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A COMBINED UPPER AND LOWER BOUND ANALYSIS AND ITS APPLICATIONS

By TAN Kiang Hwee, Akio HASEGAWA** and Fumio NISHINO****

A method of limit analysis which combines the features of the lower and upper bound approaches is outlined. In this method, the collapse load of a structure is expressed as a function of a set of kinematic and static variables which are subject to constraints resulting from an assumed collapse mechanism and yield conditions. The true collapse load is obtained by minimization with respect to the kinematic variables and maximization with respect to the static variables. The method is illustrated with simple examples. Application is made to evaluate the shear strength of reinforced concrete beams. It is concise as compared to separate upper and lower bound analyses. It also allows the minimization and maximization processes to be carried out in any order, thus enabling the collapse load to be calculated by the easier way.

1. INTRODUCTION

In structural analysis, the load carrying capacity of a structure is often of the main concern. This can be evaluated by using the theorems of limit analysis which can be stated as follows :

Theorem 1 : At limit load, all stresses remain constant and all deformations are plastic.

Theorem 2 : If a distribution of stresses which satisfies equilibrium and the stress boundary conditions exists in the structure and is everywhere below yield, then the structure will not collapse.

Theorem 3 : The structure will collapse if there is any compatible pattern of plastic deformation for which the rate of work associated with the external forces equals or exceeds the rate of internal energy dissipation.

Theorem 1 implies that the theorems of limit analysis can only be applicable for structures made of ductile materials in an exact sense. Consequently, limit analysis has proved very successful for metal structures such as steel frames, in which ductility of the material and re-distribution of stresses within the structure can be expected[1),2)]. However, it should be noted that ductility of materials does not play an important role if the geometry of the structure is such that re-destribution of stresses and a resulting increase in ultimate strength are not expected. In addition, like any other mathematical tools, the results obtained by using the theorems of limit analysis have to be substantiated by experimental evidence. Hence, it is felt that the theorems of limit analysis should also be applicable for structures which consist of non-ductile materials if the strength of the materials is reduced appropriately in the analysis[3)]. Examples of such structures are reinforced or prestressed concrete structures and soil structures.

* Student Member of JSCE M. Eng., Graduate Student, Dept. of Civil Eng., Univ. of Tokyo. (Bunkyo-ku, Tokyo)
** Member of JSCE Dr. Eng., Asso. Prof., Dept. of Civil Eng., Univ. of Tokyo. (Bunkyo-ku, Tokyo)
*** Member of JSCE Ph.D., Vice President for Academic Affairs, Asian Institute of Technology (Bangkok, Thailand), on leave from Univ. of Tokyo

Theorem 2, which corresponds to the lower bound theorem, indicates that the ultimate load derived by considering a statically admissible sress field is always either smaller or, at the most, equal to the actual collapse load. In a lower bound analysis, it is therefore usual to evaluate the ultimate strength or collapse load as a function of static variables (that is, stresses or stress resultants) and, without violating the yield conditions, maximize the value of the ultimate strength so as to obtain the best lower bound solution.

On the other hand, the upper bound theorem (Theorem 3) indicates that the ultimate load obtained by considering a kinematically admissible velocity field or collapse mechanism is always either greater or equal to the actual collapse load. In an upper bound analysis, the ultimate strength of the structure is frequently expressed as a function of kinematic variables and is minimized, without violating the compatibility conditions, to give the best upper bound solution.

In ordinary limit analysis, it is common to consider an upper bound analysis separately from a lower bound analysis. In the event that the best lower bound solution coincides with the best upper bound solution, the exact or true collapse load is said to have been obtained since the conditions of equilibrium, yield and compatibility are simultaneously satisfied for such a case. It is noted, however, that both the upper bound and lower bound approaches involve the consideration of the equilibrium state of the structure. This leads to a repetition in the derivation of the equilibrium equations (or virtual work equations) if upper and lower bound analyses are to be performed separately. The aim of this paper is, therefore, to introduce a method of limit analysis which combines the features of the upper and lower bound analyses so that such a repetition can be avoided. The method is illustrated with simple examples. Then, for an example of practical purpose, application is made to evaluate the shear strength of reinforced concrete beams. The advantages of the proposed method over separate upper and lower bound analyses are also being discussed.

2. COMBINED UPPER AND LOWER BOUND ANALYSIS

The proposed method of limit analysis can be explained by considering a structure subject to a system of external forces which are proportional to one another at any level of loading (Fig.1). From the equilibrium conditions of the structure, the representative force, P, can be expressed by a function of a set of kinematic variables $\boldsymbol{k}$ and static variables $\boldsymbol{s}$ as

$$\left.\begin{array}{l} P=P(\boldsymbol{k}, \boldsymbol{s}) \\ \text{where} \\ \boldsymbol{k}=\{k_1, k_2, \cdots, k_m\} \\ \boldsymbol{s}=\{s_1, s_2, \cdots, s_n\} \end{array}\right\} \quad \cdots\cdots (1)$$

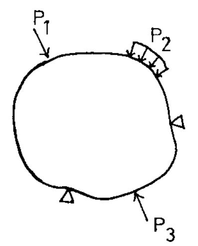

Fig. 1 Structure under a System of Forces.

The kinematic variables are simply parameters which describe the kinematically admissible velocity field or collapse mechanism of the structure. On the other hand, the static variables are stresses or stress resultants which define the statically admissible stress field of the structure. It is obvious that these variables must be subject to constraints as

$$C_i(\boldsymbol{k}, \boldsymbol{s}) \geq 0, \quad (i=1, 2, \cdots) \quad \cdots\cdots (2)$$

for the collapse mechanism to be valid and for the yield conditions to be satisfied.

By the upper bound and lower bound theorems of limit analysis, the ultimate strength or true collapse load, P_u, of the structure is then given by the value of P which is minimized with respect to the kinematic variables $\boldsymbol{k}$ and maximized with respect to the static variables $\boldsymbol{s}$. For a structure with given geometrical and material properties and loading condition, the value of P_u is uniquely determined and can thus be expressed as

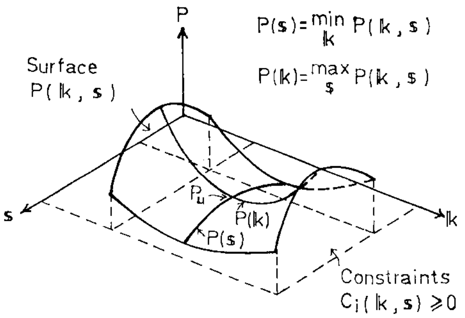

Fig. 2 Concept of Combined Upper and Lower Bound Analysis.

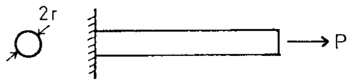

Fig. 3 Bar under Axial Force P.

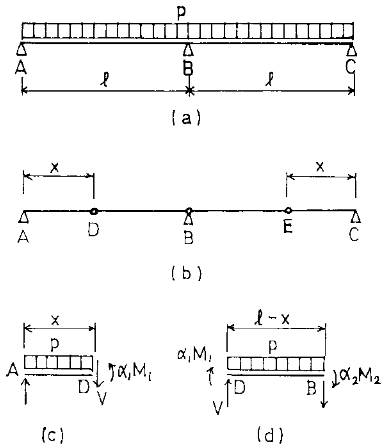

Fig. 4 Two-span Continuous Beam.

$$
\left.\begin{aligned}
P_u &= \max_{s}\left\{\min_{k} P(k, s)\right\} \\
&= \min_{k}\left\{\max_{s} P(k, s)\right\}
\end{aligned}\right\} \quad \cdots\cdots (3)
$$

where k and s are subject to constraints at (2).

The concept of the combined upper and lower bound analysis is shown diagrammatically in Fig.2. The true collapse load, P_u, corresponds to the 'saddle point' of the surface represented by the function $P(k, s)$. It can be seen that the combined upper and lower bound analysis is essentially a constrained optimization problem.

3. SIMPLE EXAMPLES

To illustrate the use of the proposed combined upper and lower bound analysis, the following problems are considered.

(1) Bar under uniaxial tension

In Fig.3, a round bar of radius, r, is subject to an axial force P. Denoting the stress in the cross section of the bar by σ (which constitutes a static variable), the following equilibrium equation may be obtained :

$$P(k, \sigma)=(\pi r^2\sigma)k \quad \cdots\cdots (4)$$

where k is a kinematic variable defining the collapse mechanism. This kinematic variable is introduced to accomodate the theoretically possible collapse mechanisms with collapse loads higher than the true collapse load. It is clear that for collapse to occur, the following condition must hold :

$$k \geq 1 \quad \cdots\cdots (5)$$

On the other hand, if the yield stress of the material of the bar is denoted by σ_y, then yield condition stipulates

$$0 \leq \sigma \leq \sigma_y \quad \cdots\cdots (6)$$

By the proposed method of limit analysis, the true collapse load, P_u,is given by

$$
\left.\begin{aligned}
P_u &= \max_{\sigma}\left\{\min_{k} (\pi r^2 k\sigma)\right\} \\
&= \min_{k}\left\{\max_{\sigma} (\pi r^2 k\sigma)\right\}
\end{aligned}\right\} \quad \cdots\cdots (7)
$$

where k and σ are subject to constraints given at (5) and (6) respectively. The required value is obtained when $k=1$ and $\sigma=\sigma_y$. In other words, the true collapse load is

$$P_u=\pi r^2\sigma_y \quad (8)$$

(2) Two-span continuous beam under uniformly distributed load

Fig.4(a) shows a two-span continuous steel beam subject to a uniformly load, p. The beam has a uniform cross-section and the only possible collapse mechanism is as shown in Fig.4(b) where plastic hinges are formed at D, B and E. Denoting the bending moments at D and B by $\alpha_1 M_1$ and $\alpha_2 M_2$, where α_1 and α_2 are introduced for the sake of generality, the free body diagrams of members AD and DB can be shown as in Figs.4(c) and (d). From these diagrams, the following equilibrium equations are obtained :

$$\left.\begin{array}{l} Vx+px^2/2=\alpha_1 M_1 \\ \text{and} \\ V(l-x)+\alpha_1 M_1+\alpha_2 M_2-p(l-x)^2/2=0 \end{array}\right\} \quad (9)$$

where $x=\overline{AD}$.

Eliminating the shear stress resultant V from Eqs.(9) leads to

$$p\equiv p(\alpha_1, \alpha_2, x, M_1, M_2)=\frac{2\{\alpha_1 M_1 l+\alpha_2 M_2 x\}}{lx(l-x)} \quad (10)$$

Here, the kinematic variables, α_1, α_2 and x are subject to the following constraints so as to ensure the validity of the collapse mechanism shown in Fig.4(b) :

$$\left.\begin{array}{l} \alpha_1\geq 1 \\ \alpha_2\geq 1 \\ 0<x<l \end{array}\right\} \quad (11)$$

On the other hand, in order to satisfy yield conditions, the static variables, M_1 and M_2, are subject to the following constraints :

$$\left.\begin{array}{l} 0\leq M_1\leq M_p \\ 0\leq M_2\leq M_p \end{array}\right\} \quad (12)$$

where M_p is the full plastic moment of the beam section.

By the proposed combined upper and lower bound analysis, the true collapse load, p_u, is given by

$$\left.\begin{array}{l} p_u=\max\limits_{\{M_1,\, M_2\}}\left\{\min\limits_{\{\alpha_1,\, \alpha_2,\, x\}} p(\alpha_1, \alpha_2, x, M_1, M_2)\right\} \\ \quad=\min\limits_{\{\alpha_1,\, \alpha_2,\, x\}}\left\{\max\limits_{\{M_1,\, M_2\}} p(\alpha_1, \alpha_2, x, M_1, M_2)\right\} \end{array}\right\} \quad (13)$$

To obtain the required solution, the value of p is first maximized with respect to static variables, M_1 and M_2. This gives

$$p(\alpha_1, \alpha_2, x)=\max\limits_{\{M_1,\, M_2\}} p(\alpha_1, \alpha_2, x, M_1, M_2)=2\, M_p\frac{\{\alpha_1 l+\alpha_2 x\}}{lx(l-x)}$$

Next, the value of $p(\alpha_1, \alpha_2, x)$ is to be minimized with respect to the kinematic variables, α_1, α_2 and x. It is clear that $p(\alpha_1, \alpha_2, x)$ is minimized with respect to α_1 and α_2 when $\alpha_1=\alpha_2=1$. Hence, the following is obtained :

$$p_u=\min\limits_{x}\left\{\min\limits_{\{\alpha_1,\, \alpha_2\}} p(\alpha_1, \alpha_2, x)\right\}=\min\limits_{x}\left\{2\, M_p\frac{(l+x)}{lx(l-x)}\right\}$$

Defining

$$p(x)\equiv 2\, M_p\frac{(l+x)}{lx(l-x)}$$

then for $\dfrac{dp(x)}{dx}$ to be equal to zero, the following must hold :

$$x=-2.414\, l \text{ or } 0.414\, l$$

As the kinematic variable x is subject to the constraints at (11) and $d^2p(x)/dx^2>0$ for $x=0.414\, l$, it is concluded that the true collapse load is

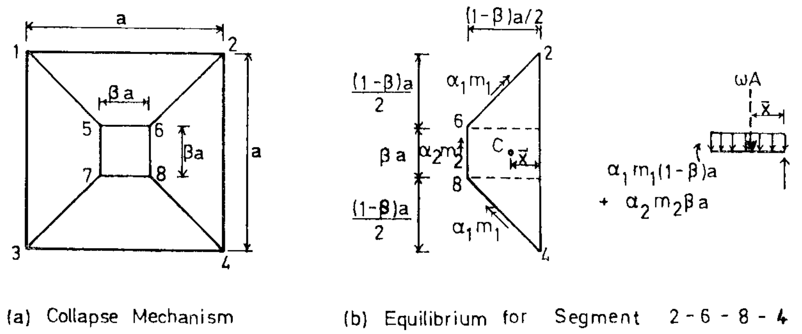

Fig. 5 A Square Plate Simply Supported on All Four Edges.

$$p_u = p(x)|_{x=0.414l} = 11.66\, M_p/l^2 \quad (14)$$

(3) Simply Supported Square Plate

Consider a square plate of side dimension a, simply supported on its four edges as shown in Fig.5. The plate has a yield moment of m_p per unit length in both the positive and negative directions. The corners of the plate are held down and the plate is subject to a uniformly distributed load ω. In this case, the collapse mechanism of the plate can be assumed as shown in Fig.5(a), in which β is a kinematic variable defining the collapse mechanism.

Considering the segment 2-6-8-4, on which the positive moments acting along the edges 2-6 and 8-4 are $\alpha_1 m_1$ per unit length and that along edge 6-8 is $\alpha_2 m_2$ per unit length as shown in Fig.5(b), where the kinematic variables α_1 and α_2 are introduced for the purpose of generality, the following equilibrium equation may be obtained :

$$\left.\begin{array}{l} \omega A\bar{x} = \alpha_1 m_1(1-\beta)a + \alpha_2 m_2 \beta a \\ \text{where} \\ \quad A = \text{area of segment 2-6-8-4} = (1-\beta^2)a^2/4 \\ \quad \bar{x} = \text{distance of centroid } C \text{ of segment 2-6-8-4 from} \\ \qquad \text{edge 2-4} = [(1-\beta)^2(1+2\beta)a]/[6(1-\beta^2)] \end{array}\right\} \quad (15)$$

From Eqs.(15), the value of ω can be written as

$$\omega = \frac{24[\alpha_1 m_1(1-\beta) + \alpha_2 m_2 \beta]}{(1-\beta)^2(1+2\beta)a^2} \quad (16)$$

The ultimate uniform load is then given by

$$\omega_u = \min_{\{\alpha_1,\, \alpha_2,\, \beta\}} \left\{ \max_{\{m_1,\, m_2\}} \omega \right\} \quad (17)$$

where the kinematic variables α_1, α_2 and β and static variables m_1 and m_2 are subject to the following constraints :

$$\left.\begin{array}{l} \alpha_1 \geq 1 \\ \alpha_2 \geq 1 \\ 0 \leq \beta \leq 1 \\ 0 \leq m_1 \leq m_p \\ 0 \leq m_2 \leq m_p \end{array}\right\} \quad (18)$$

It can be seen that ω is maximized with respect to $\{m_1, m_2\}$ when $m_1 = m_2 = m_p$ and minimized with respect to $\{\alpha_1, \alpha_2\}$ when $\alpha_1 = \alpha_2 = 1$, regardless of the value of β. Hence,

$$\omega_u = \min_{\beta} \left\{ \frac{24\, m_p}{(1-\beta)^2(1+2\beta)a^2} \right\} \quad (19)$$

Denoting the term in the denominator by $f(\beta) \equiv (1-\beta)^2(1+2\beta)$, it can be easily verified that $df(\beta)/d\beta = 0$ for $\beta = 0$ or 1 and that $d^2f(0)/d\beta^2 < 0$ and $d^2f(1)/d\beta^2 > 0$. Hence, $f(\beta)$ is maximum or, in other words, ω is minimum with respect to β when $\beta=0$. Consequently,

$$\omega_u = \left. \frac{24\, m_p}{(1-\beta)^2(1+2\beta)a^2} \right|_{\beta=0} = \frac{24\, m_p}{a^2} \qquad (20)$$

The result given by Eq.(20) has been known to be an exact solution[4].

In the above simple examples, it is clear that the values of some of the kinematic variables (such as k in example (1) and α_1, α_2 in examples (2) and (3)) which result in the true collapse load are trivial and are each equal to unity. These kinematic variables have been introduced for the purpose of consistency with the general concept of the proposed combined upper and lower bound analysis (surface P in Fig.2). They are required as the associated static variables (σ in example (1), M_1, M_2 in example (2) and m_1, m_2 in example (3)) are subject to constraints, which limit their maximum values. For more complex problems (such as the one in the next section), such kinematic variables can be conveniently taken as equal to unity for simplicity.

4. APPLICATION FOR REINFORCED CONCRETE BEAMS

In order to demonstrate the advantage of the proposed combined upper and lower bound procedure, application is made to evaluate the ultimate shear strength of reinforced concrete beams without web reinforcement. Fig.6(a) shows a simply supported reinforced concrete beam under a symmetrical two-point load. The beam has a prismatic, rectangular cross-section with both top and bottom longitudinal reinforcement. Failure of the beam is assumed to occur by the formation of a shear crack in the shape of a parablola (Fig.6(b)). To simplify the analysis, plane stress condition is considered and the area of concrete above the shear crack is assumed to be under uniform compression and shear. Dowel actions of the longitudinal bars are neglected and the force per unit area, f, due to aggregate interlock action along the crack is taken to be constant. In addition, the concrete and top longitudinal steel are assumed to carry compression forces only whereas the bottom longitudinal steel is assumed to carry tension only.

Using the co-ordinate system as shown in Fig.6(b), the equation of the parabola (symmetrical about the x-axis) which represents the shear crack can be written as

$$y^2 = \frac{\alpha^2 h^2 (x - a\gamma)}{a(1-\gamma)} \qquad (21)$$

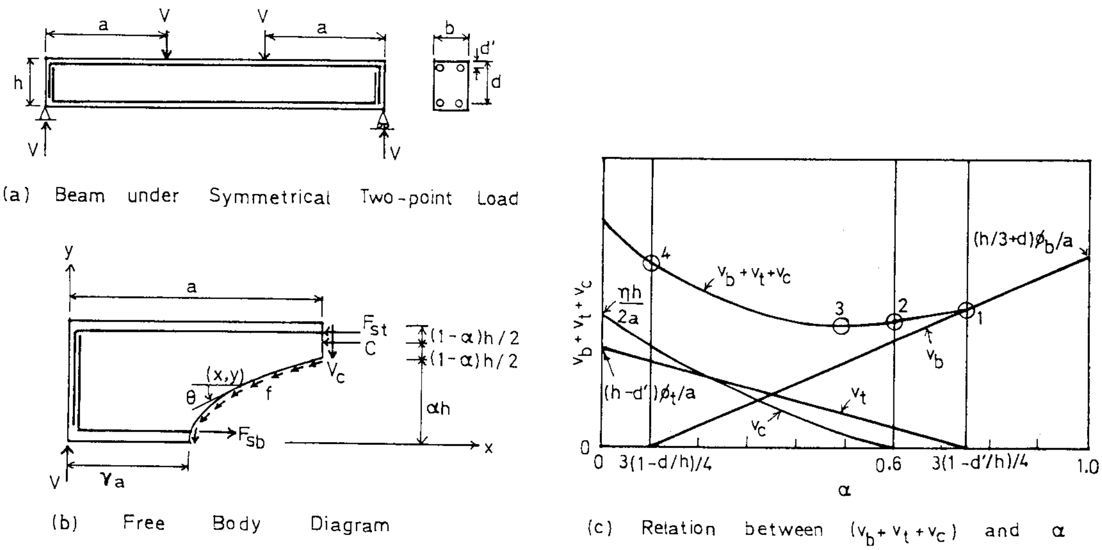

Fig. 6 Shear Strength of Reinforced Concrete Beams.

from which the slope of the crack at any point (x, y) can be calculated as

$$\frac{dy}{dx}=\tan\theta=\frac{1}{2}\cdot\frac{\alpha h}{\sqrt{\alpha(1-\gamma)}}\cdot\frac{1}{\sqrt{x-\alpha\gamma}} \qquad (22)$$

The equilibrium equations are given as

$$\left.\begin{aligned}&F_{sb}-F_{st}-C-b\int_s f\cos\theta\,ds=0\\&V-V_c-b\int_s f\sin\theta\,ds=0\\&V_c a-b\int_s f\cos\theta\,y ds+b\int_s f\sin\theta\,x ds-\frac{1}{2}(1+\alpha)hC-F_{st}(h-d')+F_{sb}(h-d)=0\end{aligned}\right\} \qquad (23)$$

where the various terms are as defined in Fig.6(b) and the integration is taken over the whole length of the crack. Substituting Eqs. (21) and (22) into (23) leads to

$$\left.\begin{aligned}&F_{sb}-F_{st}-C-\alpha bf(1-\gamma)=0\\&V-V_c-\alpha bhf=0\\&V_c a-\frac{1}{3}\alpha a bhf(1-4\gamma)-\frac{1}{2}(1+\alpha)hC-F_{st}(h-d')+F_{sb}(h-d)=0\end{aligned}\right\} \qquad (24)$$

Eliminating $(1-\gamma)f$ [i.e., γ and f] and V_c from Eqs.(24) gives

$$V=\left\{\left(\frac{4}{3}\alpha-1\right)\frac{h}{a}+\frac{d}{a}\right\}F_{sb}-\left\{\left(\frac{4}{3}\alpha-1\right)\frac{h}{a}+\frac{d'}{a}\right\}F_{st}+\frac{h}{a}\left(\frac{1}{2}-\frac{5}{6}\alpha\right)C \qquad (25)$$

Introducing non-dimensional terms

$$v\equiv\frac{V}{bhf'_c},\quad c\equiv\frac{C}{bhf'_c},\quad \phi_{sb}\equiv\frac{F_{sb}}{bhf'_c},\quad \phi_{st}\equiv\frac{F_{st}}{bhf'_c} \qquad (26)$$

where f'_c is the compressive strength of concrete, Eq.(25) can be re-written as

$$v\equiv v(\alpha,\phi_{sb},\phi_{st},c)=\left\{\left(\frac{4}{3}\alpha-1\right)\frac{h}{a}+\frac{d}{a}\right\}\phi_{sb}-\left\{\left(\frac{4}{3}\alpha-1\right)\frac{h}{a}+\frac{d'}{a}\right\}\phi_{st}+\frac{h}{a}\left(\frac{1}{2}-\frac{5}{6}\alpha\right)c \qquad (27)$$

Here, α is a kinematic variable which defines the collapse mechanism and $(\phi_{sb}, \phi_{st}, c)$ from a set of static variables which must satisfy yield conditions.

By the combined upper and lower bound analysis, the ultimate strength of the beam is given as

$$v_u=\min_{\alpha}\left\{\max_{\phi_{sb},\phi_{st},c}\{v(\alpha,\phi_{sb},\phi_{st},c)\}\right\} \qquad (28)$$

where α, ϕ_{sb}, ϕ_{st} and c are subject to the following constraints

$$\left.\begin{aligned}&0\le\alpha\le1\\&0\le\phi_{sb}\le\frac{A_{sb}f_{yb}}{bhf'_c}\equiv\phi_b\\&0\le\phi_{st}\le\frac{A_{st}f_{yt}}{bhf'_c}\equiv\phi_t\\&0\le c\le\frac{(1-\alpha)hb\eta f'_c}{bhf'_c}\equiv(1-\alpha)\eta\end{aligned}\right\} \qquad (29)$$

in which A_{sb}, f_{yb} and A_{st}, f_{yt} are the area, yield stress of the bottom and top reinforcement respectively and η is an effectiveness factor to take into account of the lack of ductility in concrete.

To further simplify the problem, both steel and concrete are assumed to be rigid perfectly plastic materials. Then, for a given value of α, the static variables ϕ_{sb}, ϕ_{st} and c must take their maximum or minimum values according to whether their coefficients are positive or negative in order that v is maximized. That is, from Eq.(28) and constraints at (29)

$$v_u=\min_{\alpha}\left\{<\left(\frac{4}{3}\alpha-1\right)\frac{h}{a}+\frac{d}{a}>\phi_b+<-\left(\frac{4}{3}\alpha-1\right)\frac{h}{a}-\frac{d'}{a}>\phi_t+<\frac{h}{a}\left(\frac{1}{2}-\frac{5}{6}\alpha\right)>(1-\alpha)\eta\right\}$$

where

$$<x>=\begin{cases}x & \text{for}\quad x>0\\0 & \text{for}\quad x<0\end{cases} \qquad (30)$$

Introducing

$$\left.\begin{aligned} v_b &= \left\langle \left(\frac{4}{3}\alpha - 1\right)\frac{h}{a} + \frac{d}{a} \right\rangle \phi_b \\ v_t &= \left\langle -\left(\frac{4}{3}\alpha - 1\right)\frac{h}{a} - \frac{d'}{a} \right\rangle \phi_t \\ v_c &= \left\langle \frac{h}{a}\left(\frac{1}{2} - \frac{5}{6}\alpha\right) \right\rangle (1-\alpha)\eta \end{aligned}\right\} \quad \cdots\cdots (31)$$

the following is obtained

$$v_u = \min_{\alpha} (v_b + v_t + v_c) \quad \cdots\cdots (32)$$

The relation between $(v_b + v_t + v_c)$ and α is shown in Fig.6(c). It can be seen that the minimum value of $(v_b + v_t + v_c)$ occurs at either points 1,2,3 or 4, depending on the values of $(\phi_b - \phi_t)/\eta$. By comparing the values of $(v_b + v_t + v_c)$ at these possible minimum points, it is found that v_u is given as

$$\left.\begin{aligned} &\text{for } (\phi_b-\phi_t)/\eta \leq 0, \\ &\quad v_u = \left(\frac{d}{a} - \frac{d'}{a}\right)\phi_b \qquad \text{(point 1)} \\ &\text{for } 0 < (\phi_b-\phi_t)/\eta \leq 1/4, \\ &\quad v_u = \left(\frac{d}{a} - \frac{h}{5a}\right)\phi_b - \left(\frac{d'}{a} - \frac{h}{5a}\right)\phi_t \qquad \text{(point 2)} \\ &\text{for } 1/4 < (\phi_b-\phi_t)/\eta \leq (1+15\,d/h)/16, \\ &\quad v_u = \left(\frac{h}{15a} + \frac{d}{a}\right)\phi_b - \left(\frac{h}{15a} + \frac{d'}{a}\right)\phi_t - \frac{\eta h}{30a} - \frac{8\eta h}{15a}\left(\frac{\phi_b-\phi_t}{\eta}\right)^2 \qquad \text{(point 3)} \\ &\text{for } (1+15\,d/h)/16 < (\phi_b-\phi_t)/\eta, \\ &\quad v_u = \frac{h\eta}{32a}\left(5\frac{d}{h} - 1\right)\left(1 + 3\frac{d}{h}\right) + \left(\frac{d}{a} - \frac{d'}{a}\right)\phi_t \qquad \text{(point 4)} \end{aligned}\right\} \quad \cdots\cdots (33)$$

It is noted that the values of the kinematic and static variables are determined at the same time as the true collapse load, v_u. These values give the shape and position of the shear crack and the state of steel and concrete (that is, whether yielding of the material occurs or not) at collapse, as summarized in Table 1. The contributions of the aggregate interlock forces, f, and shear force across the compression zone, V_c, towards the ultimate strength of the beam can also be determined from equilibrium equations.

It should be reminded, however, that this problem only serves as an example of the application of the proposed combined upper and lower bound analysis to concrete structures. In the derivation of the results at (33), some assumptions, which have been introduced, may not be realistic and the constraints for the kinematic variable, γ, and static variables, f and V_c, which have been eliminated in the process of deriving Eq.(25), have been omitted. No comparison of the solutions with experimental results is therefore intended in this paper.

Table 1 State of Reinforcement and Concrete at Failure of Beam.

Value of $(\phi_b-\phi_t)/\eta$	Yielding of Bottom Reinft.	Yielding of Top Reinft.	Crushing of Concrete
Below 0	Yes	No	No
Between 0 and 1/4	Yes	Yes	No
Between 1/4 and (1+15d/h)/16	Yes	Yes	Yes
Above (1+15d/h)/16	No	Yes	Yes

5. DISCUSSIONS AND CONCLUSIONS

The combined upper and lower analysis has been shown here to be applicable not only for structures which consist of ductile materials, but also for those with non-ductile materials. The attractiveness of the proposed method lies in the fact that the problem can be formulated as long as the collapse mechanism, yield conditions and equilibrium conditions are known. The displacement field of the structure need not be required as in an upper bound approach using the virtual work equation. The method is therefore more concise as compared to separate upper and lower bound analyses.

The use of the proposed combined upper and lower bound analysis has been illustrated by several examples. The merits of the proposed method of limit analysis may not be apparent from the examples in Section 3 of this paper. In contrast, the example given in Section 4 indicates the advantages of the proposed method when the number of unknown variables increases. In fact, the method is especially useful when the exact mode of the collapse mechanism (such as the position of the plastic hinge within the beam in Fig.4(b) and the position and shape of the shear crack in Fig.6(b)) is not exactly known in advance. In addition, since the values of the kinematic and static variables are determined together with the value of the true collapse load, the method not only allows the exact collapse mode to be determined but it also enables the various internal forces and hence their contributions to the ultimate strength to be calculated simultaneously.

A further advantage of the combined analysis is that the order of maximization and minimization is arbitrary. Hence, if there are multiple kinematic or static variables, the corresponding optimization processes can be performed in a manner such that the true collapse load can be computed easily. The maximization and minimization processes in the combined upper and lower bound analysis may be complicated for some cases. In such cases, it is usually true that the formulation is also complicated even for a separate upper bound or lower bound analyses. The problem then lies in the nature of the model used and a modification of the model may be needed instead. On the other hand, with the advent of computer softwares, it is believed that a combined upper and lower bound analysis may be more advantageous due to its easier formulation as compared to the separate upper bound and lower bound analyses.

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REFERENCES

1) P.G. Hodge, Jr. : Limit Analysis of Rotationally Symmetric Plates and Shells, Prentice-Hall, Inc./Englewood Cliffs, N.J., 1963.

2) P.G. Hodge, Jr. : Plastic Analysis of Structures, McGraw-Hill Book Co., Inc., New York, 1959.

3) for example, M.P. Nielsen and M.W. Braestrup : Shear Strength of Prestressed Concrete Beams without Web Reinforcement, Magazine of Concrete Research, Vol. 30, No.104, pp.119～128, September, 1978.

4) for example, M.P. Nielsen : The Theory of Plasticity for Reinforced Concrete Slabs, IABSE Colloquium on Plasticity in Reinforced Concrete, Copenhagen, pp.93～114, 1979.

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