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ACCURACY AND CONVERGENCE OF THE SEPARATION OF RIGID BODY DISPLACEMENTS FOR PLANE CURVED FRAMES

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The method with the separation of rigid body displacements is a powerful tool for the finite displacement analysis of structures particularly with curved members, with variable cross sections, and/or with inelastic behaviors. Nevertheless, its theoretical equivalence to the solutions of the direct Lagrangian equations has not been examined so far.

This paper presents the theoretical convergence and accuracy of the method, where an original curved element of variable cress section is considered as well as an approximate straight element of constant cross section.

1. INTRODUCTION

The method with the sepearation of rigid body displacements[1)~3)], called here in acronym the SRBD method, is a practical numerical procedure for the finite displacement analysis of structures, since it seems a very difficult task directly to solve rather complicated and highly nonlinear governing equations in terms of the coordinate system fixed in a space called here the direct Lagrangian method[4),5)]. Nevertheless, its theoretical equivalence and convergence to the solutions of the direct Lagrangian equations have only been examined for plane straight members[6)], although numerical techniques are widely available for a variety of structures only to obtain plausible solutions without any mathematical proof. Rather than for simple straight uniform members, the SRBD method appears to be a more powerful tool for the analysis of structures with curved members, with variable cross sections, and/or with the inelastic behaviors, because the member components have to be divided into small elements, whatever displacements of concern are large or not. The separation of rigid body displacements can be applied for each of those divided elements, effectively incorporating the nonlinearity which results from finite displacements.

Even if a finite element procedure is applied to solve the nonlinear direct Lagrangian differential equations[3),7)] for curved members, without any limitation on the magnitude of displacements, it is prohibitively difficult to derive the accurate stiffness equation[5)], compared with that for straight members, because of the difficulty to find a simple adequate interpolation function, as well as the high nonlinearity involved in the strain and displacement relation[5),8),9)].

Therefore the SRBD method is often used effectively for the finite displacement analysis of curved members particularly when displacement becomes large. In this method, a curved member possibly with variable cross section

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is usually approximated by the assemblage of straight elements with constant cross sections[10),11)].

For the above reasons, the SRBD method appears most appropriate for the analysis of structures composed of curved members with variable cross sections. However its theoretical convergence and accuracy to the solutions of the direct Lagrangian differential equations have not been examined so far, although limited studies are found for the approximations only by straight elements with small displacements[12),13)].

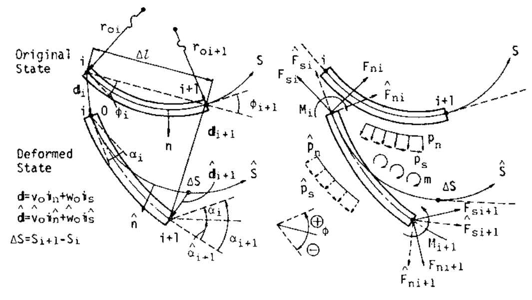
This paper presents the theoretical convergence and accuracy of the SRBD method for the finite displacement analysis of plane curved members, and generalizes the results for straight members[6)] reported before. The direct Lagrangian differential equations for arbitrarily curved members are presented, including the equation considered most general and strict within the framework of the Bernoulli-Euler hypothesis and no change of cross sectional shapes. Next, for the use of the SRBD method, the simplified local differential equations after the separation of rigid body displacements are given, which consist of the equations not only for the original curved element but also for the approximated straight element. It should be noted that the straight element has exclusively been used by others for numerical computations of the SRBD method, but the curved element is also introduced for the first time in this paper. The curved element seems not only to be more rigorous and general for the analysis of a curved member, but also to be adequate to examine the theoretical basis for convergence and accuracy in contrast to that for the straight element. Futhermore, from the view of numerical procedure, the curved element have a possibility to exhibit more effective computations, so far as an adequate interpolation function is found.

The discrete forms of the governing differential equations in terms of forces and displacements at both ends of a finite element are derived both for the direct Lagrangian and the SRBD methods by making use of the Taylor expansions with respect to the element length. After transforming the discrete form obtained for the local equations into the form in terms of the same coordinates as the direct Lagrangian equations, the convergence and accuracy for the SRBD method are examined by comparing the coincidence of the coefficients of the derived power series[6)].

2. COORDINATES AND VARIABLES

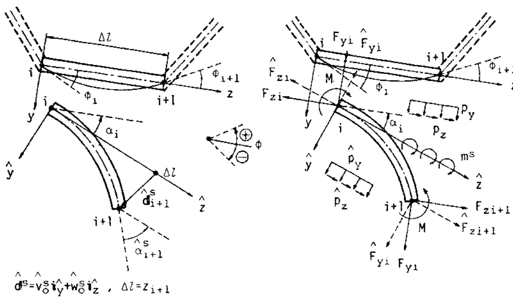
Consider a curved member element subject to distributed external forces in addition to nodal forces as shown in Fig. 1. Two reference coordinate systems are introduced, one is an orthogonal curvilinear coordinate system (n, s) with the coordinate s along the centroidal axis[*1] of a curved member at the initial configuration, and the other exclusively used for the SRBD method is a local orthogonal curvilinear coordinate system $(\hat{n}, \hat{s})$ defined for a curved element with the nodes of i and $i+1$ which moves with rigid body displacements of node i.

As an approximation for a curved element, similar two coordinate systems are introduced for a straight element of Fig. 2, as used in Reference[6)] which are the Cartesian coordinate system (y, z) with the coordinate z along the element centroidal axis at the initial configuration, and the local Cartesian coordinate system $(\hat{y}, \hat{z})$ defined after



(a) Displacement Components (b) Force Components

Fig. 1 Coordinate Systems of a Curved Beam Element : (a) and (b).



(a) Displacement Components (b) Force Components

Fig. 2 Coordinate Systems of a Straight Beam Element : (a) and (b).

*1 For a curved member, the centroidal axis taken as the origin of the coordinate n is defined as $\int_A r_0/r \cdot n dA = 0$, in which r_0 and r are the radii of curvature of the centroid and an arbitrary point of section respectively.

the separation of rigid body displacements with the coordinate $\hat{z}$ tangential to the deformed axis at node i.

The displacement components of the centroidal axis of a member are given as shown in Fig. 1 with the positive directions indicated, where the total displacements and rotation defined for the (n, s) coordinates are denoted by (v_0, w_0, α), and those with the separation of rigid body displacements defined for the $(\hat{n}, \hat{s})$ coordinates by $(\hat{v}_0, \hat{w}_0, \hat{\alpha})$.

Similarly shown in Fig. 2 are the displacements and rotation with the separation of rigid body displacements for a straight element defined for the $(\hat{y}, \hat{z})$ coordinates designated by $(\hat{v}_0^s, \hat{w}_0^s, \hat{\alpha}^s)$.

Geometrical observations for displacements and rotations among the (n, s), $(\hat{n}, \hat{s})$ and $(\hat{y}, \hat{z})$ components lead to the following relations as

$$\boldsymbol{D}_{i+1}=[T(\phi_{i+1}-\phi_i)]\,\boldsymbol{D}_i-[T(\phi_{i+1})][[I]-[T(\alpha_i)]]\,\boldsymbol{L}+[T(\alpha_i)]\,\hat{\boldsymbol{D}}_{i+1} \quad (1)$$

for a curved element, and

$$\boldsymbol{D}_{i+1}=[T(\phi_{i+1}-\phi_i)]\,\boldsymbol{D}_i-[T(\phi_{i+1})][[I]-[T(\alpha_i)]]\,\boldsymbol{L}+[T(\phi_{i+1})][T(\alpha_i)]\,\hat{\boldsymbol{D}}^s_{i+1} \quad (2)$$

for a straight element, in which

$$^t\boldsymbol{D}\equiv(v_0,\ w_0,\ \alpha),\quad {}^t\hat{\boldsymbol{D}}\equiv(\hat{v}_0,\ \hat{w}_0,\ \hat{\alpha}),\quad {}^t\hat{\boldsymbol{D}}^s\equiv(\hat{v}_0^s,\ \hat{w}_0^s,\ \hat{\alpha}^s) \quad (3\cdot \text{a}\sim\text{c})$$

$$^t\boldsymbol{L}\equiv(0,\ \Delta l,\ 0),\quad \Delta l\equiv\hat{z}_{i+1}=z_{i+1} \quad (4\cdot \text{a, b})$$

$$[T(\cdot)]\equiv\begin{bmatrix}[t(\cdot)] & 0\\ 0 & 1\end{bmatrix} \quad (5),\qquad [t(\cdot)]\equiv\begin{bmatrix}\cos(\cdot), \sin(\cdot)\\ -\sin(\cdot), \cos(\cdot)\end{bmatrix} \quad (6)$$

Subscript i indicates values at node i, ϕ_i and ϕ_{i+1} as given in Fig. 1 with the positive directions indicated are angles between the line connecting the nodes i with $i+1$ and the tangents of the centroidal axis at node i and $i+1$ respectively at the initial configuration, and $[I]$ is unit matrix.

The (n, s) and $(\hat{n}, \hat{s})$ components of nodal forces with applied distributed loads as shown in Fig. 1, are denoted by (F_n, F_s, M) and $(\hat{F}_n, \hat{F}_s, M)$ with (p_n, p_s, m) and $(\hat{p}_n, \hat{p}_s, m)$ respectively. Likewise for a straight element, the (y, z) and $(\hat{y}, \hat{z})$ components are denoted by (F_y, F_z, M) and $(\hat{F}_y, \hat{F}_z, M)$ with (p_y, p_z, m^s) and $(\hat{p}_y, \hat{p}_z, m^s)$ respectively as given in Fig. 2. All the distributed forces are assumed to be conservative and acting on the centroidal axis of the member.

The vector transformation among the components of the nodal and distributed forces and moments defined for the (n, s), $(\hat{n}, \hat{s})$ and $(\hat{y}, \hat{z})$ coordinates are given by

$$\boldsymbol{F}_a=[T(\alpha_i)]\,\hat{\boldsymbol{F}}_a \quad (7),\qquad \boldsymbol{F}_a=[T(\phi_a)][T(\alpha_i)]\,\hat{\boldsymbol{F}}^s_a \quad (8)$$

$(a=i,\ i+1)$

$$\boldsymbol{p}(s_i+\hat{s})=[T(\alpha_i)]\,\hat{\boldsymbol{p}}(\hat{s}) \quad (9)$$

in which

$$^t\boldsymbol{F}\equiv(F_n,\ F_s,\ M),\quad {}^t\hat{\boldsymbol{F}}\equiv(\hat{F}_n,\ \hat{F}_s,\ M),\quad {}^t\hat{\boldsymbol{F}}^s\equiv(\hat{F}_y,\ \hat{F}_z,\ M) \quad (10\cdot \text{a}\sim\text{c})$$

$$^t\boldsymbol{p}\equiv(p_n,\ p_s,\ m),\quad {}^t\hat{\boldsymbol{p}}\equiv(\hat{p}_n,\ \hat{p}_s,\ m) \quad (11\cdot \text{a, b})$$

excepting the transformation between the distributed force components defined for the (n, s) and $(\hat{y}, \hat{z})$ coordinates. Since the distributed forces are defined per unit length of the member axis, the following assumption is introduced as

$$\boldsymbol{p}^s(z)=\boldsymbol{p}(s_i+\lambda_i z) \quad (12)$$

in which

$$^t\boldsymbol{p}^s\equiv(p_y,\ p_z,\ m^s),\quad \lambda_i=(s_{i+1}-s_i)/\Delta l \quad (13\cdot \text{a, b})$$

for the components of the distributed force $\boldsymbol{p}^s$ defined for the straight element, considering that the curved coordinates (n, s) is well approximated by the local straight coordinates (y, z) for each small element.

Similar to Eq. (9), the vector transformation combined with Eq. (12) for the distributed forces between the (n, s) and $(\hat{y}, \hat{z})$ Coordinates leads to

$$\boldsymbol{p}(s_i+\lambda_i z)=[T(\alpha_i)]\,\hat{\boldsymbol{p}}^s(z) \quad (14)$$

in which

$$^t\hat{\boldsymbol{p}}^s\equiv(\hat{p}_y,\ \hat{p}_z,\ m^s) \quad (15)$$

3. NONLINEAR DIFFERENTIAL EQUATIONS IN DIRECT LAGRANGIAN EXPRESSIONS

With the assumptions of the Euler-Bernoulli hypothesis and no change of cross sectional shapes, the nonlinear governing equations for a curved member are derived using the theorem of virtual work in a similar way as reported in Reference 14). The direct Lagrangian equations we expressed, using the force and displacement components of the (n, s) reference coordinates defined for the initial configurations, with the basic unknowns of v_0 and w_0 in terms of the independent variable of s. The results of the derivations for the governing equations are summarized in Table 1, in which it should be noted that the rotation α is given as

$$\sin\alpha=(v_0'-w_0/r_0)/\sqrt{g_0},\quad \cos\alpha=(1+w_0'+v_0/r_0)/\sqrt{g_0} \quad\cdots\cdots (16\cdot \mathrm{a,b})$$

using the basic unknowns v_0 and w_0.

As presented in Table 1., the force component vs. stress resultant relations as well as the stress resultant vs. displacement relations are classified into three levels of nonlinearity. First is called the theory of a) finite strains with finite displacements, which has no limitations on the magnitude of displacements, only based on the constitutive equation for axial stress and strain relation as

$$\sigma_s=Ee_s \quad\cdots\cdots (17)$$

in which

$$e_s=\frac{r_0}{r}(\sqrt{g_0}-1-n\alpha') \quad\cdots\cdots (18)$$

Table 1 Direct Lagrangian Expressions.

Equilibrium Equations	Boundary Conditions: Mechanical	Boundary Conditions: Geometrical
$F_n'-\frac{1}{r_0}F_s+p_n=0$ $F_s'+\frac{1}{r_0}F_n+p_s=0$	$F_n=\bar{F}_n$ $F_s=\bar{F}_s$ $M=\bar{M}$	$v_0=\bar{v}_0$ $w_0=\bar{w}_0$ $\alpha=\bar{\alpha}$
Theories	**F_n, F_s**	**Stress Resultants vs. Displacements**
a) Finite Strains with Finite Displacements	$F_n=N\sin\alpha+\frac{(M'-m)}{\sqrt{g_0}}\cos\alpha$ $F_s=N\cos\alpha-\frac{(M'-m)}{\sqrt{g_0}}\sin\alpha$	$N=E\tilde{A}(\sqrt{g_0}-1)$ $M=-E\tilde{I}\alpha'$
b) Small Strains with Finite Displacements	$F_n=N\sin\alpha+(M'-m)\cos\alpha$ $F_s=N\cos\alpha-(M'-m)\sin\alpha$	$N=E\tilde{A}(\sqrt{g_0}-1)$ $M=-E\tilde{I}\alpha'$
c) Inextensional Finite Displacements	$F_n=N\sin\alpha+(M'-m)\cos\alpha$ $F_s=N\cos\alpha-(M'-m)\sin\alpha$	$\sqrt{g_0}=1$ $M=-E\tilde{I}\alpha'$

Remarks : The following notations are used throughout Tables.
E=Young's Modulus, $\tilde{A}=\int_A \frac{r_0}{r}dA$, $\tilde{I}=\int_A \frac{r_0}{r}n^2dA$ (A=Cross Sectional Area,
r_0=Radius of Curvature of Centroidal Axis, $r=r_0+n$), N=Axial Stress Resultant,
$g_0=(v_0'-w_0/r_0)^2+(1+v_0/r_0+w_0')^2$, $'=d/ds$

Second and third are obtained by introducing restrictions on displacements, respectively called the theory of b) small strains with finite displacements, which introduces the restrictions of $g\approx g_0\approx 1$, and the theory of c) inextensional finite displacements which further restricts the displacements by the condition of $g_0=1$.

4. SIMPLIFIED LOCAL DIFFERENTIAL EQUATIONS FOR THE SRBD METHOD

(1) A Curved Element

The local differential equations applied for a finite element after eliminating the rigid body displacements are expressed by the force and displacement components of the local $(\hat{n}, \hat{s})$ reference coordinates with the basic unknowns $\hat{v}_0$, $\hat{w}_0$. It is noted, however, that the coordinate of s has been chosen as the independent variable[6]. The results of the derivations are summarized in Table 2. These equations are understood as further simplifications of b) for direct Lagrangian expressions, and called the theories of d) beam-column, and e) small displacements.

Table 2 Expressions for Curved Element with Separation of Rigid Body Displacements.

Equilibrium Equations	Boundary Conditions	
	Mechanical	Geometrical
$\hat{F}_n' - \frac{1}{r_0}\hat{F}_s + \hat{p}_n = 0$ $\hat{F}_s' + \frac{1}{r_0}\hat{F}_n + \hat{p}_s = 0$	$\hat{F}_n = \bar{\hat{F}}_n$ $\hat{F}_s = \bar{\hat{F}}_s$ $M = \bar{M}$	$\hat{v}_0 = \bar{\hat{v}}_0$ $\hat{w}_0 = \bar{\hat{w}}_0$ $\hat{v}_0' - \frac{\hat{w}_0}{r_0} = \bar{\alpha}$
Theories	$\hat{F}_n$, $\hat{F}_s$	Stress Resultant vs. Displacements
d) Beam-Column	$\hat{F}_n = N\left(\hat{v}_0' - \frac{\hat{w}_0}{r_0}\right) + M' - m$ $\hat{F}_s = N$	$N = E\tilde{A}\left\{\hat{w}_0' + \frac{\hat{v}_0}{r_0} + \frac{1}{2}\left(\hat{v}_0' - \frac{\hat{w}_0}{r_0}\right)^2\right\}$ $M = -E\tilde{I}\left(\hat{v}_0' - \frac{\hat{w}_0}{r_0}\right)'$
e) Small Displacements	$\hat{F}_n = M' - m$ $\hat{F}_s = N$	$N = E\tilde{A}\left(\hat{w}_0' + \frac{\hat{v}_0}{r_0}\right)$ $M = -E\tilde{I}\left(\hat{v}_0' - \frac{\hat{w}_0}{r_0}\right)'$

The equations of d) are obtained from the conditions not only of small strains as explained for the equations of b) but also of relatively small displacements. The latter condition is expressed mathematically by

$$|\hat{v}_0' - \hat{w}_0/r_0|^2 \ll 1, \quad |\hat{w}_0' + \hat{v}_0/r_0| \ll 1 \quad \cdots\cdots (19 \cdot a, b)$$

which simplify the rigorous axial strain-displacement relation of Eq. (18) as

$$e_s = \frac{r_0}{r}\{\hat{w}_0' + \hat{v}_0/r_0 + (\hat{v}_0' - \hat{w}_0/r_0)^2/2 - n(\hat{v}_0' - \hat{w}_0/r_0)'\} \quad \cdots\cdots (20)$$

It is worthwhile to mention that the equation of d) of the beam-column is obtained by the theorem of virtual work which directly utilizes Eq. (20) for the virtual strain, but not from the theory of b) with the condition of Eqs. (19).

The linear equations e) of small displacements are derived simply by eliminating the nonlinear terms in the equatinos d) of the beam-column.

The governing equations for inextensional deformations which correspond to d) and e) in Table 2 can be obtained through replacing the axial stress resultant-displacement relations by

$$\left.\begin{array}{ll} \hat{w}_0' + \hat{v}_0/r_0 + (\hat{v}_0' - \hat{w}_0/r_0)^2/2 = 0 & \text{for beam-column, and} \\ \hat{w}_0' + \hat{v}_0/r_0 = 0 & \text{for small displacements} \end{array}\right\} \quad \cdots\cdots (21 \cdot a, b)$$

(2) A Straight Element

The local differential equations for a straight element have already given in Reference 6) for the theories both of f) beam-column and g) small displacements. However, it should be noted that, with the use of the cross sectional properties for straight members in this element, the solutions do not converge to those for the curved members. But this difficulty can easily be resolved by replacing the cross sectional properties for straight members denoted by A and I by those for curved members as denoted in Table 1 by $\tilde{A}$ and $\tilde{I}$. Assuming uniform cross sectional properties for each straight element in this study as in the usual analysis, the cross sectional properties at node i for a curved member denoted by $\tilde{A}_i$, and $\tilde{I}_i$ are used throughout the element with nodes of i and $i+1$.

5. DISCRETE EQUATIONS FOR THE NODAL FORCES AND DISPLACEMENTS

(1) Derivations of Discrete Equations

In order to derive the discrete equations for the general solutions of the basic differential equations, the nodal vector consisting of the (n, s) components of the forces and displacements is introduced as

$${}^t\{Q_j\} = \{F_n, \ F_s, \ M, \ v_0, \ w_0, \ \alpha\} \quad (j = 1 \sim 6) \quad \cdots\cdots (22)$$

The discrete equations of interest can be obtained by expanding the vector components of Eq. (22) into the power series with respect to the element length $\Delta s = s_{i+1} - s_i$. As has been given in Reference 6), these discrete equations have the form of transferring the vector components of Eq. (22) from node i to $i+1$ as

$$\left.\begin{array}{l} Q_j|_{i+1} = Q_j|_i + \sum_{n=1}^{\infty} \overset{(n)}{Q}_j|_i \Delta s^n / n\,! \\ \overset{(n)}{Q}_j|_i = f_j^n(\{Q_k|_i\}) \end{array}\right\} \quad \cdots\cdots (23 \cdot a, b)$$

in which $\overset{(n)}{Q_j}|_i$ is the n-th order derivative of Q_j at node i, and $f_j^n(\{Q_k|_i\})$ is the function of $Q_k|_i$ with $k=1\sim 6$.

The discrete equations for nodel forces (F_n, F_s) are derived simply from the consideration of force equilibrium, irrelevant of moment components. Since the equilibrium equations for a curved element in Table 2 is exactly transformed into the Lagrangian equations in Table 1 as evident from Eqs. (7) and (9), it is clear that the coincidence of the discrete equations is always assured for the nodal forces (F_n, F_s). However, the equilibrium equations transformed from those of the straight element for the SRBD method differ from the corresponding equations in Table 1, because the assumption of Eq. (12) is used for distributed forces. Although it is easliy proved that the equilibrium equations transformed even from a straight element coincide with those of the direct Lagrangian method in the case of uniformly distributed forces along the axis of element, the convergence and accuracy must be examined for a general case of loading by the comparisons of the discrete equations.

On the other hand, since unknown displacements are necessarily involved in the remaining vector components (M, v_0, w_0, α) of Eq. (22), the basic differential equations need to be used to derive the discrete equations. Thus, the convergence and accuracy for the SRBD method become a major concern for those components.

(2) The Discrete Equations for the Direct Lagrangian Method

In order to solve the basic differential equations by the Taylor expansions, the direct Lagrangian equations in Table 1 have to be transformed into the first order differential equations with unknowns of Eq. (22) as

$$dQ_j/ds=f_j(\{Q_k\}) \quad \cdots\cdots (24)$$

The results obtained for the respective terms of Eq. (22) are summarized in Table 3. The coefficients for $\Delta s^n/n!$ of the power series of Eq. (23·a) which is the n-th derivatives $\overset{(n)}{Q_j}|_i$ of the unknown components Q_j at node i can

Table 3 First Order Differential Equations in Direct Lagrangian Expressions.

Differential Equations		a) Finite Strains with Finite Displacements	b) Small Strains with Finite Displacements	c) Inextensional Finite Displacements
$f_j(\{Q_k\})$ $(j=1\sim 6)$	f_1	F_s/r_o-p_n	F_s/r_o-p_n	F_s/r_o-p_n
	f_2	$-F_n/r_o-p_s$	$-F_n/r_o-p_s$	$-F_n/r_o-p_s$
	f_3	$\sqrt{g_0}(F_n\cos\alpha-F_s\sin\alpha)+m$	$(F_n\cos\alpha-F_s\sin\alpha)+m$	$(F_n\cos\alpha-F_s\sin\alpha)+m$
	f_4	$\sqrt{g_0}\sin\alpha+w_o/r_o$	$\sqrt{g_0}\sin\alpha+w_o/r_o$	$\sin\alpha+w_o/r_o$
	f_5	$\sqrt{g_0}\cos\alpha-(1+v_o/r_o)$	$\sqrt{g_0}\cos\alpha-(1+v_o/r_o)$	$\cos\alpha-(1+v_o/r_o)$
	f_6	$-M/E\tilde{I}$	$-M/E\tilde{I}$	$-M/E\tilde{I}$

Remarks: $dQ_j/ds=f_j(\{Q_k\})$ and $\sqrt{g_0}=(F_s\cos\alpha+F_n\sin\alpha)/E\tilde{A}+1$

Table 4 First Order Differential Equations for Curved Elements with Separation of Rigid Body Displacements.

Differential Equations		d) Beam-Column	e) Small Displacements
$\hat{f}_j(\{\hat{Q}_k^*\})$ $(j=1\sim 6)$	$\hat{f}_1$	$\hat{F}_s/r_o-\hat{p}_n$	$\hat{F}_s/r_o-\hat{p}_n$
	$\hat{f}_2$	$-\hat{F}_n/r_o-\hat{p}_s$	$-\hat{F}_n/r_o-\hat{p}_s$
	$\hat{f}_3$	$\hat{F}_n+m-N\hat{\alpha}^*$	$\hat{F}_n+m$
	$\hat{f}_4$	$\hat{\alpha}^*+\hat{w}_o/r_o$	$\hat{\alpha}^*+\hat{w}_o/r_o$
	$\hat{f}_5$	$\hat{F}_s/E\tilde{A}-\hat{v}_o/r_o-\hat{\alpha}^{*2}/2$	$\hat{F}_s/E\tilde{A}-\hat{v}_o/r_o$
	$\hat{f}_6$	$-M/E\tilde{I}$	$-M/E\tilde{I}$

Remarks: $d\hat{Q}_j^*/ds=\hat{f}_j(\{\hat{Q}_k^*\})$, $\hat{\alpha}^*=\hat{v}_o'-\hat{w}_o/r_o$

be expressed as Eq. (23·b) only by $\{Q_k|_i\}$ by repeated use of differentiations and substitutions of Eq. (24). The derivatives of the vector components of Eq. (22) are summarized for the general case and the particular case of the inextensional deformation of axis, respectively in Table 5, 6 in comparison with those derived from the SRBD method. Discussions for the results are presented later.

(3) The Discrete Equations for the SRBD Method

The discrete equations for the SRBD method are obtained in a similar way as those for the direct Lagrangian method. The first order differential equations with unknowns of Eq. (22) expressed as the local components are

Table 5 Derivatives of Physical Quantities : (a) ~ (d)

Theories			$F'_n,\ F'_s$	$F''_n,\ F''_s$	$F'''_n,\ F'''_s$
Direct Lagrangians		a)			$F'''_n=-(f_n+p'_s r_o^2)/r_o^3$ $F'''_s=-(f_s+p'_n r_o^2)/r_o^3$
		b)	$F'_n=\frac{F_s}{r_o}-p_n,$	$F''_n=-\frac{1}{r_o^2}(F_n+F_s r'_o+p_s r+p'_n r_o^2)$	
SRBD	Curved Element	d)			$f_n=F_s(1+r_o r''_o-2r'^2_o)-3F_n r'_o-p_n r_o+p''_n r_o^3-2p_s r_o r'_o$
		e)	$F'_s=-\frac{F_n}{r_o}-p_s$	$F''_s=-\frac{1}{r^2}(F_s-F_n r'_o-p_n r_o+p'_s r_o^2)$	$f_s=F_n(1+r_o r''_o-2r'^2_o)+3F_s r'_o+p_s r_o-p''_s r_o^3-2p_n r_o r'_o$
	Straight Element	f)			$F'''_n=-(f_n+3p'_s r_o^2/2)/r_o^3$
		g)			$F'''_s=-(f_s+3p'_n r_o^2/2)/r_o^3$

Remarks : The following notations are used throughout Tables 5 and 6.

$E\tilde{A}=a,\ E\tilde{I}=b,\ \kappa=M/b,\ \rho=\kappa+1/r_o,\ \eta=\kappa-1/r_o,\ \lambda=\sqrt{g_o},\ s=\sin\alpha,\ c=\cos\alpha,\ d\hat{p}_n=p'_n c-p'_s s,\ d\hat{p}_s=p'_s c+p'_n s$

Theories			M'	M''	M'''
Direct Lagrangians		a)	$m+\lambda\hat{F}_n$	$m'-\lambda\hat{p}_n+\lambda\hat{F}_s\rho-(\hat{p}_s+\hat{F}_n\rho-\lambda a')\hat{F}_n/a$	$m''-\lambda d\hat{p}_n-\lambda\hat{p}_s\kappa-\lambda\rho(\rho\hat{F}_n+\hat{p}_s)+(\lambda\hat{F}_n+m-\kappa b')\hat{F}_s/b-\hat{F}_s r'_o/r_o^2-D_M$
		b)		$m'-\hat{p}_n+\hat{F}_s\rho$	C_M ; $C_M=m''-d\hat{p}_n-\hat{p}_s\kappa-\rho(\rho\hat{F}_n+\hat{p}_s)+(\hat{F}_n+m-\kappa b')\hat{F}_s/b-\hat{F}_s r'_o/r_o^2$
SRBD	Curved Element	d)			$C_M+\kappa^2\hat{F}_n$
		e)	$m+\hat{F}_n$	$m'-\hat{p}_n+\hat{F}_s/r_o$	$m''-d\hat{p}_n-(\hat{F}_n/r_o+\hat{p}_s)/r_o-\hat{F}_s r'_o/r_o^2$
	Straight Element	f)		$m'-\hat{p}_n+\hat{F}_s\rho$	$m''-d\hat{p}_n-\hat{p}_s\kappa-(\rho+\kappa/2)\hat{F}_n/r_o+(\hat{F}_n+m-\kappa b')\hat{F}_s/b-\hat{F}_s r'_o/r_o^2-m/4r_o^2$
		g)		$m'-\hat{p}_n+\hat{F}_s/r_o$	$m''-d\hat{p}_n-\hat{F}_n/r_o^2-\hat{F}_s r'_o/r_o^2-m/4r_o^2$

$D_M=2(\rho\hat{F}_s-\hat{p}_n)(\rho\hat{F}_n+\hat{p}_s+\hat{F}_s a'/a)/a-\hat{F}_n[2a'(\rho\hat{F}_n+\hat{p}_s)/a-\rho(\rho\hat{F}_s-\hat{p}_n)+\hat{F}_n\{r'_o/r_o^2-(\lambda\hat{F}_n-m)/b+\kappa b'/b-\hat{F}_s(aa''-2a'^2)/a^2-d\hat{p}_s+\kappa\hat{p}_n]/a$

Theories			$v'_o,\ w'_o$	$v''_o\,/\,w''_o$
Direct Lagrangians		a)		C_v
		b)		C_w
SRBD	Curved Element	d)	$v'_o=\lambda s+w_o/r_o,\ w'_o=\lambda c-1-v_o/r_o$	$C_v+(\lambda-1)\kappa c+\hat{F}_n\kappa s/a$
		e)		$C_w-(\lambda-1)\kappa s+\hat{F}_s\kappa c/a$
	Straight Element	f)		$C_v+(\lambda-1)\kappa c+\{\hat{F}_n\kappa+\hat{F}_s a'/a\}s/a$
		g)		$C_w-(\lambda-1)\kappa s+\{\hat{F}_n\kappa+\hat{F}_s a'/a\}c/a$

$C_v=-\lambda\eta c-(\hat{p}_s+\hat{F}_n\rho+\hat{F}_s a'/a)s/a-(1+v_o/r_o+r'_o w_o/r_o)/r_o$

$C_w=\lambda\eta s-(\hat{p}_s+\hat{F}_n\rho+\hat{F}_s a'/a)c/a-(w_o-r'_o v_o)/r_o^2$

Theories			α'	α''	α'''
Direct Lagrangians		a)		$-(m+\lambda\hat{F}_n+\kappa b')/b$	$\{-m'+\lambda(\hat{p}_n-\hat{F}_s\rho)+2(\hat{F}_n\lambda+m)b'/b+\kappa(bb''-2b'^2)/b+\hat{F}_n(\hat{F}_n\rho+\hat{p}_s-\hat{F}_s a'/a)/a\}/b$
		b)		$-(m+\hat{F}_n+\kappa b')/b$	C_α ; $C_\alpha=\{-m'+\hat{p}_n-\hat{F}_s\rho+2(\hat{F}_n+m)b'/b+\kappa(bb''-2b'^2)/b\}/b$
SRBD	Curved Element	d)	κ		
		e)			$C_\alpha+\hat{F}_s\kappa/b$
	Straight Element	f)		$-(m+\hat{F}_n)/b$	C'_α ; $C'_\alpha=\{-m'+\hat{p}_n-\hat{F}_s(\rho+1/r_o)\}/b+\kappa/4r_o^2$
		g)			$C'_\alpha+\hat{F}_s\kappa/b$

Table 6 Derivatives of $v_0,\ w_0$ with Inextensional Deformations.

Theories			$v'_o,\ w'_o$	$v''_o,\ w''_o$	$v'''_o\,/\,w'''_o$
Direct Lagrangians		c)			C_v / C_w
SRBD	Curved Element	d)			
		e)	$v'_o=s+w_o/r_o,\ \ w'_o=c-1-v_o/r_o$	$v''_o=-\eta c-(1+v_o/r_o)/r_o-r'_o w_o/r_o^2$ $w''_o=\eta s-(w_o-v_o r'_o)/r_o^2$	$C_v+\kappa^2 s$ / $C_w+\kappa^2 c$
	Straight Element	f)			$C_v+\kappa s/2r_o-\kappa b'c/b$ / $C_w+\kappa c/2r_o+\kappa b's/b$
		g)			$C_v+\kappa s/2r_o-\kappa b'c/b+\kappa^2 s$ / $C_w+\kappa c/2r_o+\kappa b's/b+\kappa^2 c$

$C_v=-(F_n+m-\kappa b')c/b-\kappa^2 s+\eta s/r_o-2r'_o c/r_o^2+3v_o r'_o/r_o^3-w_o(1+r_o r''_o-2r'^2_o)/r_o^3+2r'_o/r_o^2$

$C_w=(F_n+m-\kappa b')s/b-\kappa^2 c+\eta c/r_o+2r'_o s/r_o^2+3w_o r'_o/r_o^3+v_o(1+r_o r''_o-2r'^2_o)/r_o^3+1/r_o^2$

derived from the local differential equations given in Table 2 for a curved element. The results are summarized in Table 4, in which the unknowns are expressed for the $(\hat{n}, \hat{s})$ components as

$${}^{t}\{\hat{Q}_j^*\}=\{\hat{F}_n, \hat{F}_s, M, \hat{v}_0, \hat{w}_0, \hat{\alpha}^*\} \quad (25)$$

with

$$\hat{\alpha}^*=\hat{v}_0'-\hat{w}_0/r_0 \quad (26)$$

The corresponding first order differential equations for a straight element are already given in Reference 6) with the unknowns in this paper expressed for the $(\hat{y}, \hat{z})$ components as

$${}^{t}\{\hat{Q}_j^{s*}\}=\{\hat{F}_y, \hat{F}_z, M, \hat{v}_0^s, \hat{w}_0^s, \hat{\alpha}^{s*}\} \quad (27)$$

with

$$\hat{\alpha}^{s*}=d\hat{v}_0^s/dz \quad (28)$$

As described for the direct Lagrangian method, the local first order differential equations are expanded into the Taylor series with respect to Δs for a curved element, and Δl for a straight element respectively with the boundary conditions at node i as

$$\left.\begin{array}{ll}(\hat{v}_0, \hat{w}_0, \hat{\alpha}^*)=0 & \text{for Eq. (25)}\\ (\hat{v}_0^s, \hat{w}_0^s, \hat{\alpha}^{s*})=0 & \text{for Eq. (27)}\end{array}\right\} \quad (29 \cdot \text{a, b})$$

resulting finally in the local discrete equations for the SRBD method.

In order to compare the results with those for the direct Lagrangian method, the local components derived for the SRBD method need to be transformed into the (n, s) components, with the use of Eqs. (1), (7) and (9) for a curved element, while Eqs. (2), (8) and (14) for a straight element. It should be remarked for the above procedure that the angles $\hat{\alpha}$ and $\hat{\alpha}^s$ after eliminating the rigid body rotation for an element are considered so small that the linear approximations as

$$\hat{\alpha}=\hat{\alpha}^*, \quad \hat{\alpha}^s=\hat{\alpha}^{s*} \quad (30 \cdot \text{a, b})$$

do not lose the accuracy as proved for a straight element in Reference 6).

The discrete equations for the SRBD method expressed in the (n, s) components still include functions of Δs which are $\sin(\phi_{i+1}-\phi_i)$, $\cos(\phi_{i+1}-\phi_i)$, $\sin\phi_{i+1}$, $\cos\phi_{i+1}$, and Δl as evident from Eqs. (1) and (2). Those have further to be expanded into the power series of Δs to make it possible to compare with the direct Lagrangian discrete equations which have already been expanded completely with respect to Δs as in Tables 5 and 6. With somewhat complexed calculations, the Taylor expansion at node i leads to the expressions of the power series as

$$\left.\begin{array}{l}\sin(\phi_{i+1}-\phi_i)=1/r_0\cdot\Delta s-r_0'/r_0^2\cdot\Delta s^2/2!-(r_0r_0''-2r_0'^2+1)/r_0^3\cdot\Delta s^3/3!\\ \qquad +(6r_0'-6r_0'^3+6r_0r_0'r_0''-r_0^2r_0''')/r_0^4\cdot\Delta s^4/4!+O(\Delta s^5)\\ \cos(\phi_{i+1}-\phi_i)=1-1/r_0^2\cdot\Delta s^2/2!+3r_0'^2/r_0^3\cdot\Delta s^3/3!+(1-11r_0'^2+4r_0r_0'')/r_0^4\cdot\Delta s^4/4!+O(\Delta s^5)\\ \sin(\phi_{i+1})=1/2r_0\cdot\Delta s-2r_0'/3r_0^2\cdot\Delta s^2/2!-(6r_0r_0''-2r_0'^2+1)/8r_0^3\cdot\Delta s^3/3!\\ \qquad +(29r_0'-144r_0'^2+144r_0r_0'r_0''-2r_0^2r_0''')/30r_0^4\cdot\Delta s^4/4!+O(\Delta s^5)\\ \cos(\phi_{i+1})=1-1/4r_0^2\cdot\Delta s^2/2!+r_0'/r_0^3\cdot\Delta s^3/3!+(3+72r_0r_0''-208r_0'^2)/48r_0^4\cdot\Delta s^4/4!+O(\Delta s^5)\\ \Delta l=\Delta s-1/4r_0^2\cdot\Delta s^3/3!+r_0'/r_0^3\cdot\Delta s^4/4!+O(\Delta s^5)\end{array}\right\} \quad (31 \cdot \text{a}\sim\text{e})$$

in which r_0 indicates the radius of centroidal axis curvature at node i, and the maximum order in the expressions is determined from the necessity of comparison.

As for a straight element, the derivatives of the distrubuted forces (p_y, p_z, m^s) with respect to z are used in the local discrete equations. Thus, these derivatives have also to be transformed into those of the (n, s) components with respect to s. By making use of Eq. (12) combined with the differentiation for compound functions, the following relations are obtained as

$$\left.\begin{array}{l}dp_y/dz=p_n'\Delta s/\Delta l, \quad dp_z/dz=p_s'\Delta s/\Delta l, \quad dm^s/dz=m'\Delta s/\Delta l\\ d^2p_y/dz^2=p_n''(\Delta s/\Delta l)^2, \quad d^2p_z/dz^2=p_s''(\Delta s/\Delta l)^2, \quad d^2m^s/dz^2=m''(\Delta s/\Delta l)^2\end{array}\right\} \quad (32 \cdot \text{a}\sim\text{f})$$

in which

$$\left.\begin{array}{l}\Delta s/\Delta l=1+1/12r_0^2\cdot\Delta s^2/2!+O(\Delta s^3)\\ (\Delta s/\Delta l)^2=1+1/6r_0^2\cdot\Delta s^2/2!+O(\Delta s^3)\end{array}\right\} \quad (33 \cdot \text{a, b})$$

Use and rearrangement of Eq. (31) for a curved element with additional use of Eqs. (32) and (33) for a straight element finally lead to the discrete equations for the SRBD method in the form of power series with respect to Δs which correspond to those for the direct Lagrangian method. The n-th order coefficients for $\Delta s^n/n!$ of the power series obtained correspond to the n-th order derivatives for the unknowns of Eq. (22) at node i in the direct Lagrangian expressions. Thus, those coefficients are tabulated in the same columns as the derivatives for the direct Lagrangian method in Table 5 and 6, classified respectively into the extensional and hnextensional deformation of member axis, in which all the values are those at node i, although subscript i is omitted here for simplicity. The maximum order of the derivatives in the Tables is determined such that the coincidence among the orders of the derivatives can completely be examined for each method. Since the derivatives for the unknowns of F_n, F_s, M and α do not depend on whether the extensional or inextensional deformations of axis is considered, the expressions for those derivatives are only given in Table 5.

6. DISCUSSIONS

(1) Convergence of the SRBD Methed

When convergence is produced for the results by the SRBD method, the solutions must satisfy the simultaneous first order differential equations for physical quantities as Eq. (24) which can be obtained by reducing the element length of the discrete equations of Eq. (23) infinitesimally close to zero[6]. The forms of differential equations can be determined from the first order coefficients of the Taylor expansion of Eq. (23) with respect to Δs.

Tables 5 and 6 indicate that all the first order coefficients for the SRBD method perfectly coincide with those derived from the direct Lagrangian equations b) and c) classified in Table 1 respectively for the extensional and inextensional deformations of axis whichever the curved or straight element is applied, and also whichever the beam-column or small displacement theory is used for the SRBD method. Hence, it is concluded that the converged solutions for the SRBD method become identical to the analytical solutions of the direct Lagrangian differential equations for b) small strains with finite displacements. As for straight elements, it should be noted that, unless the cross sectional properties are replaced by those for curved members, solutions do not converge to those of the direct Lagrangian differential equations, only producing the approximate solutions for the conditions of $r_0/r \approx 1$.

In order to obtain the more precise converged solutions identical to those of the direct Lagrangian equation for a) finite strains with finite displacements, the local equilibrium equations for the SRBD method have to reflect the extension of axis, because it cannot be separated by the rigid body displacement introduced in this paper. Introducing this extensional deformation even for the most simplified local equations brings nonlinear terms. Thus, $\hat{F}_n$ in equations e) of Table 2 needs be replaced by

$$\hat{F}_n=(M'-m)/(1+\hat{w}_0') \cdots\cdots (34)$$

as has been discussed for a straight member[6].

(2) Accuracy of the SRBD method

It is important from the view of numerical computations to examine the accuracy of the SRBD method where the magnitude of element length can not be infinitesimally small, but is finite. This can be examined by comparing the coefficients of higher orders in the Taylor expansions of Eq. (23) between the SRBD method and the direct Lagrangian method. From the results of Table 5 and 6, the coincidence of the maximum order of derivatives is summarized in Table 7.

For a general case of the extensional deformation of axis, the most accurate local differential equation for the SRBD method is found that of d) beam-column for a curved element, where all the physical quantities but displacements v_0 and w_0 are proved at least the second order approximation. When a straight element is used, the approximation for nodal forces F_n and F_s turns worse abruptly compared with a curved element. Under the condition of uniformly distributed forces, however, it is remarked that the maximum order of coincidence for a straight element becomes infinite, that is, the same as for a curved element. It is further noted that the order of coincidence of rotation α for a straight element has increased by one from the result of Table 7, as indicated in parenthesis, when the actual $E\tilde{I}$ of a concerned member is constant along the axis. On the whole, for the extensional deformation

Table 7 Coincidence of the Order of Derivatives for Small Strains with Finite Displacements.

Theories			F_n	F_s	M	v_0	w_0	α
SRBD	Curved Element	d)	∞	∞	2	1 [3]	1 [3]	3
		c)	∞	∞	1	1 [2]	1 [2]	2
	Straight Element	f)	2	2	2	1 [2]	1 [2]	1 (2)
		g)	2	2	1	1 [2]	1 [2]	1 (2)

1) ∞: All the derivatives coincide
2) [·] and (·) indicate only those which are different from the general case entitled in the caption, where
[·]: For inextensional deformation of member axis
(·): For constant $\tilde{EI}$ along the member axis

of axis the minimum order of coincidence for all the physical quantities is only one, whichever local equations presented in Table 2 is used. It is concluded from a mathematical standpoint that the SRBD method can only be said a method of the first order approximation.

As for the case of the inextensional deformation of axis, the coincidence of the derivatives for F_n, F_s, M and α is exactly the same as those for the extensional deformation of axis stated above, and hence the coincidence is discussed only for displacement v_0 and w_0 as indicated in the bracket of Table 7. It shows that the order of approximation is improved on the whole so that the order of coincidence increases up to three for a curved element with the theory of d) beam-column, and up to two for the other elements. Thus, the SRBD method with the theory of beam-column for a curved element is found at least the method of the second order approximation. It should be remarked for a straight element that the SRBD method with the theory of f) beam-column has been found the method of the second order approximation under the conditions of constant $\tilde{EI}$ along the member axis in the case of inextensional deformation of axis.

Last but not least, it is noted that the accuracy of the SRBD method for a curved member has been proved exactly coincident with that for a straight member[6] when a curved element is used. Hence, the results obtained here for a curved member with a curved element is found an extension from the results for a straight member.

7. CONCLUDING REMARKS

The convergence and accuracy of the SRBD method for the finite displacement analysis of a curved member with variable cross section are examined theoretically, in which a curved element of variable section is considered as well as a straight element of constant $\tilde{EA}$ and $\tilde{EI}$.

The converged solutions of the SRBD method for infinitesimally small length of element have been found identical to the analytical solutions of the direct Lagrangian differential equations for the theory of small strains with finite displacements or inextensional finite displacements in the case of the extensional or the inextensional deformation of axis respectively.

Regarding the accuracy in the case of the element length being finite, it can only be said for members with the extensional deformation of axis that the SRBD method is of the first order approximation whichever a curved or a straight element with local differential equations either of beam-column or small displacements is used. When the elongation of the member axis is negligibly small, however, the order of approximation increases so that the SRBD method with the theory of beam-column for a curved element assures at least the second order approximation, and that for a straight element also assures the second order approximation under the condition of the constant $\tilde{EI}$ along the member axis.

REFERENCES

1) Saafan, S. A. : Nonlinear Behavior of Structural Plane Frames. Journal of Structural Division, ASCE, Vol. 89, No. ST 4, pp 557～579, Aug., 1963.
2) Powell, G. H. : Theory of Nonlinear Elastic Structures, Journal of Structural Division, ASCE, Vol. 95, No. ST 12, pp. 2687～2701, Dec., 1969.
3) Ai, M. and F. Nishino : Mechanics in Geometrically Nonlinear Problem of Discrete System and Application to Plane Frame-

Works, Proc. of JSCE, No. 304, pp. 17～32, Dec., 1980 (in Japanese).

4) Epstein, M. and D. W. Murray : Large Deformation In Plane Analysis of Elastic Beams, Computers and Structures, Vol. 6, pp. 1～9, 1976.

5) Calhoun, P. R. and D. A. DaDeppo : Nonlinear Finite Element Analysis of Clamped Arches, Jounal of Strudtural Division, ASCE, Vol. 109, No. ST 3, pp. 599～621, Mar., 1983.

6) Goto, Y., A. Hasegawa and F. Nishino : Accuracy of Finite Displacement Analysis of Plane Frames, Proc. of JSCE, No. 331, pp. 33～44, Mar., 1983 (in Japanese).

7) Reissner, E. : On One-Dimensional Finite-Strain Beam Theory, ZAMP, Vol. 23, pp. 795～804, 1972.

8) Sabir, A. B., and A. C. Lock : Large Deflection, Geometrically Non-linear Finite Element Analysis of Circular Arches, International Journal for Numerical Methods in Engineering., Vol. 15, pp. 37～47, 1973.

9) Dawe, D. T. : A Finite-Deflection Analysis of Shallw Arches by the Discrete Element Method, International Journal for Numerical Methods in Engineering, Vol. 3, pp. 529～552, 1971

10) Kuranish, S. and T. Yabuki : Ultimate In-plane Strength of 2-hinged Arches subject to Lateral Loads, Proc. of JSCE, No. 272, pp. 1～12, Apr., 1978 (in Japanese).

11) Maeda, Y. and M. Hayashi : Geometrical Nonlinear Behavior of Rib Arches, Proc. of JSCE, No. 299, pp. 35～47, July, 1980 (in Japanese).

12) Kikuchi, F. : On the Validity of the Finite Element Amalysis of Circular Arches Represented by an Assemblage of Beam Element, Computer Methods in Applied Mechanics and Engineering, Vol. 5, pp. 253～276, 1975.

13) Yamamoto, Y. and H. Ohtsubo : A Qualitative Accuracy Consideration of Arch Element, International Journal for Numerical Methods in Engineering, Vol. 18, pp. 1179～1195, 1982.

14) Nishino, F., Y. Kurakata and Y. Goto : Finite Displacement Beam Theory, Proc. of JSCE, No. 237, pp. 11～26, May, 1975 (in Japanese).

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