[PROC. OF JSCE No. 344／I-1 (Structural Eng.／Eanrthquake Eng.) April 1984]

LAGRANGIAN NONLINEAR THEORY OF THIN ELASTIC SHELLS WITH FINITE ROTATIONS

By Masashi IURA and Masaharu HIRASHIMA***

Accurate equilibrium equations and appropriate static and geometric boundary conditions are derived for the geometrically nonlinear theory of shells undergoing finite rotations without restriction to small strains. The principle of virtual work is used to obtain the shell equations in which a nonrational tensor of change of curvature is employed. The introduction of variations of displacement vectors instead of those of displacement components makes it possible to reduce computational efforts for deriving the shell equations. The effects of finite rotations at the shell boundary are strictly taken into account utilizing the total finite rotation vector for the boundary.

1. INTRODUCTION

In the nonlinear theory of thin elastic shells, it is often desirable to employ the Lagrangian formulation rather than the Eulerian formulation. The Lagrangian equilibrium equations and appropriate geometric and static boundary conditions are, in general, written with reference to the undeformed shell midsurface, the geometry of which is known. In the Eulerian approach, on the other hand, all quantities are usually reffered to the unknown deformed shell configuration. When we obtain the numerical solutions of the boundary value problems of nonlinear theory of shells, it is preferable to use the boundary values reffered to the known geometrical quantities. In this way the Lagrangian nonlinear shell equations have been widely utilized. However, to the best of our knowledge, nobody has succeeded as yet in deriving the consistent fully Lagrangian nonlinear theory of shells undergoing finite (unrestricted) rotations without using small strain assumptions.

A consistent Eulerian nonlinear theory of shells with finite rotations has been developed under the Kirchhoff-Love hypotheses[1)~3)]. In these investigations, the parameter β_{ν}, which is defined with respect to the deformed boundary, has been used as the geometric boundary condition for the couple. On the basis of the Eulerian shell theory, the appropriate transformation rules, in which the Lagrangian quantities are related to the Eulerian ones, have been applied to derive the Lagrangian nonlinear theory of shells. This procedure[4),5)], however, may not lead to the consistent Lagrangian shell theory, since the proper formulation of the fourth boundary condition for the resultant boundary couple can not be made[9)]. With the aid of the principle of virtual work, Pietraszkiewicz[1),2)] has obtained the Lagrangian nonlinear theory of shells, in which the parameter β_{ν} defined with respect to the deformed boundary

* Member of JSCE, Dr. Eng., Assistant Professor, Department of Civil and Structural Engineering, Tokyo Denki University (Hatoyama, Hiki-gun, Saitama)
** Member of JSCE, Dr. Eng., Professor, Department of Civil Engineering, Waseda University (Okubo 3-4-1, Shinjuku-ku, Tokyo)

has been used as the fourth boundary condition, while the displacement components defined with respect to the undeformed boundary has been used as displacement conditions on the shell boundary. Such a form of the boundary conditions is incompatible with other fully Lagrangian shell equations.

The appearance of finite rotations is one of the important features of any nonlinear theory of shells[6)~11)]. Most of the existing litareture[12)~17)], however, have dealt with the nonlinear theory of shells with small or moderate rotations. The resulting boundary conditions for the couple have been expressed by the linear terms with respect to displacements and their surface derivatives, since some approximations are made on the nonlinear terms of the tensor of change of curvature. Within the theory of shells undergoing finite rotations, the tensor of change of curvature, defined as a difference between the curvature tensor of the deformed and undeformed shell midsurface, is essentially a nonrational function of displacements and their surface derivatives. Pietraszkiewicz[8)] has derived the two-dimensionally exact nonlinear equilibrium equations under the Kirchhoff-Love hypotheses by using the exact tensor of change of curvature defined above, however he failed to obtain the associated geometric and static boundary conditions. No one has developed as yet the consistent nonlinear theory of shells with finite rotations by utilizing the nonrational tensor of change of curvature. Using the modified tensor of change of curvature, Pietraszkiewicz and Szwabowicz[9)] have obtained the entirely Lagrangian nonlinear theory of thin shells.

In the present paper, a consistent fully Lagrangian nonlinear theory of thin elastic shells with finite rotations is developed under the Kichhoff-Love hypotheses. When we obtain the equilibrium equations and the associated geometric and static boundary conditions of the shell with the use of the principle of virtual work, we do not use the small strain assumptions, nor restrict the magnitude of rotations of the shell. It appears that a cumbersome caluculation is hard to be avoided if we take the variations of the surface strain tensor and the tensor of change of curvature with respect to the displacement components. In this paper, however, the variations of displacement vectors instead of those of displacement components are introduced in the principle of virtual work, so that it is a straight forward matter to caluculate the internal and external virtual work. Thus the present tensor of change of curvature remains to be a nonrational function. The small strain assumptions are introduced only at the constitutive equations. In the published papers, the small strain assumptions have been introduced at a too early stage of derivation of shell equations. It is widely accepted that neglecting the higher terms in the nonlinear equations at a early stage leads to the inconsistent or inaccurate results. In order to avoid inconsistencies of this kind, we derive the two-dimensionally exact nonlinear shell equations without restriction to small strains. On the basis of the resulting exact shell equations, the various variants of consistent approximate theories may be obtained.

The rotations have been conventionally described by a proper orthogonal tensor or a finite rotation vector. Simmonds and Danielson[3)] have formulated a general nonlinear theory of thin shells in terms of a finite rotation vector. Pietraszkiewicz[1) 2)] has obtained the general formulae for the finite rotation tensor and the finite rotation vector in terms of displacements. In this paper the external virtual work for the couple on the shell boundary is expressed by the inner product of the total finite rotation vector and the boundary couple vector. Consequently the effects of finite rotations at the shell boundary are exactly taken into account.

The boundary line integrals in the internal and external virtual work are expressed in terms of the variations of the displacement vector and the fourth parameter defined with respect to the undeformed boundary. The resulting equilibrium equations are found accurate within the range of the two-dimensionally consistent theory. The associated boundary conditions, which consist of four equations both for static and geometric boundary conditions, are compatible with the consistent Lagrangian nonlinear shell theory. Throughout this paper, the summation convention will apply to repeated Greek indexes (in mixed position) with range 2.

2. PRELIMINARIES

In this work, we adopt wherever feasible the notation used by Pietraszkiewicz[11)]. The position vector to a point on the undeformed midsurface M with surface coordinates θ^α is denoted by $\boldsymbol{r}(\theta^\alpha)$. With the reference surface M we associate covariant surface base vectors $\boldsymbol{a}_\alpha=\boldsymbol{r}_{,\alpha}$ and a unit normal vector $\boldsymbol{n}=\varepsilon^{\alpha\beta}\boldsymbol{a}_\alpha\times\boldsymbol{a}_\beta/2$. These and all other vector fields considered are assumed sufficiently smooth to justify all differentiation operations that follow. The notation ( ) , α denotes partial differentiation with respect to surface coordinates θ^α, while $\varepsilon^{\alpha\beta}$ denotes the permutation tensor of the undeformed midsurface. As usual we define covariant components of the metric tensor $a_{\alpha\beta}=\boldsymbol{a}_\alpha\cdot\boldsymbol{a}_\beta$ with determinant $a=|a_{\alpha\beta}|$, and of the curvature tensor $b_{\alpha\beta}=\boldsymbol{a}_{\alpha,\beta}\cdot\boldsymbol{n}$.

Let $\boldsymbol{u}$ denote the displacement field mapping each point of M into a point on the deformed shell midsurface $\overline{M}$. Likewise, with $\overline{M}$ we associate a position vector $\overline{\boldsymbol{r}}=\boldsymbol{r}+\boldsymbol{u}$, base vectors $\overline{\boldsymbol{a}}_\alpha=\overline{\boldsymbol{r}}_{,\alpha}$, a unit normal vector $\overline{\boldsymbol{n}}=\overline{\varepsilon}^{\alpha\beta}\,\overline{\boldsymbol{a}}_\alpha\times\overline{\boldsymbol{a}}_\beta/2$, metric and curvature tensor components $\overline{a}_{\alpha\beta}$ and $\overline{b}_{\alpha\beta}$, respectively. Then we have[11),14)]

$$\overline{\boldsymbol{a}}_\alpha=l_{\lambda\alpha}\,\boldsymbol{a}^\lambda+\phi_\alpha\,\boldsymbol{n}=\boldsymbol{a}_\alpha+\boldsymbol{u}_{,\alpha} \tag{1·a}$$

$$\overline{\boldsymbol{n}}=n_\alpha\,\boldsymbol{a}^\alpha+n\boldsymbol{n} \tag{1·b}$$

$$l_{\alpha\beta}=a_{\alpha\beta}+\theta_{\alpha\beta}-\omega_{\alpha\beta} \tag{1·c}$$

$$l^\alpha_\beta=\delta^\alpha_\beta+u^\alpha|_\beta-b^\alpha_\beta\,w \tag{1·d}$$

$$\theta_{\alpha\beta}=\frac{1}{2}(u_\alpha|_\beta+u_\beta|_\alpha)-b_{\alpha\beta}\,w \tag{1·e}$$

$$\phi_\alpha=w_{,\alpha}+b^\lambda_\alpha\,u_\lambda \tag{1·f}$$

$$\omega_{\alpha\beta}=\frac{1}{2}(u_\beta|_\alpha-u_\alpha|_\beta) \tag{1·g}$$

$$n_\mu=\sqrt{\frac{a}{\overline{a}}}\,\varepsilon^{\alpha\beta}\,\varepsilon_{\lambda\mu}\,\phi_\alpha\,l^\lambda_\beta \tag{1·h}$$

$$n=\frac{1}{2}\sqrt{\frac{a}{\overline{a}}}\,\varepsilon^{\alpha\beta}\,\varepsilon_{\lambda\mu}\,l^\lambda_\alpha\,l^\mu_\beta \tag{1·i}$$

where u_α and w denote the components of the displacement vector defined by

$$\boldsymbol{u}=u_\alpha\,\boldsymbol{a}^\alpha+w\boldsymbol{n} \tag{2}$$

and δ^α_β denotes Kronecker's delta and $(\ )|_\alpha$ the surface covariant differentiation at M. The displacement vector at an arbitrary point of the shell with distance ζ from the midsurface is presented by

$$\boldsymbol{V}=\boldsymbol{u}+\zeta\boldsymbol{\beta} \tag{3}$$

where

$$\boldsymbol{\beta}=\overline{\boldsymbol{n}}-\boldsymbol{n} \tag{4}$$

The boundary contour C of M is defined by the equations $\theta^\alpha=\theta^\alpha(S)$, in which S is the length parameter along C. With each point M of C we associate the unit tangent vector $\boldsymbol{t}=\boldsymbol{r}_{,S}$, where $(\ )_{,S}$ denotes $d(\ )/dS$, and the outward unit normal vector $\boldsymbol{\nu}=\boldsymbol{t}\times\boldsymbol{n}$. For the orthonormal triad $\boldsymbol{\nu}$, $\boldsymbol{t}$ and $\boldsymbol{n}$, we have the following systems of differential formulae :

$$\begin{bmatrix}\boldsymbol{\nu}\\ \boldsymbol{t}\\ \boldsymbol{n}\end{bmatrix}_{,S}=\begin{bmatrix}0 & \varkappa_t & -\tau_t\\ -\varkappa_t & 0 & \sigma_t\\ \tau_t & -\sigma_t & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{\nu}\\ \boldsymbol{t}\\ \boldsymbol{n}\end{bmatrix} \tag{5}$$

where σ_t is a normal curvature, τ_t a geodesic torsion and $\varkappa_t$ a geodesic curvature of the surface boundary contour C. For the later convienience, we decompose the vector $\boldsymbol{\beta}$ with respect to the orthonormal triad $\boldsymbol{\nu}$, $\boldsymbol{t}$ and $\boldsymbol{n}$

$$\boldsymbol{\beta}=\beta^*_\nu\,\boldsymbol{\nu}+\beta^*_t\,\boldsymbol{t}+\beta^*\,\boldsymbol{n} \tag{6}$$

Within the nonlinear theory of shells under the Kirchhoff-Love hypotheses, the shell deformation can be described by the surface strain tensor $\gamma_{\alpha\beta}$ and the tensor of change of curvature $\varkappa_{\alpha\beta}$ defined by

$$\gamma_{\alpha\beta}=\frac{1}{2}(\overline{\boldsymbol{a}}_\alpha\cdot\overline{\boldsymbol{a}}_\beta-\boldsymbol{a}_\alpha\cdot\boldsymbol{a}_\beta) \tag{7·a}$$

$$\varkappa_{\alpha\beta}=-(\overline{\boldsymbol{a}}_{\alpha,\beta}\cdot\overline{\boldsymbol{n}}-\boldsymbol{a}_{\alpha,\beta}\cdot\boldsymbol{n}) \tag{7·b}$$

Substituting Eqs. (1) into Eqs. (7) yields

$$\gamma_{\alpha\beta}=\frac{1}{2}(l^\lambda_\alpha\,l_{\lambda\beta}+\phi_\alpha\,\phi_\beta-a_{\alpha\beta}) \tag{8·a}$$

$$\varkappa_{\alpha\beta}=-[n(\phi_\alpha|_\beta+b_{\lambda\beta}\,l^\lambda_\alpha)+n_\lambda(l^\lambda_\alpha|_\beta-b^\lambda_\beta\,\phi_\alpha)-b_{\alpha\beta}] \tag{8·b}$$

In general, $\gamma_{\alpha\beta}$ are quadratic polynomials in u_α, w and their derivatives, while $\varkappa_{\alpha\beta}$ are nonrational functions of those variables since they contain an invariant $\sqrt{a/\overline{a}}$, where[14)]

$$\bar{a}/a=1+2\gamma^{\alpha}_{\alpha}+2(\gamma^{\alpha}_{\alpha}\gamma^{\beta}_{\beta}-\gamma^{\alpha}_{\beta}\gamma^{\beta}_{\alpha}) \qquad (9)$$

In the existing literature, based on the small strain assumptions, various variants of approximate strain-displacement relations have been introduced. With the use of an approximation $\sqrt{a/\bar{a}}=1-\gamma^{\alpha}_{\alpha}$, the tensor of change of curvature becomes a polynomial of fifth degree in displacements u_{α}, w and their surface derivatives. Because of complex form of $\varkappa_{\alpha\beta}$, Pietraszkiewicz and Szwabowicz[9] have introduced the modified tensor of change of curvature, which is a third degree polynomial in u_{α}, w and their derivatives, defined as

$$\chi_{\alpha\beta}=-\left(\sqrt{\frac{\bar{a}}{a}}\,\bar{b}_{\alpha\beta}-b_{\alpha\beta}\right)+b_{\alpha\beta}\,\gamma^{\varkappa}_{\varkappa} \qquad (10)$$

where the simmilar expression has appeared in 14).

In this work, the variations of displacement vectors are used effectively instead of those of displacement components. The advantage of the usage of such variations is that the variations of $\gamma_{\alpha\beta}$ and $\varkappa_{\alpha\beta}$ in terms of the displacement vector take the more simple form, as shown in Eqs. (11), than those in terms of the displacement components and that computational efforts for deriving the shell equations are significantly reduced. The variation of the displacement vector has been also introduced in 8), however the variation of derivatives of displacement vectors in the outward normal direction could not have been eliminated through integration by parts along the boundary C. As a result appropriate boundary conditions have not been obtained in 8).

The variations of the surface strain tensor and the tensor of change of curvature in terms of the displacement vector are obtained from Eqs. (7) as

$$\delta\gamma_{\alpha\beta}=\frac{1}{2}(\delta\boldsymbol{u}_{,\alpha}\cdot\bar{\boldsymbol{a}}_{\beta}+\bar{\boldsymbol{a}}_{\alpha}\cdot\delta\boldsymbol{u}_{,\beta}) \qquad (11\cdot a)$$

$$\delta\varkappa_{\alpha\beta}=-[(\delta\boldsymbol{u}_{,\alpha})|_{\beta}-\bar{a}^{\varkappa\lambda}\gamma_{\lambda\alpha\beta}\,\delta\boldsymbol{u}_{,\varkappa}]\cdot\bar{\boldsymbol{n}} \qquad (11\cdot b)$$

where[1]

$$\gamma_{\lambda\alpha\beta}=\gamma_{\lambda\alpha}|_{\beta}+\gamma_{\lambda\beta}|_{\alpha}-\gamma_{\alpha\beta}|_{\lambda} \qquad (12)$$

These relations are obviously the function of $\boldsymbol{u}$ and are linear in $\delta\boldsymbol{u}$.

3. DEFORMATION OF SHELL BOUNDARY

During the shell deformation under the Kirchhoff-Love hypotheses, the orthonormal triad $\boldsymbol{\nu}$, $\boldsymbol{t}$ and $\boldsymbol{n}$ is transformed into an orthogonal triad $\bar{\boldsymbol{a}}_{\nu}$, $\bar{\boldsymbol{a}}_{t}$ and $\bar{\boldsymbol{n}}$, defined by

$$\bar{\boldsymbol{a}}_{t}=\bar{\boldsymbol{r}}_{,s}=t^{\alpha}\bar{\boldsymbol{a}}_{\alpha} \qquad (13\cdot a)$$

$$\bar{\boldsymbol{a}}_{\nu}=\bar{\boldsymbol{a}}_{t}\times\bar{\boldsymbol{n}}=\sqrt{\frac{\bar{a}}{a}}\,\nu_{\alpha}\,\bar{\boldsymbol{a}}^{\alpha} \qquad (13\cdot b)$$

$$|\bar{\boldsymbol{a}}_{t}|=|\bar{\boldsymbol{a}}_{\nu}|=\bar{a}_{t}=\sqrt{1+2\gamma_{tt}} \qquad (13\cdot c)$$

$$\gamma_{tt}=\gamma_{\alpha\beta}\,t^{\alpha}t^{\beta} \qquad (13\cdot d)$$

According to the polar decomposition theorem[1)~3)], the deformation near a particle can be decomposed into a rigid-body translation, a pure strain along principal directions of strain and a rigid-body rotation of the principal directions. The directions defined by $\boldsymbol{\nu}$ and $\boldsymbol{t}$ do not coincide, in general, with the principal directions of strain at $M\in C$. During the pure strain the principal directions are only stretched without rotation, while the vectors $\boldsymbol{\nu}$ and $\boldsymbol{t}$ not only change their lengths but, in general, yield rotations. Accordingly the total rotation vector $\boldsymbol{\Omega}_{t}$ of the orthonormal vectors $\boldsymbol{\nu}$, $\boldsymbol{t}$ and $\boldsymbol{n}$ is composed of the finite rigid-body rotation vector $\boldsymbol{\Omega}$ and the finite rotation vector $\check{\boldsymbol{\Omega}}_{t}$ of the boundary caused by the pure strain. In the Lagrangian description, the transformation of $\boldsymbol{\nu}$, $\boldsymbol{t}$ and $\boldsymbol{n}$ into $\bar{\boldsymbol{a}}_{\nu}$, $\bar{\boldsymbol{a}}_{t}$ and $\bar{\boldsymbol{n}}$ consists of extension by the factor $\bar{a}_{t}$, which causes no extention in $\boldsymbol{n}$, and the two successive rotations : first through $\check{\boldsymbol{\Omega}}_{t}$, then through $\boldsymbol{\Omega}$. The relationships of the vectors $\boldsymbol{\nu}$, $\boldsymbol{t}$ and $\boldsymbol{n}$, and the vectors $\bar{\boldsymbol{a}}_{\nu}$, $\bar{\boldsymbol{a}}_{t}$ and $\bar{\boldsymbol{n}}$ are written by[1)~3)],

$$\bar{\boldsymbol{a}}_{\nu}=\bar{a}_{t}[\boldsymbol{\nu}+\boldsymbol{\Omega}_{t}\times\boldsymbol{\nu}+\boldsymbol{\Omega}_{t}\times(\boldsymbol{\Omega}_{t}\times\boldsymbol{\nu})/2\cos^{2}\omega_{t}/2] \qquad (14\cdot a)$$

$$\bar{\boldsymbol{a}}_{t}=\bar{a}_{t}[\boldsymbol{t}+\boldsymbol{\Omega}_{t}\times\boldsymbol{t}+\boldsymbol{\Omega}_{t}\times(\boldsymbol{\Omega}_{t}\times\boldsymbol{t})/2\cos^{2}\omega_{t}/2] \qquad (14\cdot b)$$

$$\bar{\boldsymbol{n}}=[\boldsymbol{n}+\boldsymbol{\Omega}_{t}\times\boldsymbol{n}+\Omega_{t}\times(\boldsymbol{\Omega}_{t}\times\boldsymbol{n})/2\cos^{2}\omega_{t}/2] \qquad (14\cdot c)$$

where

$$\sin\omega_t = |\boldsymbol{\Omega}_t| \quad (15)$$

From Eqs. (14) we have

$$2\boldsymbol{\Omega}_t\cdot\boldsymbol{\nu} = \bar{\boldsymbol{t}}\cdot\boldsymbol{n} - \bar{\boldsymbol{n}}\cdot\boldsymbol{t} \quad (16\cdot a)$$

$$2\boldsymbol{\Omega}_t\cdot\boldsymbol{t} = \bar{\boldsymbol{n}}\cdot\boldsymbol{\nu} - \bar{\boldsymbol{\nu}}\cdot\boldsymbol{n} \quad (16\cdot b)$$

$$2\boldsymbol{\Omega}_t\cdot\boldsymbol{n} = \bar{\boldsymbol{\nu}}\cdot\boldsymbol{t} - \bar{\boldsymbol{t}}\cdot\boldsymbol{\nu} \quad (16\cdot c)$$

where the unit vectors $\bar{\boldsymbol{\nu}}$ and $\bar{\boldsymbol{t}}$ are defined as

$$\bar{\boldsymbol{\nu}} = \bar{\boldsymbol{a}}_\nu/\bar{a}_t \quad (17\cdot a)$$

$$\bar{\boldsymbol{t}} = \bar{\boldsymbol{a}}_t/\bar{a}_t \quad (17\cdot b)$$

These relations will be used in evaluating the external virtual work.

4. INTERNAL VIRTUAL WORK

Let the shell be in equilibrium under the external surface and boundary loads, the directions of which are assumed to remain constant during deformation. For any additional virtual displacement vector $\delta\boldsymbol{u} = \delta u^\alpha \boldsymbol{a}_\alpha + \delta w\,\boldsymbol{n}$ subject to geometric constraints, the principle of virtual work states that the internal virtual work IVW, performed by the internal stress and couple resultant tensors on variations of corresponding strain measures, should be equal to the external virtual work EVW, performed on variations of appropriate displacemental variables by the external surface and boundary loads.

Under the Kirchhoff-Love hypotheses, the Lagrangian internal virtual work can be put in the form[1),2),11)]

$$IVW = \iint_M (N^{\alpha\beta}\,\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\,\delta\chi_{\alpha\beta})dA \quad (18)$$

where $N^{\alpha\beta}$ and $M^{\alpha\beta}$ denote components of the symmetric (2nd Piola-Kirchhoff type) internal stress and couple resultant tensors. The introduction of Eqs. (11) into Eq. (18) yields

$$IVW = -\iint_M [(N^{\alpha\beta}\bar{\boldsymbol{a}}_\alpha)|_\beta + (M^{\alpha\beta}\bar{a}^{\kappa\lambda}\gamma_{\lambda\alpha\beta}\bar{\boldsymbol{n}})|_\kappa + (M^{\alpha\beta}\bar{\boldsymbol{n}})|_{\alpha\beta}]\cdot\delta\boldsymbol{u}\,dA + \int_C [N^{\alpha\beta}\bar{\boldsymbol{a}}_\beta + (M^{\alpha\beta}\bar{\boldsymbol{n}})|_\beta + M^{\kappa\beta}\bar{a}^{\alpha\lambda}\gamma_{\lambda\kappa\beta}\bar{\boldsymbol{n}}]\nu_\alpha$$

$$\cdot\delta\boldsymbol{u}\,dS - \int_C M^{\alpha\beta}t_\beta\nu_\alpha\bar{\boldsymbol{n}}\cdot\delta\boldsymbol{u}_{,s}\,dS - \int_C M^{\alpha\beta}\nu_\alpha\nu_\beta\underline{\bar{\boldsymbol{n}}\cdot\delta\boldsymbol{u}_{,\nu}}\,dS \quad (19)$$

Because of variations of derivatives of the displacement vector with respect to the outward normal direction corresponding to the underlined term in Eq. (19), the internal virtual work could not have been expressed in terms of the variations of four appropriate variables. In this paper, it is shown after some transformation discussed later that the internal virtual work is represented in terms of the variations of four independent variables.

Consider the derivatives of the position vector after deformation with respect to the outward normal direction. Using the definition $\bar{\boldsymbol{n}} = \bar{\varepsilon}^{\alpha\beta}\bar{\boldsymbol{a}}_\alpha\times\bar{\boldsymbol{a}}_\beta/2$ and observing that[9)]

$$\bar{\boldsymbol{a}}_\alpha = \nu_\alpha\bar{\boldsymbol{r}}_{,\nu} + t_\alpha\bar{\boldsymbol{r}}_{,s} \quad (20)$$

we can write the normal vector after deformation in the form

$$\bar{\boldsymbol{n}} = \sqrt{\frac{a}{\bar{a}}}\,(\bar{\boldsymbol{r}}_{,\nu}\times\bar{\boldsymbol{r}}_{,s}) \quad (21)$$

Introducing Eq. (21) into Eq. (13·b) leads to

$$\bar{\boldsymbol{a}}_\nu = \sqrt{\frac{a}{\bar{a}}}\,(\bar{a}_t^2\bar{\boldsymbol{r}}_{,\nu} - \bar{b}_t\bar{\boldsymbol{r}}_{,s}) \quad (22)$$

where

$$\bar{b}_t = \nu^\alpha t^\beta\bar{a}_{\alpha\beta} \quad (23)$$

From Eq. (22) we have

$$\bar{\boldsymbol{r}}_{,\nu} = \frac{\bar{b}_t}{\bar{a}_t^2}\bar{\boldsymbol{r}}_{,s} + \frac{1}{\bar{a}_t^2}\sqrt{\frac{\bar{a}}{a}}\,\bar{\boldsymbol{a}}_\nu \quad (24)$$

With the use of Eq. (24) and the condition of orthogonality of the vectors $\bar{\boldsymbol{a}}_\nu$, $\bar{\boldsymbol{a}}_t$ and $\bar{\boldsymbol{n}}$, the inner product indicated by the underline in Eq. (19) is presented by

$$\overline{\boldsymbol{n}}\cdot\delta\boldsymbol{u}_{,\nu}=\overline{\boldsymbol{n}}\cdot\delta\overline{\boldsymbol{r}}_{,\nu}=\frac{\overline{b}_t}{\overline{a}_t^2}\overline{\boldsymbol{n}}\cdot\delta\boldsymbol{u}_{,s}-\frac{1}{\overline{a}_t}\sqrt{\frac{\overline{a}}{a}}\,\overline{\boldsymbol{\nu}}\cdot\delta\overline{\boldsymbol{n}} \quad (25)$$

With the aid of the displacement vector at the boundary

$$\boldsymbol{u}=u_\nu\,\boldsymbol{\nu}+u_t\,\boldsymbol{t}+w\boldsymbol{n} \quad (26)$$

the tangent vector after deformation is expressed by

$$\overline{\boldsymbol{a}}_t=c_\nu\,\boldsymbol{\nu}+c_t\,\boldsymbol{t}+c\boldsymbol{n} \quad (27)$$

where

$$c_\nu=u_{\nu,s}+\tau_t\,w-\kappa_t\,u_t \quad (28\cdot\text{a})$$

$$c_t=1+u_{t,s}+\kappa_t\,u_\nu-\sigma_t\,w \quad (28\cdot\text{b})$$

$$c=w_{,s}+\sigma_t\,u_t-\tau_t\,u_\nu \quad (28\cdot\text{c})$$

From the definition $\overline{\boldsymbol{\nu}}=\overline{\boldsymbol{t}}\times\overline{\boldsymbol{n}}$ and Eq. (27), we can express the inner product $\overline{\boldsymbol{\nu}}\cdot\delta\overline{\boldsymbol{n}}$ in the form

$$\overline{\boldsymbol{\nu}}\cdot\delta\overline{\boldsymbol{n}}=d_\nu\,\delta\beta_\nu^*+d_t\,\delta\beta_t^*+d\delta\beta^* \quad (29)$$

where

$$d_\nu=\{c_t(1+\beta^*)-c\beta_t^*\}/\overline{a}_t \quad (30\cdot\text{a})$$

$$d_t=\{c\beta_\nu^*-c_\nu(1+\beta^*)\}/\overline{a}_t \quad (30\cdot\text{b})$$

$$d=(c_\nu\beta_t^*-c_t\beta_\nu^*)/\overline{a}_t \quad (30\cdot\text{c})$$

It should be noted that the variations $\delta\beta_t^*$ and $\delta\beta^*$ are dependent variables, which may be experssed in terms of $\delta\boldsymbol{u}_{,s}$ and $\delta\beta_\nu^*$.

Since

$$\boldsymbol{\beta}\cdot\boldsymbol{\beta}=-2\,\boldsymbol{n}\cdot\boldsymbol{\beta} \quad (31)$$

we have

$$(\beta_\nu^*)^2+(\beta_t^*)^2+(\beta^*)^2=-2\beta^* \quad (32)$$

Forming the variation of Eq. (32) we obtain, after a simple transformation,

$$\delta\beta^*=-(\beta_\nu^*\,\delta\beta_\nu^*+\beta_t^*\,\delta\beta_t^*)/(1+\beta^*) \quad (33)$$

On the other hand, the variation $\delta\beta_t^*$ is written by

$$\delta\beta_t^*=\delta\boldsymbol{\beta}\cdot\boldsymbol{t} \quad (34\cdot\text{a})$$

$$=-\overline{\boldsymbol{n}}\cdot\delta\boldsymbol{u}_{,s}-c_\nu\,\delta\beta_\nu^*-(c_t-1)\,\delta\beta_t^*-c\delta\beta^* \quad (34\cdot\text{b})$$

Substituting Eq. (33) into Eq. (34·b) we obtain for the variation $\delta\beta_t^*$ the following relation :

$$\delta\beta_t^*=f_t\,\overline{\boldsymbol{n}}\cdot\delta\boldsymbol{u}_{,s}+f_\nu\,\delta\beta_\nu^* \quad (35)$$

where

$$f_t=-(1+\beta^*)/\{c_t(1+\beta^*)-c\beta_t^*\} \quad (36\cdot\text{a})$$

$$f_\nu=\{c\beta_\nu^*-c_\nu(1+\beta^*)\}/\{c_t(1+\beta^*)-c\beta_t^*\} \quad (36\cdot\text{b})$$

Introducing Eq. (35) into Eq. (33) leads to

$$\delta\beta^*=g_t\,\overline{\boldsymbol{n}}\cdot\delta\boldsymbol{u}_{,s}+g_\nu\,\delta\beta_\nu^* \quad (37)$$

where

$$g_t=-f_t\,\beta_t^*/(1+\beta^*) \quad (38\cdot\text{a})$$

$$g_\nu=-(\beta_\nu^*+f_\nu\,\beta_t^*)/(1+\beta^*) \quad (38\cdot\text{b})$$

From Eqs. (29), (35) and (37) the inner product $\overline{\boldsymbol{\nu}}\cdot\delta\overline{\boldsymbol{n}}$ can be expressed in terms of the variations $\delta\boldsymbol{u}_{,s}$ and $\delta\beta_\nu^*$ as

$$\overline{\boldsymbol{\nu}}\cdot\delta\overline{\boldsymbol{n}}=h_t\,\overline{\boldsymbol{n}}\cdot\delta\boldsymbol{u}_{,s}+h_\nu\,\delta\beta_\nu^* \quad (39)$$

where

$$h_t=d_t\cdot f_t+d\cdot g_t \quad (40\cdot\text{a})$$

$$h_\nu=d_\nu+d_t\cdot f_\nu+d\cdot g_\nu \quad (40\cdot\text{b})$$

Utilizing Eqs. (25) and (39), the internal virtual work can be rewritten in the form

$$IVW=-\iint_M \{(N^{\alpha\beta}\bar{\boldsymbol{a}}_\alpha)|_\beta+(M^{\alpha\beta}\bar{a}^{\kappa\lambda}\gamma_{\lambda\alpha\beta}\bar{\boldsymbol{n}})|_\kappa+(M^{\alpha\beta}\bar{\boldsymbol{n}})|_{\alpha\beta}\}\cdot\delta\boldsymbol{u}\,dA$$
$$+\int_C\{N^{\alpha\beta}\bar{\boldsymbol{a}}_\beta+(M^{\alpha\beta}\bar{\boldsymbol{n}})|_\beta+M^{\kappa\beta}\bar{a}^{\alpha\lambda}\gamma_{\lambda\kappa\beta}\bar{\boldsymbol{n}}\}\nu_\alpha\cdot\delta\boldsymbol{u}\,dS$$
$$+\int_C[M^{\alpha\beta}\nu_\alpha\{t_\beta+\Big(\frac{\bar{b}_t}{\bar{a}_t^2}-h_t\frac{1}{\bar{a}_t}\sqrt{\frac{\bar{a}}{a}}\Big)\nu_\beta\}\bar{\boldsymbol{n}}]_{,s}\cdot\delta\boldsymbol{u}\,dS+\int_C M^{\alpha\beta}\nu_\alpha\nu_\beta\frac{1}{\bar{a}_t}\sqrt{\frac{\bar{a}}{a}}h_\nu\,\delta\beta_\nu^*\,dS+\sum_k\boldsymbol{R}_k^*\cdot\delta\boldsymbol{u}_k \qquad (41)$$

where

$$\boldsymbol{R}^*=M^{\alpha\beta}\nu_\alpha\left\{\nu_\beta\left(h_t\frac{1}{\bar{a}_t}\sqrt{\frac{\bar{a}}{a}}-\frac{\bar{b}_t}{\bar{a}_t^2}\right)-t_\beta\right\}\bar{\boldsymbol{n}} \qquad (42\cdot a)$$

$$\boldsymbol{R}_k^*=\boldsymbol{R}^*(S_k+0)-\boldsymbol{R}^*(S_k-0) \qquad (42\cdot b)$$

and M_k, $k=1,2,\cdots,K$, are corner points of the boundary contour C and S_k denotes the coordinate corresponding to the corner points M_k.

It should be emphasized that the internal virtual work can be expressed in terms of the independent variations $\delta\boldsymbol{u}$ and $\delta\beta_\nu^*$ without using small strain assumptions.

5. EXTERNAL VIRTUAL WORK

When the shell is subject to the surface force $\boldsymbol{p}=p^\alpha\boldsymbol{a}_\alpha+p\boldsymbol{n}$, the boundary force $\tilde{\boldsymbol{F}}=\tilde{F}_\nu\boldsymbol{\nu}+\tilde{F}_t\boldsymbol{t}+\tilde{F}\boldsymbol{n}$ and the boundary couple $\tilde{\boldsymbol{K}}=-\tilde{K}_t\boldsymbol{\nu}+\tilde{K}_\nu\boldsymbol{t}+\tilde{K}\boldsymbol{n}$, the external virtual work are written in the form[1),2),11)]

$$EVW=\iint_M\boldsymbol{p}\cdot\delta\boldsymbol{u}\,dA+\int_C(\tilde{\boldsymbol{F}}\cdot\delta\boldsymbol{u}+\underline{\tilde{\boldsymbol{K}}\cdot\delta\boldsymbol{\Omega}_t})dS \qquad (43)$$

Consider the inner product of $\boldsymbol{\nu}$, $\boldsymbol{t}$ and $\boldsymbol{n}$, and $\delta\boldsymbol{\Omega}_t$ in order to express the underlined term in Eq. (43) in terms of the variations $\delta\boldsymbol{u}$ and $\delta\beta_\nu^*$. From Eqs. (16), (35) and (37), and the condition of orthogonality of the vectors $\bar{\boldsymbol{\nu}}$, $\bar{\boldsymbol{t}}$ and $\bar{\boldsymbol{n}}$, we obtain the following results:

$$\boldsymbol{\nu}\cdot\delta\boldsymbol{\Omega}_t=\frac{1}{2}\delta(\bar{\boldsymbol{t}}\cdot\boldsymbol{n}-\bar{\boldsymbol{n}}\cdot\boldsymbol{t})=\boldsymbol{q}_\nu\cdot\delta\boldsymbol{u}_{,s}+q_\nu\,\delta\beta_\nu^* \qquad (44\cdot a)$$

$$\boldsymbol{t}\cdot\delta\boldsymbol{\Omega}_t=\frac{1}{2}\delta(\bar{\boldsymbol{n}}\cdot\boldsymbol{\nu}-\bar{\boldsymbol{\nu}}\cdot\boldsymbol{n})=\boldsymbol{q}_t\cdot\delta\boldsymbol{u}_{,s}+q_t\,\delta\beta_\nu^* \qquad (44\cdot b)$$

$$\boldsymbol{n}\cdot\delta\boldsymbol{\Omega}_t=\frac{1}{2}\delta(\bar{\boldsymbol{\nu}}\cdot\boldsymbol{t}-\bar{\boldsymbol{t}}\cdot\boldsymbol{\nu})=\boldsymbol{q}\cdot\delta\boldsymbol{u}_{,s}+q\,\delta\beta_\nu^* \qquad (44\cdot c)$$

where

$$\boldsymbol{q}_\nu=q_{\nu\nu}^*\boldsymbol{\nu}+q_{\nu t}^*\boldsymbol{t}+q_{\nu n}^*\boldsymbol{n} \qquad (45\cdot a)$$

$$\boldsymbol{q}_t=q_{t\nu}^*\boldsymbol{\nu}+q_{tt}^*\boldsymbol{t}+q_{tn}^*\boldsymbol{n} \qquad (45\cdot b)$$

$$\boldsymbol{q}=q_\nu^*\boldsymbol{\nu}+q_t^*\boldsymbol{t}+q_n^*\boldsymbol{n} \qquad (45\cdot c)$$

$$q_t=(\bar{a}_t-c_\nu f_\nu+c_t)/2\bar{a}_t \qquad (45\cdot d)$$

$$q_\nu=-f_\nu/2 \qquad (45\cdot e)$$

$$q=(c-c_\nu g_\nu)/2\bar{a}_t \qquad (45\cdot f)$$

$$q_{\nu\nu}^*=(-c\,c_\nu/\bar{a}_t^3-f_t n^\alpha\nu_\alpha)/2 \qquad (45\cdot g)$$

$$q_{\nu t}^*=(-c\,c_t/\bar{a}_t^3-f_t n^\alpha t_\alpha)/2 \qquad (45\cdot h)$$

$$q_{\nu n}^*=(1/\bar{a}_t-c^2/\bar{a}_t^3-f_t n)/2 \qquad (45\cdot i)$$

$$q_{t\nu}^*=\{-c_\nu f_t n^\alpha\nu_\alpha/\bar{a}_t-\beta_t^*/\bar{a}_t+c_\nu(c_\nu\beta_t^*-c_t\beta_\nu^*)/\bar{a}_t^3\}/2 \qquad (45\cdot j)$$

$$q_{tt}^*=\{-c_\nu f_t n^\alpha t_\alpha/\bar{a}_t+\beta_\nu^*/\bar{a}_t+c_t(c_\nu\beta_t^*-c_t\beta_\nu^*)/\bar{a}_t^3\}/2 \qquad (45\cdot k)$$

$$q_{tn}^*=\{-c_\nu f_t n/\bar{a}_t+c(c_\nu\beta_t^*-c_t\beta_\nu^*)/\bar{a}_t^3\}/2 \qquad (45\cdot l)$$

$$q_\nu^*=[-(2+\beta^*)\bar{a}_t^2-\{c\beta_\nu^*-c_\nu(2+\beta^*)\}c_\nu-\bar{a}_t^2 c_\nu g_t n^\alpha\nu_\alpha]/2\bar{a}_t^3 \qquad (45\cdot m)$$

$$q_t^*=[-\{c\beta_\nu^*-c_\nu(2+\beta^*)\}c_t-\bar{a}_t^2 c_\nu g_t n^\alpha t_\alpha]/2\bar{a}_t^3 \qquad (45\cdot n)$$

$$q_n^*=[\beta_\nu^*\bar{a}_t^2-\{c\beta_\nu^*-c_\nu(2+\beta^*)\}c-c_\nu g_t n\,\bar{a}_t^2]/2\bar{a}_t^3 \qquad (45\cdot o)$$

and the following releations are introduced:

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}=(\boldsymbol{b}\times\boldsymbol{c})\cdot\boldsymbol{a}=(\boldsymbol{c}\times\boldsymbol{a})\cdot\boldsymbol{b} \qquad (46\cdot a)$$

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d})=(\boldsymbol{a}\cdot\boldsymbol{c})\cdot(\boldsymbol{b}\cdot\boldsymbol{d})-(\boldsymbol{a}\cdot\boldsymbol{d})\cdot(\boldsymbol{b}\cdot\boldsymbol{c}) \quad (46\cdot b)$$

Substituting Eqs. (44) into Eq. (43) and intergrating by parts lead to

$$EVW=\iint_M \boldsymbol{p}\cdot\delta\boldsymbol{u}\,dA+\int_{C_1}(\tilde{\boldsymbol{F}}^*\cdot\delta\boldsymbol{u}+\tilde{M}^*\,\delta\beta_\nu^*)dS+\sum_j\tilde{\boldsymbol{R}}_j^*\cdot\delta\boldsymbol{u}_j \quad (47)$$

where

$$\tilde{\boldsymbol{F}}^*=\tilde{\boldsymbol{F}}-(\tilde{K}_\nu\,\boldsymbol{q}_t)_{,s}+(\tilde{K}_t\,\boldsymbol{q}_\nu)_{,s}-(\tilde{K}\,\boldsymbol{q})_{,s} \quad (48\cdot a)$$

$$\tilde{M}^*=\tilde{K}_\nu\,q_t-\tilde{K}_t\,q_\nu+\tilde{K}\,q \quad (48\cdot b)$$

$$\tilde{\boldsymbol{R}}^*=\tilde{K}_\nu\,\boldsymbol{q}_t-\tilde{K}_t\,\boldsymbol{q}_\nu+\tilde{K}\,\boldsymbol{q} \quad (48\cdot c)$$

$$\tilde{\boldsymbol{R}}_j^*=\tilde{\boldsymbol{R}}^*(S_j+0)-\tilde{\boldsymbol{R}}^*(S_j-0) \quad (48\cdot d)$$

and C_1 is the part of C on which at least one components of $\tilde{\boldsymbol{F}}^*$ or $\tilde{M}^*$ is prescribed, while M_j, $j=1,2,\cdots,J$, are those corner points of C where at least one components of $\tilde{\boldsymbol{R}}_j^*$ is prescribed. Thus the external virtual work is obtained with the help of the total finite rotation vector $\boldsymbol{\Omega}_t$ for the boundary, and expressed in terms of the variations of $\delta\boldsymbol{u}$ and $\delta\beta_\nu^*$.

6. LAGRANGIAN SHELL EQUATIONS

Since the internal and external virtual works have been expressed in terms of the variations $\delta\boldsymbol{u}$ and $\delta\beta_\nu^*$, the Lagrangian equilibrium equations are obtained in the form

$$\boldsymbol{T}^\beta|_\beta+\boldsymbol{p}=0 \quad (49)$$

where

$$\boldsymbol{T}^\beta=T^{\alpha\beta}\,\bar{\boldsymbol{a}}_\alpha+Q^\beta\,\bar{\boldsymbol{n}} \quad (50\cdot a)$$

$$T^{\alpha\beta}=N^{\alpha\beta}-M^{\kappa\beta}\,\bar{b}^\alpha_\kappa \quad (50\cdot b)$$

$$Q^\beta=M^{\alpha\beta}|_\alpha+M^{\alpha\kappa}\,\bar{a}^{\beta\lambda}\,\gamma_{\lambda\alpha\kappa} \quad (50\cdot c)$$

The associated static boundary conditions take the form

$$\boldsymbol{F}^*=\tilde{\boldsymbol{F}}^* \quad \text{and} \quad M^*=\tilde{M} \quad \text{on} \quad C_1 \quad (51\cdot a)$$

$$\boldsymbol{R}_j^*=\tilde{\boldsymbol{R}}_j^* \quad \text{at each} \quad M_j\in C_1 \quad (51\cdot b)$$

where

$$\boldsymbol{F}^*=\boldsymbol{T}^\beta\,\nu_\beta+\left[M^{\alpha\beta}\,\nu_\alpha\left\{t_\beta+\nu_\beta\left(\frac{\bar{b}_t}{\bar{a}_t^2}-h_t\frac{1}{\bar{a}_t}\sqrt{\frac{\bar{a}}{a}}\right)\right\}\bar{\boldsymbol{n}}\right]_{,s} \quad (52\cdot a)$$

$$M^*=M^{\alpha\beta}\,\nu_\alpha\,\nu_\beta\,\frac{1}{\bar{a}_t}\sqrt{\frac{\bar{a}}{a}}\,h_\nu \quad (52\cdot b)$$

and the geometric boundary conditions are given by

$$\boldsymbol{u}=\tilde{\boldsymbol{u}} \quad \text{and} \quad \beta_\nu^*=\tilde{\beta}_\nu^* \quad \text{on} \quad C_2 \quad (53\cdot a)$$

$$\boldsymbol{u}_i=\tilde{\boldsymbol{u}}_i \quad \text{at each} \quad M_i\in C_2 \quad (53\cdot b)$$

where C_2 is the part of C on which at least one component of $\tilde{\boldsymbol{u}}$ or $\tilde{\beta}_\nu^*$ is prescribed while M_i, $i=1,2,\cdots,I\leqq K$, are those corner points of C_2 where at least one component of $\tilde{\boldsymbol{u}}_i$ is prescribed.

To complete the shell theory, some two-dimensional constitutive equations should be given. For an elastic shell there exists a strain energy function Σ per unit area of M. In the case of small strain everywhere and isotropic material behavior it can be consistently approximated[9),11),16)~21)]. For a consistent first-approximation theory of shells the strain energy function is given by the following quadratic expression :

$$\Sigma=\frac{h}{2}H^{\alpha\beta\lambda\mu}\left(\gamma_{\alpha\beta}\,\gamma_{\lambda\mu}+\frac{h^2}{12}\,\kappa_{\alpha\beta}\,\kappa_{\lambda\mu}\right) \quad (54)$$

where

$$H^{\alpha\beta\lambda\mu}=\frac{E}{2(1+\nu)}\left(a^{\alpha\lambda}\,a^{\beta\mu}+a^{\alpha\mu}\,a^{\beta\lambda}+\frac{2\nu}{1-\nu}\,a^{\alpha\beta}\,a^{\lambda\mu}\right) \quad (55)$$

and E denotes Young's modulus, ν Poisson's ratio and h the thickness of shells. With the strain energy function (54), we obtain the constitutive equations as follows :

$$N^{\alpha\beta}=\frac{\partial\Sigma}{\partial\gamma_{\alpha\beta}}=h\,H^{\alpha\beta\lambda\mu}\,\gamma_{\lambda\mu} \qquad (56\cdot a)$$

$$M^{\alpha\beta}=\frac{\partial\Sigma}{\partial\chi_{\alpha\beta}}=\frac{h^3}{12}\,H^{\alpha\beta\lambda\mu}\,\chi_{\lambda\mu} \qquad (56\cdot b)$$

For a consistent second-approximation theory of shells[19)~21)], it has been shown that the two-dimensional equations including the effect of bending terms cannot be constructed correctly within the framework of the Kirchhoff-Love hypotheses. However, comparison of eigenvalues of the cylindrical shell equations[20)] indicates that the difference between the numerical results of the consistent second-approximation theory including the effects of the normal stress and strain, and those of the second-approximation theory derived under the Kirchhoff-Love hypotheses is negligiblly small. Thus the second-approximation theory under the Kirchhoff-Love hypotheses may be applicable to an analysis of shell structures without significant error. In the case of a second-approximation theory under the Kirchhoff-Love hypotheses, a strain energy function Σ^* for an isotropic elastic shell takes the form[21)]

$$\Sigma^*=\frac{h}{2}\,H^{\alpha\beta\lambda\mu}\left(\gamma_{\alpha\beta}\,\gamma_{\lambda\mu}+\frac{h^2}{12}\,\chi_{\alpha\beta}\,\chi_{\lambda\mu}-\frac{h^2}{6}\,\gamma_{\alpha\beta}\,b^{\kappa}_{\kappa}\,\chi_{\lambda\mu}\right)+\frac{h^3}{12}\,H_1^{\alpha\beta\lambda\mu}\,\gamma_{\alpha\beta}\,\chi_{\lambda\mu} \qquad (57)$$

where

$$H_1^{\alpha\beta\lambda\mu}=\frac{E}{2(1+\nu)}\{2(a^{\alpha\lambda}\,b^{\beta\mu}+b^{\alpha\lambda}\,a^{\beta\mu})+2(a^{\alpha\mu}\,b^{\beta\lambda}+b^{\alpha\mu}\,a^{\beta\lambda})+\frac{4\nu}{1-\nu}(a^{\alpha\beta}\,b^{\lambda\mu}+b^{\alpha\beta}\,a^{\lambda\mu})\} \qquad (58)$$

The constitutive equations for a second-approximation theory are obtained by

$$N^{\alpha\beta}=\frac{\partial\Sigma^*}{\partial\gamma_{\alpha\beta}} \qquad (59\cdot a)$$

$$M^{\alpha\beta}=\frac{\partial\Sigma^*}{\partial\chi_{\alpha\beta}} \qquad (59\cdot b)$$

It is noted that force and couple stress resultants in a second-approximation theory remain to be symmetric.

7. DISCUSSION AND CONCLUSIONS

It is interesting to compare the present results with the existing ones obtained from the Lagrangian formulation. When Eq. (49) is expressed by components in the bases $\boldsymbol{a}_\alpha$ and $\boldsymbol{n}$, the following equilibrium equations in the component form are obtained :

$$(l^{\alpha}_{\kappa}\,T^{\kappa\beta}+n^{\alpha}\,Q^{\beta})|_{\beta}-b^{\alpha}_{\beta}\,(\phi_{\kappa}\,T^{\kappa\beta}+nQ^{\beta})+p^{\alpha}=0 \qquad (60\cdot a)$$

$$b_{\alpha\beta}\,(l^{\alpha}_{\kappa}\,T^{\kappa\beta}+n^{\alpha}\,Q^{\beta})+(\phi_{\kappa}\,T^{\kappa\beta}+nQ^{\beta})|_{\beta}+p=0 \qquad (60\cdot b)$$

These equations coincide exactly with those obtained by Pietraszkiewicz[1),11)]. In 8), the modified tensor of change of curvature, in which the products of second fundamental tensors and surface strain tensors are contained as additional terms, has been used. However, it is easily verified that when the additional terms are neglected the resulting equilibrium equations agree with the present ones. The equilibrium equations in 9), however, differ significantly from the present equations since the modified tensor of change of curvature has been introduced in 9). In view of the fact that the present equilibrium equations are derived without using any approximation, they will be exact in the range of two-dimensional theory under the Kirchhoff-Love hypotheses.

Consider the boundary conditions which consist of four equations both for static and geometric boundary conditions. The static boundary conditions of forces in the component form with respect to the reference triad of the vectors $\boldsymbol{\nu}$, $\boldsymbol{t}$ and $\boldsymbol{n}$, and the static boundary condition of moment are represented by

$$(T^{\alpha\beta}\,l^{\kappa}_{\alpha}\,\nu_{\beta}+Q^{\beta}\,n^{\kappa}\,\nu_{\beta}+T_{n,s}\,n^{\kappa}-T_n\,\bar{b}^{\kappa}_{\mu}\,t^{\mu})\nu_{\kappa}$$
$$=\tilde{F}_{\nu}+I_{1,s}-\chi_t\,I_2+\tau_t\,I_3 \qquad (61\cdot a)$$

$$(T^{\alpha\beta}\,l^{\kappa}_{\alpha}\,\nu_{\beta}+Q^{\beta}\,n^{\kappa}\,\nu_{\beta}+T_{n,s}\,n^{\kappa}-T_n\,\bar{b}^{\kappa}_{\mu}\,t^{\mu})t_{\kappa}$$
$$=\tilde{F}_{t}+I_{2,s}+\chi_t\,I_1-\sigma_t\,I_3 \qquad (61\cdot b)$$

$$T^{\alpha\beta}\,\phi_{\alpha}\,\nu_{\beta}+Q^{\beta}\,n\nu_{\beta}+T_{n,s}\,n=\tilde{F}+I_{3,s}-\tau_t\,I_1+\sigma_t\,I_2 \qquad (61\cdot c)$$

$$M^{\alpha\beta}\,\nu_{\alpha}\,\nu_{\beta}\,\frac{1}{\bar{a}_t}\sqrt{\frac{\bar{a}}{a}}\,h_{\nu}=\tilde{K}_{\nu}\,q_t-\tilde{K}_t\,q_{\nu}+\tilde{K}\,q \qquad (61\cdot d)$$

where

$$T_n=M^{\alpha\beta}\nu_\alpha\left\{t_\beta+\nu_\beta\left(\frac{\bar{b}_t}{\bar{a}_t^2}-h_t\frac{1}{\bar{a}_t}\sqrt{\frac{\bar{a}}{a}}\right)\right\} \qquad (62\cdot a)$$

$$I_1=-\tilde{K}_\nu q^*_{t\nu}+\tilde{K}_t q^*_{\nu\nu}-\tilde{K}\, q^*_\nu \qquad (62\cdot b)$$

$$I_2=-\tilde{K}_\nu q^*_{tt}+\tilde{K}_t q^*_{\nu t}-\tilde{K}\, q^*_t \qquad (62\cdot c)$$

$$I_3=-\tilde{K}_\nu q^*_{tn}+\tilde{K}_t q^*_{\nu n}-\tilde{K}\, q^*_n \qquad (62\cdot d)$$

As the geometric boundary conditions, the components of the displacement vector $\boldsymbol{u}$ and the parameter β^*_ν are prescribed on the shell boundary C_2. The parameter β^*_ν is the nonlinear one with respect to displacements and their derivatives represented by

$$\beta^*_\nu=\sqrt{\frac{a}{\bar{a}}}\,\varepsilon^{\alpha\beta}\varepsilon_{\lambda\mu}\nu^\mu\phi_\alpha l^\lambda_\beta \qquad (63)$$

In 8), the appropriate geometric boundary conditions have not been obtained since the term $\delta\boldsymbol{u}_{,\nu}$ has not been expressed in terms of the variations of the displacement vector and the fourth parameter which represents the rotation at the boundary. And also the effects pertaining to the term $\delta\boldsymbol{u}_{,\nu}$ have not been included in the static boundary conditions. Pietraszkiewicz and Szwabowicz[9] have derived the geometric boundary conditions which agree with the present equations (53). As a fourth boundary condition the parameter β^*_ν describing the finite rotation of the shell boundary is prescribed. The static boundary conditions in 9), however, differ significantly from the present ones. This discrepancy may be caused by the difference of the tensor of change of curvature. The modified tensor of change of curvature, which is a third-degree polynomial in displacements and their derivatives, has been introduced in 9), while the present paper employs the exact tensor of change of curvature, which is a nonrational function of displacements and their derivatives.

The equilibrium equations and the associated boundary conditions obtained in this paper are derived for unrestricted midsurface strains, displacements or rotations irrespective of the constitutive equations. As discussed above the validity of the present results is confirmed by comparing them with the existing results. There exists very little literature which has derived both the equilibrium equations and the boundary conditions consistent with a Lagrangian nonlinear theory of shells undergoing unrestricted rotations without using small strain assumptions. The present paper has succeeded first to derive the consistent Lagrangian equilibrium equations and the appropriate static and geometric boundary conditions of shells without restriction to small strains.

ACKNOWLEDGMENTS

The authors would like to thank Dr. T. Okumura, Emeritus Professor of the University of Tokyo and Professor of Tokyo Denki University, for his valuable suggestions and Ph.D. K. Matsui, Professor of Tokyo Denki University, for reading and criticizing the manuscript. Thanks are also due to Professor W. Pietraszkiewicz for providing his papers.

REFERENCES

1) Pietraszkiewicz, W. : Finite Rotations in the Nonlinear Theory of Thin Shells, In : Thin Shell Theory, New Trends and Application, Ed. by W. Olszak, CISM Courses and Lectures, No. 240, Springer-Verlag, pp. 153～208, Wien-New York, 1980.

2) Pietraszkiewicz, W. : Finite Rotations in Shells, In : Theory of Shells, Ed. by W. T. Koiter and G. K. Mikhailov, Proc. 3rd IUTAM Symp., Tbilisi 1978 ;North-Holland Publ. Co., pp. 445～471, Amsterdam, 1980.

3) Novozhilov, V. V. and Shamina, V. A. : Kinematic Boundary Conditions in Nonlinear Elasticity Theory Problems, Mechanics of Solids, pp. 63～74, 5 (1975).

4) Tōsaka, N. and Tsuboi, Y. : The Nonlinear Theory of Thin Elastic Shells, Trans. of A. I. J., No. 235, pp. 27～38, Sept. 1975 (in Japanese).

5) Shrivastava, J. P. and Glockner, P. G. : Lagrangian Formulation of Static of Shells, Proc. of ASCE, Vol. 96, No. EM 5, pp. 547～563, 1970.

6) Simmonds, J. G. and Danielson, D. A. : Nonlinear Shell Theory with Finite Rotation and Stress-Function Vectors, J. Appl. Mech., Trans. ASME, E 39, 4, pp. 1085～1090, 1972.

7) Simmonds, J. G. and Danielson, D. A. : Nonlinear Shell Theory with a Finite Rotation Vector, Proc. Kon. Ned. Ak. Wet., Ser. B, 73, pp. 460~478, 1970.
8) Pietraszkiewicz, W : Lagrangian Nonlinear Theory of Shells, Arch. Mech., 26, 2, pp. 221~228, 1974.
9) Pietraszkiewicz, W. and Szwabowicz, M. L. : Entirely Lagrangian Nonlinear Theory of Thin Shells, Arch. Mech., 33, 2, pp. 273~288, 1981.
10) Sakurai, T., Hasegawa, A. and Nishino, F. : A Finite Displacement Formulation of Elastic Shells, Proc. of JSCE, No. 330, pp. 151~159, Feb. 1983.
11) Pietraszkiewicz, W. : Introduction to the Nonlinear Theory of Shells, Ruhr-Universität Bochum, Mitteilungen aus dem Institut für Mechanik 10, 1977.
12) Sanders, J. L. : Nonlinear Theories for Thin Shells, Quart. Appl. Math., Vol. 21, pp. 21~36, 1963.
13) Naghdi, P. M. and Nordgren, R. P. : On the Nonlinear Theory of Elastic Shells under the Kirchhoff Hypothesis, Quart. Appl. Math., Vol. 21, pp. 49~59, 1963.
14) Koiter, W. T. : On the Nonlinear Theory of Thin Elastic Shells, Proc. Kon. Ned. Ak. Wet., Ser. B, 69, pp. 1~54, 1966.
15) Budiansky, B. : Notes on Nonlinear Shell Theory, J. Appl. Mech., Trans. ASME, E 35, 2, pp. 393~401, 1968.
16) Stumpf, H. : The Derivation of Dual Extremum and Complementary Stationary Principles in Geometrical Nonlinear Shell Theory, Ing.-Arch., pp. 221~237, 48, 1979.
17) Schmidt, R. and Pietraszkiewicz, W. : Variational Principles in the Geometrically Nonlinear Theory of Shells Uudergoing Moderate Rotations, Ing.-Arch., pp. 187~201, 50, 1981.
18) Koiter, W. T. : A Consistent First Approximation in the General Theory of Thin Elastic Shells, In : The Theory of Thin Elastic Shells, Proc. IUTAM Symp. Delft 1959, pp. 12~33. Amsterdam : North-Holland Publ. Co., 1960.
19) Abé, H. : On Bending Terms in the Linear Thin Shell Equations in terms of the Displacement Components, Int. J. Engng. Sci., Vol. 13, pp. 1055~1065, 1975.
20) Abé, H. : A Second Approximation Theory and Simplified Equations of Cylindrical Shells, Trans. of JSME, A 45, No. 399, pp. 1356~1363, Nov. 1979 (in Japanese).
21) Pietraszkiewicz, W. : Consistent Second Approximation to the Elastic Strain Energy of a Shell, ZAMM 59, T 206-T 208, 1979.

(Recieved November 29, 1983)