{\text{C}}\text{ov}(\tau)=\bar{P}_{DD}(\tau)-\bar{P}_D^2 \quad \cdots (11)$$

$$\tilde{r}(\tau)=\{\bar{P}_{DD}(\tau)/\bar{P}_D^2-1\}/(1/\bar{P}_D-1) \quad \cdots (12)$$

where '——' represents the following average operation over T :

$$\overline{A}=\frac{1}{T}\int_0^T A(t)\,\mathrm{d}t \cdots\cdots (13)$$

It should be noted that $\tilde{\sigma}^2\neq\overline{\sigma}^2$, $\tilde{\mathrm{r}}(\tau)\neq\overline{\mathrm{r}}(\tau)$, etc.

(2) **Occurrence of a single event**

a) The probability of a dry state

Evaluated hereafter is the probability that the l-th interval is completely dry under the condition that a single event occurs within the $(l+j)$-th interval. We shall call the l-th interval the target interval. First, we shall consider the case where the single event occurs in an interval right to the target interval.

Let q' denote the probability, and t_r and $f_r(t_r)$ a half of the duration of a relaxation process and the probability density of t_r, respectively. Then, in view of Fig. 2. (a),

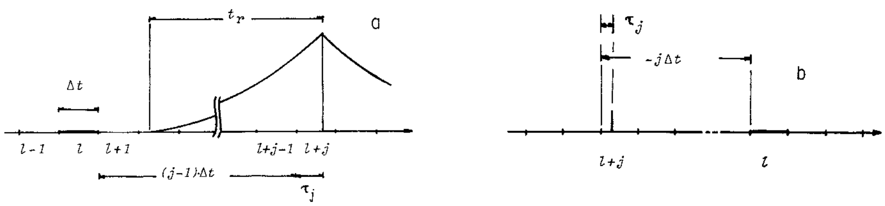

Fig. 2 The dry probability of the l-th interval with a single event in the $(l+j)$-th interval.

$$q'(l+j,\ \tau_j)=\mathrm{Prob}\{t_r\leqq(j-1)\Delta t+\tau_j\}=\int_0^{(j-1)\Delta t+\tau_j} \mathrm{f}_r(t_r)\,\mathrm{d}t_r \quad (j\geqq 1)\cdots\cdots (14)$$

where Δt is the duration of a unit interval and τ_j is the displacement of the point from the left end of the $(l+j)$-th interval.

We shall assume that Δt is small enough to satisfy the condition :

$$\lambda(t+\Delta t)=\lambda(t)+\Delta t\left.\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right|_{t=t}+\cdots \cdots\cdots (15)$$

and

$$\lambda(t)\gg\left|\Delta t\left.\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right|_{t=t}\right| \cdots\cdots (16)$$

Then, the second term of the right hand side of (15) can be neglected. This may guarantee that the time of occurrence of a single event is uniformly distributed within an interval, and the expectation of q' with respect to τ_j can be approximated by

$$q(l+j)=\frac{1}{\Delta t}\int_0^{\Delta t} q'(l+j,\ \tau_j)\,\mathrm{d}\tau_j \cdots\cdots (17)$$

If t_r is exponentially distributed with the scale parameter β_r,

$$\mathrm{f}_r(t_r)=\beta_r\mathrm{e}^{-\beta_r t_r}$$

and

$$q(l+j)=1-\frac{1}{\beta_r\Delta t}\mathrm{e}^{-\beta_r(j-1)\Delta t}(1-\mathrm{e}^{-\beta_r\Delta t})\quad (j\geqq 1)\cdots\cdots (18)$$

Similarly, when a single event takes place in an interval left to the target interval (Fig. 2. (b)),

$$q(l+j)=\frac{1}{\Delta t}\int_0^{\Delta t}\int_0^{-j\Delta t-\tau_j}\mathrm{f}_r(t_r)\,\mathrm{d}t_r\mathrm{d}\tau_j \cdots\cdots (19)$$

$$=1-\frac{1}{\beta_r\Delta t}\mathrm{e}^{-\beta_r(-j-1)\Delta t}(1-\mathrm{e}^{-\beta_r\Delta t})\quad (j\leqq -1)\cdots\cdots (20)$$

When the event occurs in the target interval, the interval is wet with probability one. Thus,

$$q(l)=0\quad (j=0)\cdots\cdots (21)$$

Table 1 The dry probability with a single event.

	Range of j	$j \leqq -1$	0	$j \geqq 1$
$q(l+j)$	General expression	(19)	0	(17)
	Exponential density	(20)	(21)	(18)

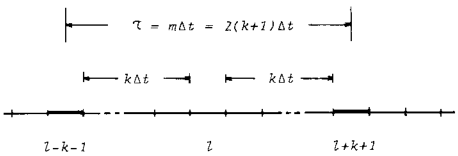

Fig. 3 The lag between target intervals is an even number.

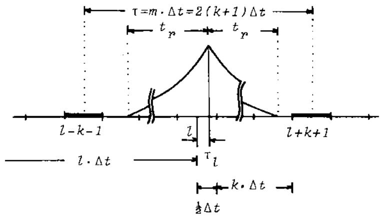

Fig. 4(a) The probability of co-occurrence of two dry states with a single event in the right half of the l-th interval.

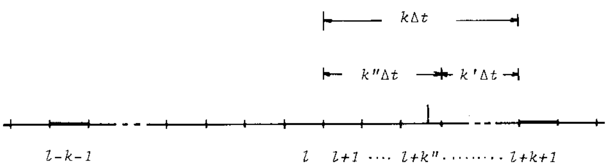

Fig. 4(b) The dry probability of co-occurrence of two dry states with a single event.

The results are tabulated in Table 1.

b) The probability of co-occurrence of two dry states

We shall evaluate the probability that both of the target intervals at a distance τ are dry under the condition that a single event occurs in an interval.

First, we shall discuss the case where the lag between the target intervals is an even number as shown in Fig. 3. Namely,

$$\tau = m\Delta t = 2(k+1)\Delta t \qquad (22)$$

$$m = 2(k+1), \quad k = \frac{m}{2} - 1 \qquad (23)$$

where τ is the time lag between the intervals, and m is the lag expressed in terms of the number of intervals.

Let k'' designate the lag between the l-th interval and the interval in which an event occurs, and $S(l+k'')$ the probability that both of the target intervals are dry.

1) $k''=0$

Namely, single event occurs in the l-th interval as shown in Fig. 4 (a). When the event occurs in the right half of the l-th interval, $\tau_l \geqq 0$, and

$$S(l) = \frac{2}{\Delta t}\int_0^{\Delta t/2} \mathrm{Prob}\left[\{\text{the } (l+k+1)\text{-th interval is dry}\} \cap \{\text{the } (l-k-1)\text{-th interval is dry}\}\right] d\tau_l$$

$$= \frac{2}{\Delta t}\int_0^{\Delta t/2} \mathrm{Prob}\left[\left\{t_r \leqq \left(k+\frac{1}{2}\right)\Delta t - \tau_l\right\} \cap \left\{t_r \leqq \left(k+\frac{1}{2}\right)\Delta t + \tau_l\right\}\right] d\tau_l \qquad (24)$$

Since $\tau_l \geqq 0$,

$$\left(k+\frac{1}{2}\right)\Delta t - \tau_l \leqq \left(k+\frac{1}{2}\right)\Delta t + \tau_l \qquad (25)$$

Therefore, if the first condition in Prob [ ] in (24) is satisfied, so is the second condition with probability one. Thus,

$$S(l) = \frac{2}{\Delta t}\int_0^{\Delta t/2} \mathrm{Prob}\left\{t_r \leqq \left(k+\frac{1}{2}\right)\Delta t - \tau_l\right\} d\tau_l$$

$$= \frac{2}{\Delta t}\int_0^{\Delta t/2}\int_0^{\left(k+\frac{1}{2}\right)\Delta t - \tau_l} f_r(t_r)\, dt_r\, d\tau_l \qquad (26)$$

When t_r is exponentially distributed,

$$S(l)=1-\frac{2}{\beta_r\Delta t}\,e^{-\beta_r k\Delta t}(1-e^{-\beta_r\frac{\Delta t}{2}})=1-\frac{2}{\beta_r\Delta t}\,e^{-(\frac{m}{2}-1)\beta_r\Delta t}(1-e^{-\beta_r\frac{\Delta t}{2}})\cdots\cdots(27)$$

2) $k''=1\sim k=1\sim\left(\frac{m}{2}-1\right)$

Hereafter, let k' be the number of intervals between the interval in which an event occurs and one of the target intervals which is closer to the interval with the event. In view of Fig. 4(b),

$$k'=-k''+k=-k''+\left(\frac{m}{2}-1\right)\cdots\cdots(28)$$

and

$$S(l+k'')=\frac{1}{\Delta t}\int_{-\Delta t/2}^{\Delta t/2}\int_0^{(k'+\frac{1}{2})\Delta t-\tau_{k'}} f_r(t_r)\,d\tau_{k'}\cdots\cdots(29)$$

When t_r is exponentially distributed,

$$S(l+k'')=1-\frac{1}{\beta_r\Delta t}\,e^{-k'\beta_r\Delta t}(1-e^{-\beta_r\Delta t})\cdots\cdots(30)$$

3) $k''=k+1=\frac{m}{2}$

A single event occurs in one of the target intervals. Since the probability that the target interval is wet is equal to unity,

$$S(l+k'')=0\cdots\cdots(31)$$

Similarly, $S(l+k'')$ can be evaluated for all the ranges of k'' as shown in Table 2.

Similarly, for the case where the lag is an odd number as shown in Fig. 5, the probability can be evaluated as shown in Table 3.

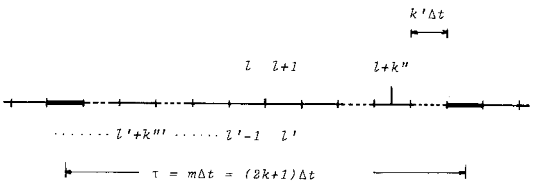

Fig. 5 The lag between target intervals is an odd number.

(3) Dry probabilities and autocorrelation function

a) General expressions

Since the original point process is assumed to be a nonhomogeneous Poisson process, the number of events n which occur in an interval $(t, t+\Delta t)$

Table 2 The probability of co-occurrence of two dry states with a single event (lag m : an even number).

Range of k''		$-\infty\sim-\frac{m}{2}-1$	$-\frac{m}{2}$	$-\frac{m}{2}+1\sim-1$	0 ($\tau_i<0$)	0 ($\tau_i\geq0$)	$1\sim\frac{m}{2}-1$	$\frac{m}{2}$	$\frac{m}{2}+1\sim\infty$
k', k		$-k''-\frac{m}{2}-1$		$k''+\frac{m}{2}-1$	$(k=)\ \frac{m}{2}-1$		$-k''+\frac{m}{2}-1$		$k''-\frac{m}{2}-1$
$S(l+k'')$	General expression	(29)	0	(29)	(26)	(26)	(29)	0	(29)
	Exponential density	(30)	(31)	(30)	(27)	(27)	(30)	(31)	(30)

Table 3 The probability of co-occurrence of two dry states with a single event (lag m : an odd number).

Range of							
	k''				$1\sim\frac{m-1}{2}$	$\frac{m+1}{2}$	$\frac{m+3}{2}\sim\infty$
	k'''	$-\infty\sim-\frac{m+3}{2}$	$-\frac{m+1}{2}$	$-\frac{m-1}{2}\sim-1$			
k'		$-k'''-\frac{m+3}{2}$		$k'''+\frac{m-1}{2}$	$-k''+\frac{m-1}{2}$		$k''-\frac{m+3}{2}$
$S(l+k'')$	(a)				(29)	0	(29)
	(b)				(30)	(31)	(30)
$S(l+k''')$	(a)	(29)	0	(29)			
	(b)	(30)	(31)	(30)			

(a) General expression (b) Exponential density follows Poisson distribution with the average number of occurrence $\{\Lambda(t+\Delta t)-\Lambda(t)\}$, where $\Lambda(t)=\int_0^t \lambda(s)\,ds$. Let t denote the left end of the $(l+j)$-th interval. If (16) is satisfied,

$$\mathrm{Prob}\,(n=\nu)=\frac{(\lambda_{l+j}\cdot\Delta t)^\nu}{\nu!}\cdot e^{-\lambda_{l+j}\Delta t} \quad (32)$$

where

$$\lambda_{l+j}=\lambda\{(l+j)\Delta t\} \quad (33)$$

and

$$\lambda\{(l+j)\Delta t\}\,\Delta t\approx\lambda(t)\Delta t\approx\Lambda(t+\Delta t)-\Lambda(t) \quad (34)$$

$q(l+j)$ in Table 1 represents the probability that the l-th interval is dry for $n=1$. Since the number n may take 0 or any positive integer, the probability that the l-th interval is dry under the condition that some events occur within the $(l+j)$-th interval is

$$\sum_{\nu=0}^{\infty}\frac{(\lambda_{l+j}\cdot\Delta t)^\nu}{\nu!}\cdot e^{-\lambda_{l+j}\Delta t}\{q(l+j)\}^\nu=\exp[-\lambda_{l+j}\Delta t\,\{1-q(l+j)\}] \quad (35)$$

Therfore, $P_D(l)$, i. e., the probability that events in all intervals do not make the target interval wet is expressed as follows:

$$P_D(l)=\prod_{j=-\infty}^{\infty}\exp[-\lambda_{l+j}\Delta t\,\{1-q(l+j)\}]$$

$$=\exp\left[-\sum_{j=-\infty}^{\infty}\lambda_{l+j}\Delta t\,\{1-q(l+j)\}\right] \quad (36)$$

where $P_D(l\Delta t)$ is represented by $P_D(l)$ for the simplicity in expressions.

Similarly, $P_{DD}(l,m)\approx P_{DD}(l\Delta t, m\Delta t)$ is expressed in terms of λ_{l+j} and $S(l+k'')$ as follows:

$$P_{DD}(l,m)=\exp\left[-\sum_{k''=-\infty}^{\infty}\lambda_{l+j}\Delta t\,\{1-S(l+k'')\}\right] \quad (37)$$

where $S(l+k'')$ is the dry probability of both of the target intervals for the occurrence of a single event in the k''-th interval as shown in Table 2 or 3.

Coupling (36) and (37) with the equations in Table 1 through 3 produces the general expressions of the dry probabilities $P_D(l)$ and $P_{DD}(l,m)$ of a binary relaxed Poisson process observed with a discrete time interval Δt. Then, substituting (36) and (37) into (7), the deduction of the autocorrelation function is straightforward.

It should be noted that the BRP process discussed in this paper is based on the stationarity and the independency of duration of relaxation process, and so the discussion on a BRP process may go further by employing more general assumptions on the basic processes.

b) Continuous observation (Δt is infinitesimal)

We shall assume the exponential density for the distribution of duration of relaxation process. Then, substituting the expressions in Table 1 into (36),

$$P_D(l)=\exp\left[-\left\{\sum_{j=-\infty}^{-1}\lambda_{l+j}\Delta t\,\frac{1}{\beta_r\Delta t}\,e^{-\beta_r(-j-1)\Delta t}(1-e^{-\beta_r\Delta t})+\lambda_l\Delta t\right.\right.$$

$$\left.\left.+\sum_{j=1}^{\infty}\lambda_{l+j}\Delta t\,\frac{1}{\beta_r\Delta t}\,e^{-\beta_r(j-1)\Delta t}(1-e^{-\beta_r\Delta t})\right\}\right] \quad (38)$$

When Δt is infinitely small,

$$(1-e^{-\beta_r\Delta t})/\Delta t\rightarrow\beta_r \quad (39)$$

and $\lambda(t)\Delta t\rightarrow 0$.

Replacing summation symbols by integrals, and $l\Delta t$ and $j\Delta t$ by t and s',

$$P_D(t)=\exp\left[-\left\{\int_0^\infty\lambda(t+s')e^{-\beta_r s'}ds'+\int_{-\infty}^0\lambda(t+s')e^{-\beta_r(-s')}ds'\right\}\right]$$

$$=\exp\left[-\int_0^\infty\{\lambda(t+s)+\lambda(t-s)\}e^{-\beta_r s}ds\right] \quad (40)$$

Similarly, by substituting the expressions of $S(l+k'')$ in Table 2 into (37), for an even number m,

$$P_{DD}(l,m)=\exp\Big[-\Big\{\sum_{k''=-\infty}^{-(\frac{m}{2}+1)}\Delta t\lambda_{l+k''}\frac{1}{\beta_r\Delta t}e^{-\beta_r(-k''-\frac{m}{2}-1)\Delta t}(1-e^{-\beta_r\Delta t})+\Delta t\lambda_{l-\frac{m}{2}}$$

$$+\sum_{k''=-\frac{m}{2}+1}^{-1}\Delta t\lambda_{l+k''}\frac{1}{\beta_r\Delta t}e^{-\beta_r(k''+\frac{m}{2}-1)\Delta t}(1-e^{-\beta_r\Delta t})+\Delta t\lambda_l\frac{2}{\beta_r\Delta t}e^{-\beta_r(\frac{m}{2}-1)\Delta t}(1-e^{-\beta_r\Delta t/2})$$

$$+\sum_{k''=1}^{\frac{m}{2}-1}\Delta t\lambda_{l+k''}\frac{1}{\beta_r\Delta t}e^{-\beta_r(-k''+\frac{m}{2}-1)\Delta t}(1-e^{-\beta_r\Delta t})+\Delta t\lambda_{l+\frac{m}{2}}$$

$$+\sum_{k''=\frac{m}{2}+1}^{\infty}\Delta t\lambda_{l+k''}\frac{1}{\beta_r\Delta t}e^{-\beta_r(k''-\frac{m}{2}-1)\Delta t}(1-e^{-\beta_r\Delta t})\Big\}\Big] \quad (41)$$

For an infinitesimal Δt, replacing $l\Delta t$, $k''\Delta t$ and $m\Delta t$ by t, s' and τ, and after some mathematical manipulation,

$$P_{DD}(t,\tau)=\exp\Big[-\Big[\int_0^\infty\Big\{\Big(\lambda\Big(t-s-\frac{\tau}{2}\Big)+\lambda\Big(t+s+\frac{\tau}{2}\Big)+\lambda\Big(t+s-\frac{\tau}{2}\Big)+\lambda\Big(t-s+\frac{\tau}{2}\Big)\Big)$$

$$-(\lambda(t+s)+\lambda(t-s))e^{-\beta_r\tau/2}\Big\}e^{-\beta_r s}ds\Big]\Big] \quad (42)$$

c) A periodic binary relaxed Poisson process

Assume the rate of occurrence of events $\lambda(t)$ changes periodically with the primal period T. Then,

$$\lambda(t)=b_0+\sum_{n=1}^{\infty}b_n\cos\{\omega_n(t-t_n)\} \quad (\geqq 0) \quad (43)$$

where b_n, t_n, ω_n, and T_n are amplitude, phase lag, angular frequency, and period of the n-th periodic component. It should be noted that some conditions to satisfy $\lambda(t)\geqq 0$ be imposed on b_n, ω_n, and t_n.

$$\lambda(t+s)+\lambda(t-s)=2\Big[b_0+\sum_{n=1}^{\infty}b_n\cos\{\omega_n(t-t_n)\}\cos(\omega_n s)\Big] \quad (44)$$

When Δt is infinitesimal, substituting (44) into (40) and integrating it,

$$P_D(t)=\exp\Big[-2\Big\{\frac{b_0}{\beta_r}+\sum_{n=1}^{\infty}b_n\cos(\omega_n(t-t_n))\frac{\beta_r}{\omega_n^2+\beta_r^2}\Big\}\Big] \quad (45)$$

When the primal periodic component is dominant in the periodic components,

$$\overline{P}_D=\frac{1}{T_1}\int_0^{T_1}P_D(t)\,dt=e^C I_0(G) \quad (46)$$

where

$$C=-2b_0/\beta_r,\quad G=2b_1\beta_r/(\omega_1^2+\beta_r^2),\ \text{and} \quad (47)$$

$$I_0(G)=\frac{1}{\pi}\int_0^\pi e^{G\cos s}ds \quad (48)$$

$I_0(G)$ in (48) is the first kind modified Bessel's function of the 0-th order. To satisfy $\lambda(t)\geqq 0$, $b_1\leqq b_0$.

We shall evaluate $P_{DD}(\tau)$.

$$\lambda\Big(t-s-\frac{\tau}{2}\Big)+\lambda\Big(t+s+\frac{\tau}{2}\Big)+\lambda\Big(t+s-\frac{\tau}{2}\Big)+\lambda\Big(t-s+\frac{\tau}{2}\Big)$$

$$=4\Big[b_0+\sum_{n=1}^{\infty}b_n\cos(\omega_n s)\cdot\cos\{\omega_n(t-t_n)\}\cdot\cos\Big(\omega_n\frac{\tau}{2}\Big)\Big] \quad (49)$$

By substituting (44) and (49) into (42),

$$P_{DD}(t,\tau)=\exp\Big[-2\Big\{\frac{b_0}{\beta_r}(2-e^{-\beta_r\frac{\tau}{2}})+\sum_{n=1}^{\infty}b_n\cos(\omega_n(t-t_n))\frac{\beta_r}{\omega_n^2+\beta_r^2}\Big(2\cos\Big(\omega_n\frac{\tau}{2}\Big)-e^{-\beta_r\frac{\tau}{2}}\Big)\Big\}\Big] \quad (50)$$

When the primal periodic component is dominant,

$$\overline{P}_{DD}(\tau)=e^A I_0(B) \quad (51)$$

where

$$A=-2b_0(2-e^{-\beta_r\frac{\tau}{2}})/\beta_r \quad (52)$$

$$B=2\beta_r b_1\{2\cos\left(\omega_1\frac{\tau}{2}\right)-e^{-\beta_r\frac{\tau}{2}}\}/(\omega_1^2+\beta_r^2) \quad \cdots\cdots (53)$$

By substituting (46) and (51) into (12), the autocorrelation function of a periodic binary relaxed Poisson process is derived as follows, for the case in which 1) the unit interval of observation is infinitesimal, 2) the primal periodic component is dominant in the variation of the rate of occurrence of events, and 3) the duration of relaxation process is exponentially distributed:

$$\tilde{r}(\tau)=\frac{e^A I_0(B)/\{e^C I_0(G)\}^2-1}{1/\{e^C I_0(G)\}-1} \quad \cdots\cdots (54)$$

For the homogeneous Poisson process, $b_1=0$, then the expression is reduced to

$$r(\tau)=\frac{\exp\left\{2\lambda\mu_r\exp\left(-\frac{\tau}{2\mu_r}\right)\right\}-1}{\exp(2\lambda\mu_r)-1} \quad \cdots\cdots (55)$$

where

$\lambda=b_0,\ \mu_r=1/\beta_r$

μ_r is the average duration of relaxation process. As observed from (55), $r(\tau)$ is reduced to a doubly exponential type of a function in terms of τ.

It is unwieldy to derive the expressioons of $\overline{P}_D$, $\overline{P}_{DD}(\tau)$ and, then, $\tilde{r}(\tau)$ for a finite Δt and an arbitrary periodic function of $\lambda(t)$. Presented below is the final result of the derivation of $\tilde{r}(\tau)$ for the case in which 1) the primal periodic component is dominant over higher order periodic components and 2) the interval of observation Δt is finite.

For $m=2(k+1)$, A and B in (51) are:

$$A=-2b_0\left[\Delta t+\frac{1-e^{-\beta_r\frac{\Delta t}{2}}}{\beta_r}e^{-\beta_r\left(\frac{m}{2}-1\right)\Delta t}+\frac{1-e^{-\beta_r\Delta t}}{\beta_r}\left\{\sum_{k=1}^{\frac{m}{2}-1}e^{-\beta_r\left(-k+\frac{m}{2}-1\right)\Delta t}+\sum_{k=\frac{m}{2}+1}^{\infty}e^{-\beta_r\left(k-\frac{m}{2}-1\right)\Delta t}\right\}\right] \quad \cdots\cdots (56)$$

$$B=2b_1\left[\Delta t\cos\left(\omega_1\frac{m}{2}\Delta t\right)+\frac{1-e^{-\beta_r\frac{\Delta t}{2}}}{\beta_r}e^{-\beta_r\left(\frac{m}{2}-1\right)\Delta t}+\frac{1-e^{-\beta_r\Delta t}}{\beta_r}\left\{\sum_{k=1}^{\frac{m}{2}-1}e^{-\beta_r\left(-k+\frac{m}{2}-1\right)\Delta t}\cos(\omega_1 k\Delta t)\right.\right.$$
$$\left.\left.+\sum_{k=\frac{m}{2}+1}^{\infty}e^{-\beta_r\left(k-\frac{m}{2}-1\right)\Delta t}\cos(\omega_1 k\Delta t)\right\}\right] \quad \cdots\cdots (57)$$

and for $m=2k+1$,

$$A=-2b_0\left[\Delta t+\frac{1-e^{-\beta_r\Delta t}}{\beta_r}\left\{\sum_{k=1}^{\frac{m-1}{2}}e^{-\beta_r\left(-k+\frac{m-1}{2}\right)\Delta t}+\sum_{k=\frac{m+3}{2}}^{\infty}e^{-\beta_r\left(k-\frac{m+3}{2}\right)\Delta t}\right\}\right] \quad \cdots\cdots (58)$$

$$B=2b_1\left[\Delta t\cos\left(\omega_1\frac{m}{2}\Delta t\right)+\frac{1-e^{-\beta_r\Delta t}}{\beta_r}\left\{\sum_{k=1}^{\frac{m-1}{2}}e^{-\beta_r\left(-k+\frac{m-1}{2}\right)\Delta t}\cos\left(\omega_1\left(k-\frac{1}{2}\right)\Delta t\right)\right.\right.$$
$$\left.\left.+\sum_{k=\frac{m+3}{2}}^{\infty}e^{-\beta_r\left(k-\frac{m+3}{2}\right)\Delta t}\cos\left(\omega_1\left(k-\frac{1}{2}\right)\Delta t\right)\right\}\right] \quad \cdots\cdots (59)$$

And in the expression (46),

$$C=-b_0\left\{\Delta t+2\frac{1-e^{-\beta_r\Delta t}}{\beta_r}\sum_{j=1}^{\infty}e^{-\beta_r(j-1)\Delta t}\right\} \quad \cdots\cdots (60)$$

$$G=b_1\left\{\Delta t+2\frac{1-e^{-\beta_r\Delta t}}{\beta_r}\sum_{j=1}^{\infty}e^{-\beta_r(j-1)\Delta t}\cos(\omega_1 j\Delta t)\right\} \quad \cdots\cdots (61)$$

(4) Characteristics of the autocorrelation function

We shall observe the characteristics of $r(\tau)$ for the homogeneous Poisson process, i.e., (55) When $2\lambda\mu_r$ is very large,

$$r(\tau)\simeq\exp\left[2\lambda\mu_r\left\{\exp\left(-\frac{\tau}{2\mu_r}\right)-1\right\}\right] \quad \cdots\cdots (62)$$

Since $\{\ \}<0$, when λ or μ_r is infinitely large,

$r(\tau)=1 \quad (\tau=0), \qquad r(\tau)=0 \quad (\tau>0)$ ……………… (63)

When $2\lambda\mu_r$ is close to 0, substituting the polynomial expansion for exponentials in (55) and neglecting the higher order terms,

$$r(\tau)\approx\exp\left(-\frac{\tau}{2\mu_r}\right) \qquad (64)$$

As indicated above, $2\lambda\mu_r$ is a characteristic parameter of the process. We shall consider the physical meaning of the parameter.

1) $0<2\lambda\mu_r\ll1$

Relaxation processes do not overlap each other. P_D is close to unity. The analysis of the process is rather simple, because the time of occurrence of each event can be easily detected and the characteristics of the point process and the relaxation process are analyzed separately. When a Poisson process is employed for the point process, the characteristics of the autocorrelation function of such a process directly reflects those of the relaxation process. For the BRP process discussed above, it becomes an exponential function reflecting the exponential density of the distribution of duration as shown in (64).

2) $2\lambda\mu_r\approx1$

Some of the relaxation processes overlap each other. The theories such as one presented here work powerfully to describe and analyze the case. It may be observed in intermittent turbulence in waters near boundaries, atmosphere, etc. A precipitation series is a typical example of the case.

3) $2\lambda\mu_r\gg1$

The observed sequence is almost a continuous one. The series may be characterized mainly by disturbances generated in relaxation processes. The theories based on the hypothesis of Gaussian disturbance may become appropriate to analyze the case through the Central Limit Theorem.

3. CONCLUDING REMARKS

The autocorrelation function of a binary relaxed Poisson process is derived. Some characteristics of the function are analyzed.

For a stationary BRP process, the function is reduced to a doubly exponential type. For a periodic BRP process with the dominant primal periodic component, it is also reduced to a simple expression in terms of the first kind modified Bessel's function of the 0-th order.

The present theory is applicable to describe and analyze intermittent time series such as precipitation series, intermittent turbulence in flows, etc.

Etoh et al. applied the theory presented in this paper to the analysis of daily precipitation series[1]. Going through their paper, readers may perceive the background of the development, the applicability, etc. of the theory.

REFERENCES

1) Etoh, T. and Nishimura, K. : A MPP model of daily precipitation series, Proc. of JSCE, No. 342, pp. 171~178, Feb., 1984.
2) Snyder, D. L. : Random point processes, Wiley-Interscience pub., 1975.
3) Parzen, E. : Stochastic processes, Holden-Day, 1962.

(Received August 30 1983)