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RECTANGULAR THICK PLATES ON LINEAR VISCOELASTIC FOUNDATIONS*

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1. INTRODUCTION

Most of the previous works on the quasistatic analysis of elastic plates resting on linear viscoelastic foundations have been confined within the scope of thin plate theory[1)~7)]. Only a few works based on thick plate theory can be found in literature. Pister[8)] investigated the axisymmetric bending of a viscoelastic infinite plate on the basis of Reissner's thick plate theory,[9)] using the Fourier-Bessel integral of zero order. The authors[10)] presented the general solution of elastic circular plates with various boundary conditions on the basis of Mindlin's thick plate theory[11)], using the method of eigenfunction expansions. In the both works, the correspondence principle[12),13)] was utilized as an elastic-viscoelastic analogy.

Among the various thick plate theories available at present, Mindlin theory and Reissner theory seem to be most familiar. But both the theories are not so much different. If in Reissner theory the effect of transverse normal stress is neglected and in Mindlin theory the shear coefficient is taken equal to 5/6, both the theories coincide with each other.

The present paper is concerned with the quasistatic bending of rectangular Mindlin plates resting on linear viscoelastic foundations obeying the Winkler's hypothesis. The plates are assumed to be simply supported on two opposite edges and subjected to arbitrary surface loads. Double series solutions are derived by means of eigenfunction expansions and by utilizing the correspondence principle, and the numerical results are compared with those by a thin plate theory for the viscoelastic foundations of the Kelvin (Voigt), Maxwell, and Standard linear solid types.

2. BASIC EQUATIONS FOR ELASTIC FOUNDATION PROBLEM

Mindlin's original equilibrium equations for a thick plate include both the effects of shear deformation and rotatory inertia. In the present quasistatic analysis the latter effect can be deleted, and then the governing equation of a rectangular thick plate on the Winkler-type elastic foundation becomes

$$\frac{D}{2}\left[(1-\nu)\nabla^2\psi_x+(1+\nu)\frac{\partial\phi}{\partial x}\right]+\kappa Gh\left(\frac{\partial w}{\partial x}-\psi_x\right)=0 \quad \cdots\cdots(1\cdot a)$$

$$\frac{D}{2}\left[(1-\nu)\nabla^2\psi_y+(1+\nu)\frac{\partial\phi}{\partial y}\right]+\kappa Gh\left(\frac{\partial w}{\partial y}-\psi_y\right)=0 \quad \cdots\cdots(1\cdot b)$$

$$\kappa Gh(\nabla^2 w-\phi)=kw-q \quad \cdots\cdots(1\cdot c)$$

with

$$\phi=\frac{\partial\psi_x}{\partial x}+\frac{\partial\psi_y}{\partial y} \quad \cdots\cdots(1\cdot d)$$

Here, $w(x,y)$ is the deflection at the plate middle surface; $\psi_x(x,y)$ and $\psi_y(x,y)$ are the angular rotations of the normal to the middle surface in the x- and y-coordinates directions, respectively; D is the flexural rigidity given by $D=Eh^3/12(1-\nu^2)$; E is Young's modulus; G is the shear modulus; ν is Poisson's ratio; h is the plate thickness; κ is the shear coefficient taken equal to 5/6; k is the elastic modulus of the foundation; $q(x,y)$ is the surface load intensity; and $\nabla^2(=\partial^2/\partial^2 x+\partial^2/\partial y^2)$ is the Laplace operator.

Stress couples and resultants can be related to the displacement components as

$$M_x=-D\left(\frac{\partial\psi_x}{\partial x}+\nu\frac{\partial\psi_y}{\partial y}\right)$$

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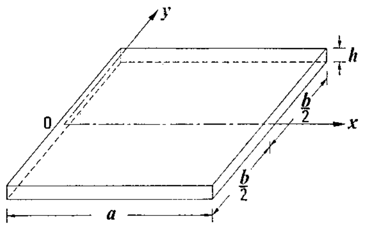

Fig. 1 Geometry and coordinate system of rectangular thick plate.

$$\left.\begin{aligned} M_y &= -D\left(\frac{\partial \psi_y}{\partial y}+\nu\frac{\partial \psi_x}{\partial x}\right) \\ M_{xy} &= -\frac{1-\nu}{2}D\left(\frac{\partial \psi_x}{\partial y}+\frac{\partial \psi_y}{\partial x}\right) \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots(2\cdot\text{a})$$

and

$$\left.\begin{aligned} Q_x &= \kappa Gh\left(\frac{\partial w}{\partial x}-\psi_x\right) \\ Q_y &= \kappa Gh\left(\frac{\partial w}{\partial y}-\psi_y\right) \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots(2\cdot\text{b})$$

Geometry and coordinates of the plate considered are shown in **Fig. 1**. The plate is simply supported on two opposite edges ($x=0, a$) and the other two edges ($y=\pm b/2$) are restrained in any manner such as simply supported, clamped or free. Boundary conditions are written as

(1) Along $x=0, a$:

$M_x=\psi_y=w=0$

for a simply supported edge ……(3)

(2) Along $y=\pm b/2$:

$M_y=\psi_x=w=0$

for a simply supported edge ……(4·a)

$\psi_x=\psi_y=w=0$

for a clamped edge ………………(4·b)

$M_y=M_{xy}=Q_y=0$

for a free edge ……………………(4·c)

3. SERIES SOLUTIONS FOR ELASTIC FOUNDATION PROBLEM

The governing equations (1) can be separated into two partial differential equations on the deflection w and the stress function ψ $(=\partial\psi_x/\partial y-\partial\psi_y/\partial x)$, respectively, one of which is of fourth order, and the other of second order[14]. Therefore, the solutions of Eq. (1) may be derived from direct integration of these equations under consideration of the prescribed boundary conditions. However, the solutions obtained in such a way are not appropriate for the application of correspondence principle between elastic solution and viscoelastic solution because they are composed of hyperbolic and trigonometric functions including the elastic modulus of the foundation in their arguments whose inverse Laplace transforms must be rely on a numerical method. Thus, to avoid the difficulties of numerical inverse Laplace transforms the elastic solutions are given in the form of eigenfunction expansions, which is successfully used in the previous paper on circular plate problem[10].

Then the solution of Eq. (1) are taken in the forms

$$\begin{bmatrix} w(x,y) \\ \psi_x(x,y) \\ \psi_y(x,y) \end{bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Z_{mn} \begin{bmatrix} W_{mn}(x,y) \\ \Psi_{x,mn}(x,y) \\ \Psi_{y,mn}(x,y) \end{bmatrix} \quad \cdots\cdots\cdots\cdots(5)$$

where Z_{mn} is the unknown coefficient and the symbols $W_{mn}(x, y)$, $\psi_{x,mn}(x, y)$, and $\psi_{y,mn}(x, y)$, respectively, represent the m,n-th normal modes (eigenfunctions) in free vibrations of the same Mindlin plate without any foundation neglecting the effect of rotatory inertia.

Such eigenfunctions satisfy the following set of equations (note that subscripts m, n, and arguments x, y are omitted):

$$\frac{D}{2}\left[(1-\nu)\nabla^2\Psi_x+(1+\nu)\frac{\partial \Phi}{\partial x}\right] + \kappa Gh\left(\frac{\partial W}{\partial x}-\Psi_x\right)=0 \quad \cdots\cdots\cdots(6\cdot\text{a})$$

$$\frac{D}{2}\left[(1-\nu)\nabla^2\Psi_y+(1+\nu)\frac{\partial \Phi}{\partial y}\right] + \kappa Gh\left(\frac{\partial W}{\partial y}-\Psi_y\right)=0 \quad \cdots\cdots\cdots(6\cdot\text{b})$$

$$\kappa Gh(\nabla^2 W-\Phi)+CW=0 \quad \cdots\cdots\cdots(6\cdot\text{c})$$

with

$$\Phi=\frac{\partial \Psi_x}{\partial x}+\frac{\partial \Psi_y}{\partial y}, \quad C=D(\lambda_{mn}/a)^4 \quad \cdots\cdots\cdots(6\cdot\text{d})$$

where λ_{mn} is eigenvalue to be determined from boundary conditions. Mindlin[11] showed that Eqs. (6·a) to (6·c) could be uncoupled by introducing three potentials w_1, w_2, and w_3 satisfying the equations

$$\left.\begin{aligned} &[\nabla^2+(\delta_1/a)^2]w_1=0 \\ &[\nabla^2-(\delta_i/a)^2]w_i=0, \quad i=2, 3 \end{aligned}\right\} \quad \cdots\cdots\cdots(7)$$

and

$$\left.\begin{aligned} \Psi_{x,mn}(x,y) &= (1-\sigma_1)\frac{\partial w_1}{\partial x}+(1-\sigma_2)\frac{\partial w_2}{\partial x}+\frac{\partial w_3}{\partial y} \\ \Psi_{y,mn}(x,y) &= (1-\sigma_1)\frac{\partial w_1}{\partial y}+(1-\sigma_2)\frac{\partial w_2}{\partial y}-\frac{\partial w_3}{\partial x} \\ W_{mn}(x,y) &= w_1+w_2 \end{aligned}\right\} \quad \cdots\cdots\cdots(8)$$

The dimensionless symbols introduced in Eqs. (7) and (8) are put as

$$\left.\begin{aligned}&\delta_1^2,\ \delta_2^2=\frac{1}{2}\lambda_{mn}^4[\pm S+(S^2+4\lambda_{mn}^{-4})^{1/2}]\\&\delta_3^2=2/[S(1-\nu)]\\&\sigma_1,\sigma_2=S(\delta_2^2,\ -\delta_1^2)\\&S=D/\kappa Gha^2\end{aligned}\right\}\quad\cdots\cdots(9)$$

Considering the simply supported condition of Eq. (3), solutions of Eq. (7) can be taken in the forms:

(1) Symmetric mode about x-axis,

$$\left.\begin{aligned}w_1&=A_1\cosh(\eta_1 y/b)\sin(m\pi x/a), &&\text{for } \delta_1<m\pi\\&=A_1\cos(\eta_1 y/b)\sin(m\pi x/a), &&\text{for } \delta_1>m\pi\\w_2&=A_2\cosh(\eta_2 y/b)\sin(m\pi x/a),\\w_3&=A_3\sinh(\eta_3 y/b)\cos(m\pi x/a)\end{aligned}\right\}\quad\cdots\cdots(10)$$

(2) Antisymmetric mode about x-axis,

$$\left.\begin{aligned}w_1&=A_1\sinh(\eta_1 y/b)\sin(m\pi x/a), &&\text{for } \delta_1<m\pi\\&=A_1\sin(\eta_1 y/b)\sin(m\pi x/a), &&\text{for } \delta_1>m\pi\\w_2&=A_2\sinh(\eta_2 y/b)\sin(m\pi x/a),\\w_3&=A_3\cosh(\eta_3 y/b)\cos(m\pi x/a)\end{aligned}\right\}\quad\cdots\cdots(11)$$

where

$$\left.\begin{aligned}\eta_1&=\xi(m^2\pi^2-\delta_1^2)^{1/2}, &&\text{for } \delta_1<m\pi\\&=\xi(\delta_1^2-m^2\pi^2)^{1/2}, &&\text{for } \delta_1>m\pi\\\eta_i&=\xi(\delta_i^2+m^2\pi^2)^{1/2}, &&i=2,\ 3\end{aligned}\right\}\quad\cdots\cdots(12)$$

and $\xi(=b/a)$ is the plate aspect ratio.

The prescribed boundary conditions along the edges $y=\pm b/2$ given by Eq. (4) lead to the homogeneous equations on the integration constants $A_i\,(i=1,\ 2,\ 3)$. Then only relative ratios of the integration constants are determined and the characteristic equation determining the eigenvalues is obtained by setting the determinant of coefficient matrix in these equations equal to zero. The explicit forms of the integration constants and the characteristic equations for symmetric modes are given in **Appendix**.

The eigenfunctions possess the orthogonality properties,

$$\left.\begin{aligned}&\int_{-b/2}^{b/2}\int_0^a W_{mn}(x,\ y)W_{ij}(x,\ y)\,dx\,dy\\&\quad=0, \qquad i\neq m \text{ or } j\neq n\\&\quad=N_{mn}, \qquad i=m \text{ and } j=n\end{aligned}\right\}\quad\cdots(13)$$

Hence, the surface load $q(x,y)$ is expanded into a double series of the eigenfunctions

$$q(x,\ y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q_{mn}W_{mn}(x,\ y)\quad\cdots\cdots\cdots\cdots(14)$$

where

$$q_{mn}=\frac{1}{N_{mn}}\int_{-b/2}^{b/2}\int_0^a q(x,\ y)W_{mn}(x,\ y)\,dx\,dy\quad\cdots\cdots(15)$$

Substituting the series (5) and (14) into the governing equations (1) and taking into account Eq. (6), we obtain

$$Z_{mn}=\frac{q_{mn}}{D(\lambda_{mn}/a)^4+k}\quad\cdots\cdots\cdots\cdots(16)$$

and thus the expressions for displacement components and the bending moments are determined as follows:

$$\begin{bmatrix}w(x,\ y)\\\psi_x(x,\ y)\\\psi_y(x,\ y)\end{bmatrix}=\frac{a^4}{D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{q_{mn}}{\lambda_{mn}^4+K^4}\begin{bmatrix}W_{mn}(x,\ y)\\\Psi_{x,mn}(x,\ y)\\\Psi_{y,mn}(x,\ y)\end{bmatrix}\quad\cdots\cdots(17\cdot\text{a})$$

$$\begin{bmatrix}M_x(x,y)\\M_y(x,y)\end{bmatrix}=-a^4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{q_{mn}}{\lambda_{mn}^4+K^4}\begin{bmatrix}M_{x,mn}(x,y)\\M_{y,mn}(x,y)\end{bmatrix}\quad\cdots\cdots(17\cdot\text{b})$$

where the dimensionless parameter K is defined by

$$K=(ka^4/D)^{1/4}\quad\cdots\cdots\cdots\cdots(18)$$

and the symbols $M_{x,mn}(x,\ y)$ and $M_{y,mn}(x,\ y)$ mean

$$\left.\begin{aligned}M_{x,mn}(x,\ y)&=\partial\Psi_{x,mn}/\partial x+\nu\partial\Psi_{y,mn}/\partial y\\M_{y,mn}(x,\ y)&=\partial\Psi_{y,mn}/\partial y+\nu\partial\Psi_{x,mn}/\partial x\end{aligned}\right\}\quad\cdots\cdots(19)$$

The foundation reaction $p(x,y)$ is obtained from the relation $p=kw$ as

$$p(x,\ y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{q_{mn}K^4}{\lambda_{mn}^4+K^4}W_{mn}(x,\ y)\quad\cdots\cdots(20)$$

4. SOLUTION FOR VISCOELASTIC FOUNDATION PROBLEM

Three types of viscoelastic models used here are presented in **Fig. 2**.

The application of the correspondence principle to the elastic solutions given by Eqs. (17) and (20) yields the Laplace transforms of the solutions of viscoelastic foundation problem with regard to time t. For brevity only the Laplace transforms for the deflection, W, and the reactive force, P, are described. Then,

$$\left.\begin{aligned}\bar{W}(x,\ y,\ s)&=a^4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\bar{q}_{mn}(s)}{D\lambda_{mn}^4+\bar{k}(s)a^4}W_{mn}(x,\ y)\\\bar{P}(x,\ y,\ s)&=a^4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\bar{q}_{mn}(s)\bar{k}(s)}{D\lambda_{mn}^4+\bar{k}(s)a^4}W_{mn}(x,\ y)\end{aligned}\right\}\quad\cdots\cdots(21)$$

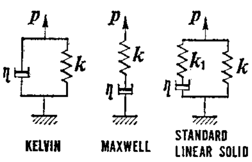

Fig. 2 Viscoelastic models.

Table 1 Viscoelastic operator $\bar{k}(s)$ and time functions $T_1(t)$ and $T_2(t)$.

	Kelvin	Maxwell	Standard linear solid
$\bar{k}(s)$	$k(1+\tau_c s)$	$k\left(\frac{\tau_r s}{1+\tau_r s}\right)$	$k\left(\frac{1+\tau_c s}{1+\tau_r s}\right)$
$T_1(t)$	$T(t)$	$\left(\frac{K}{\lambda_{mn}}\right)^4[T(t)-1]$	$\frac{(K_1/\lambda_{mn})^4}{1+(K_2/\lambda_{mn})^4}T(t)$
$T_2(t)$	$\left(\frac{\lambda_{mn}}{K}\right)^4 T(t)$	$T(t)-1$	$\frac{(K_1/K)^4}{1+(K_2/\lambda_{mn})^4}T(t)$
$T(t)$	$\exp\left\{-\left[1+\left(\frac{\lambda_{mn}}{K}\right)^4\right]\frac{t}{\tau_c}\right\}$	$\exp\left\{-\left[\frac{1}{1+(K/\lambda_{mn})^4}\right]\frac{t}{\tau_r}\right\}$	$\exp\left\{-\left[\frac{(K_2/K)^4+(K_2/\lambda_{mn})^4}{1+(K_2/\lambda_{mn})^4}\right]\frac{t}{\tau_c}\right\}$
Remarks	τ_c denotes the retardation time. $\tau_c=\frac{\eta}{k}$, $K^4=\frac{ka^4}{D}$	τ_r denotes the relaxation time. $\tau_r=\frac{\eta}{k}$, $K^4=\frac{ka^4}{D}$	τ_c denotes the retardation time. τ_r denotes the relaxation time. $\tau_c=\left(1+\frac{k}{k_1}\right)\frac{\eta}{k}$, $\tau_r=\frac{\eta}{k_1}$, $K^4=\frac{ka^4}{D}$ $K_1^4=\frac{k_1a^4}{D}$, $K_2^4=K^4+K_1^4$

where a bar over the symbols means the Laplace transform as follows:

$$\left.\begin{aligned}\bar{F}(s)&=\int_0^\infty F(t)\exp(-st)\,dt\,;\\ F(t)&=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\bar{F}(s)\exp(st)\,ds\end{aligned}\right\}\quad\cdots\cdots\cdots(22)$$

Note that $\bar{k}(s)$ represents a viscoelastic operator[13] generally expressed by a rational function of the transformed variable, s. The explicit forms of $\bar{k}(s)$ for Kelvin, Maxwell, and Standard linear solid-type models are given in **Table 1.**

Now we assume a surface load $q(x,y,t)$ expressed by the unit step function $H(t)$ as

$$q(x,y,t)=q(x,y)H(t)\quad\cdots\cdots\cdots(23)$$

The Laplace transform of it becomes

$$\left.\begin{aligned}\bar{q}(x,y,t)&=q(x,y)/s\\ \bar{q}_{mn}(s)&=q_{mn}/s\end{aligned}\right\}\quad\cdots\cdots\cdots(24)$$

Substituting $\bar{q}_{mn}(s)$ of Eq. (24) and $\bar{k}(s)$ given in **Table 1** into Eq. (21) and taking the inverse Laplace transforms of the resulting equations, we obtain the deflection and the foundation reaction for the viscoelastic foundation problems as follows:

$$\left.\begin{aligned}&W(x,y,t)\\&\quad=w(x,y)-\frac{a^4}{D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{q_{mn}T_1(t)}{\lambda_{mn}^4+K^4}W_{mn}(x,y)\\&P(x,y,t)\\&\quad=p(x,y)+K^4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{q_{mn}T_2(t)}{\lambda_{mn}^4+K^4}W_{mn}(x,y)\end{aligned}\right\}\quad\cdots\cdots\cdots(25\cdot a)$$

where $w(x,y)$ and $p(x,y)$ are the elastic solutions given by Eqs. (17·a) and (20) respectively; $T_1(t)$ and $T_2(t)$ are functions of time t alone, which are listed in **Table 1** for each viscoelastic foundation.

Finally, the bending moments are also obtained through the same procedure as

$$\begin{aligned}\begin{bmatrix}M_x(x,y,t)\\M_y(x,y,t)\end{bmatrix}&=\begin{bmatrix}M_x(x,y)\\M_y(x,y)\end{bmatrix}\\&\quad+a^4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{q_{mn}T_1(t)}{\lambda_{mn}^4+K^4}\begin{bmatrix}M_{x,mn}(x,y)\\M_{y,mn}(x,y)\end{bmatrix}\end{aligned}\quad\cdots\cdots\cdots(25\cdot b)$$

where $M_x(x,y)$ and $M_y(x,y)$ are the elastic solutions given by Eq. (17·b).

5. CONVERGENCE OF SERIES SOLUTIONS

Let the method developed in the preceding sections be applied to a square plate with the edges $x=0$, a simply supported and the other two free, which is subjected to a uniformly distributed load q over the square area in the middle of plate as shown in **Fig. 3**. The eigenvalues λ_{mn} which are the roots of the characteristic equations (A·6) and (A·8) are calculated by the Regula-Falsi method. From consideration of symmetry with respect to the axis $x=a/2$, only odd numbers 1, 3, 5,… are taken in the term 'm' of series solutions.

For numerical calculations, the series expressions (17) can be transformed to more rapidly convergent forms[10),15)] as follows:

$$\left.\begin{aligned}&w(x,y)=w^*(x,y)\\&\quad-\frac{a^4}{D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{q_{mn}K^4}{\lambda_{mn}^4(\lambda_{mn}^4+K^4)}W_{mn}(x,y)\\&\begin{bmatrix}M_x(x,y)\\M_y(x,y)\end{bmatrix}=\begin{bmatrix}M_x^*(x,y)\\M_y^*(x,y)\end{bmatrix}\\&\quad+a^4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{q_{mn}K^4}{\lambda_{mn}^4(\lambda_{mn}^4+K^4)}\begin{bmatrix}M_{x,mn}(x,y)\\M_{y,mn}(x,y)\end{bmatrix}\end{aligned}\right\}\quad\cdots\cdots\cdots(26)$$

In the above, $w^*(x,y)$ is the deflection of the plate with the same geometry and surface load but without elastic foundation, and $M_x^*(x,y)$ and

Table 2 Comparison of convergence of the series expressions (17) and (26) for maximum deflection and maximum bending moments at the center of plate ($\nu=1/6$, $b/a=1$, $c/a=1/5$, $h/a=0.15$, and $K=3$).

m, n	w		M_x		M_y	
	Using Eq. (17)	Using Eq. (26)	Using Eq. (17)	Using Eq. (26)	Using Eq. (17)	Using Eq. (26)
1, 2	0.532 68	0.587 03	0.526 33	0.815 19	0.315 54	0.563 60
3, 3	0.573 90	0,586 63	0.715 61	0.813 20	0.482 35	0.562 35
5, 4	0.584 54	0.586 61	0.793 89	0.813 02	0.550 95	0.562 21
7, 5	0.587 65	0.586 60	0.822 30	0.812 99	0.573 86	0.562 19
9, 6	0.588 06	—	0.827 42	—	0.576 16	0.562 19
11, 7	0.587 59	—	0.823 49	—	0.571 23	0.562 20
19, 11	0.586 38	—	0.810 60	—	0.559 89	—
39, 21	0.586 57	—	0.812 64	—	0.561 86	—
59, 31	0.586 59	—	0.812 88	—	0.562 09	—
79, 41	0.586 60	—	0.812 95	—	0.562 15	—
99, 51	—	—	0.812 97	—	0.562 17	—
Multiplier	$10^{-3}\,qa^4/D$		$10^{-2}\,qa^2$		$10^{-2}\,qa^2$	

Table 3 Deflection and bending moments at the center and at the middle point of free edge of plate for different values of thickness to side ratio h/a ($\nu=1/6$, $b/a=1$, $c/a=1/5$, and $K=3$).

h/a	at $x=a/2$, $y=0$			at $x=a/2$, $y=\pm b/2$	
	w	M_x	M_y	w	M_x
0.0	0.5345[a]	0.8244[a]	0.5603[a]	0.3487[a]	0.3488[a]
0.05	0.5407	0.8236	0.5615	0.3464	0.3552
0.10	0.5581	0.8198	0.5621	0.3443	0.3593
0.15	0.5866	0.8130	0.5622	0.3425	0.3613
0.20	0.6258	0.8037	0.5618	0.3409	0.3610
Multiplier	$10^{-3}\,qa^4/D$	$10^{-2}\,qa^2$		$10^{-3}\,qa^4/D$	$10^{-2}\,qa^2$

[a] The values are obtained from thin plate theory.

$M_y^*(x, y)$ also are the bending moments in the same plate. Following the approach proposed by Marguerre and Woernle,[14] we can obtain them in the form of Levy-type single series with a rapid convergency.

In order to compare the convergence of the series (26) with that of the series (17), the numerical results for maximum deflection and maximum bending moments are shown in **Table 2** for a square plate ($b/a=1$) with the following parameters:

$$\left.\begin{aligned} &\nu=1/6, \quad c/a=1/5, \\ &h/a=0.15 \quad \text{and} \quad K=3 \end{aligned}\right\} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(27)$$

It is found from this table that all the values calculated from the series (26) are very rapidy convergent and a sufficient accuracy within three significant figures can be obtained when both m and n are taken up to three terms, while in the series (17) 39 terms for m and 21 terms for n are required to obtain the same accuracy.

Table 3 shows values of the deflections and the bending moments at the center and at the middle point of free edge for the plates with different values of h/a. It is observed from the table that w and M_y at the center and M_x at the middle point of free edge increase as h/a increases, while w at the middle point of free edge and M_x at the

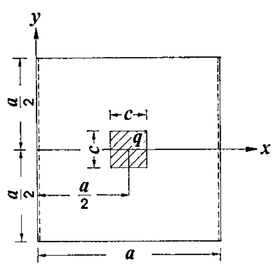

Fig. 3 Partially loaded square plate with the edges ($x=0$ and a) simply supported and the other two free.

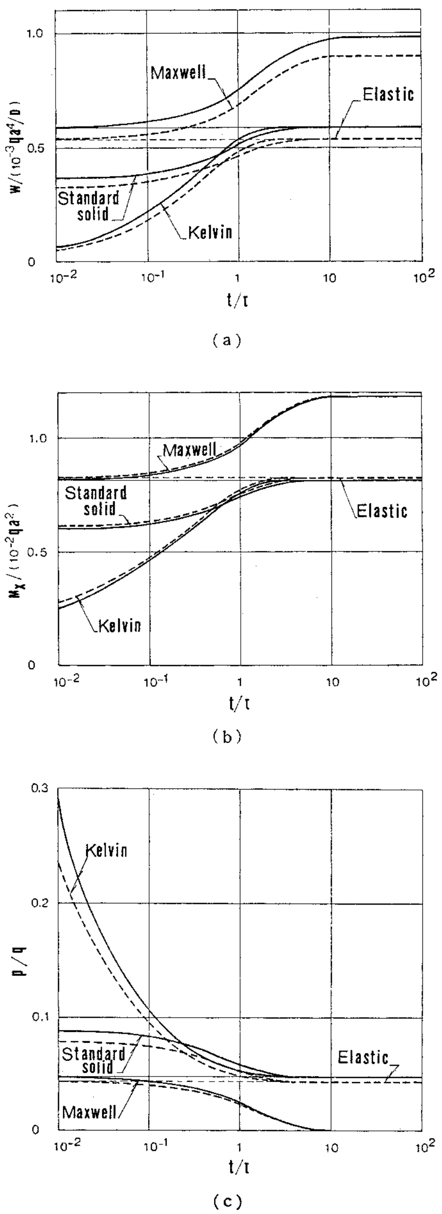

Fig. 4 (a) Center deflection, (b) Center bending moment M_x, and (c) Center foundation reaction versus time (solid line= thick plate; dotted line=thin plate; $b/a=1$; $c/a=1/5$; $h/a=0.15$; $\nu=1/6$; and $K=3$).

center decrease as h/a increases. However, the effect of shear deformation on the bending moments is not appreciable in comparison with the deflection.

6. NUMERICAL EXAMPLES

Numerical calculations for viscoelastic foundation problems are carried out using the parameters given in Eq. (27) and $k_1=2k$. In **Fig. 4**, the time histories of deflection W, bending moment M_x and foundation reaction P at the center of the plate are shown in terms of the non-dimensional time t/τ ($\tau=\eta/k$). The results from a thin plate theory are also plotted by the dotted lines for comparison.

The initial states of the plates on the Kelvin, Maxwell, and Standard linear solid-type foundations are the states of plates on the perfectly rigid foundation and on the elastic foundation with the moduli of the foundation being k and $3k$ ($=k_1+k$), respectively. The deflections, bending moments, and foundation reactions for the Kelvin and Standard linear solid-type foundations asymptotically approach to those for the elastic foundation with the modulus of the foundation being k. A state attained after about 10τ elapses becomes almost constant and therefore it may be regarded as the final state from practical view point. On the other hand the final state of the plate on the Maxwell type foundation is the state of plate without an elastic foundation, because the foundation reaction vanishes (**Fig. 4** (c)). The time for reaching to a state almost regarded as the final state is about 20τ.

7. CONCLUDING REMARKS

The general solutions of the quasistatic bending problems of rectangular Mindlin plates with two opposite edges simply supported on linear viscoelastic foundations are given in the form of a double series of the eigenfunctions derived from the free vibration problems of the plates with the same geometry but without any foundation neglecting the effect of rotatory inertia. Although only the three elementary models of viscoelastic foundation are used in this study, the method of solution developed herein has the applicability to wide range of linear viscoelastic foundation models.

APPENDIX

The integration constants A_i ($i=1, 2, 3$) in Eq. (10) and the characteristic equations for a symmetric mode about x-axis are given in the following cases as:

(1) Simply supported edges along $y=\pm b/2$;

For $\delta_1<m\pi$

there exists no eigenvalue problem.

For $\delta_1 > m\pi$

$$A_1=1, \quad A_2=A_3=0 \qquad \cdots\cdots(A\cdot1)$$

$$\cos(\eta_1/2)=0 \qquad \cdots\cdots(A\cdot2)$$

The roots of Eq. (A·2) are $\eta_1=(2n-1)\pi$, $n=1, 2, 3, \cdots, \infty$.

(2) Clamped edges along $y=\pm b/2$;

For $\delta_1 < m\pi$

there exists no eigenvalue problem.

For $\delta_1 > m\pi$

$$\left.\begin{array}{l} A_1=1/\cos(\eta_1/2), \quad A_2=-1/\cosh(\eta_2/2), \\ A_3=(\sigma_1-\sigma_2)m\pi\xi/[\eta_3\cosh(\eta_3/2)] \end{array}\right\} \qquad \cdots\cdots(A\cdot3)$$

$$(1-\sigma_1)\eta_1\eta_3\tan(\eta_1/2)+(1-\sigma_2)\eta_2\eta_3\tanh(\eta_2/2) -(\sigma_1-\sigma_2)(m\pi\xi)^2\tanh(\eta_3/2)=0 \quad \cdots\cdots(A\cdot4)$$

Equation (A·4) yields infinite number of roots corresponding to $n=1, 2, 3, \cdots, \infty$ for each 'm'.

(3) Free edges along $y=\pm b/2$;

For $\delta_1 < m\pi$

$$\left.\begin{array}{l} A_1=1/\sinh(\eta_1/2) \\ A_2=-\dfrac{\eta_1[\eta_2^2-\nu(m\pi\xi)^2]}{\eta_2[\eta_1^2-\nu(m\pi\xi)^2]\sinh(\eta_2/2)} \\ A_3=\dfrac{(1-\nu)(\sigma_1-\sigma_2)\eta_1 m\pi\xi}{[\eta_1^2-\nu(m\pi\xi)^2]\sinh(\eta_3/2)} \end{array}\right\} \cdots(A\cdot5)$$

$$(1-\sigma_1)\eta_2[\eta_1^2-\nu(m\pi\xi)^2]^2\coth(\eta_1/2) -(1-\sigma_2)\eta_1[\eta_2^2-\nu(m\pi\xi)^2]^2\coth(\eta_2/2) +(1-\nu)^2(\sigma_1-\sigma_2)\eta_1\eta_2\eta_3(m\pi\xi)^2\coth(\eta_3/2) =0 \quad \cdots\cdots(A\cdot6)$$

Equation (A·6) has only one root corresponding to $n=1$ for each 'm'.

For $\delta_1 > m\pi$

$$\left.\begin{array}{l} A_1=1/\sin(\eta_1/2) \\ A_2=-\dfrac{\eta_1[\eta_2^2-\nu(m\pi\xi)^2]}{\eta_2[\eta_1^2+\nu(m\pi\xi)^2]\sinh(\eta_2/2)} \\ A_3=\dfrac{(1-\nu)(\sigma_1-\sigma_2)\eta_1 m\pi\xi}{[\eta_1^2+\nu(m\pi\xi)^2]\sinh(\eta_3/2)} \end{array}\right\} \cdots(A\cdot7)$$

$$(1-\sigma_1)\eta_2[\eta_1^2+\nu(m\pi\xi)^2]^2\cot(\eta_1/2) +(1-\sigma_2)\eta_1[\eta_2^2-\nu(m\pi\xi)^2]^2\coth(\eta_2/2) -(1-\nu)^2(\sigma_1-\sigma_2)\eta_1\eta_2\eta_3(m\pi\xi)^2\coth(\eta_3/2) =0 \quad \cdots\cdots(A\cdot8)$$

Equation (A·8) yields infinite number of roots corresponding to $n=2, 3, 4, \cdots, \infty$ for each 'm'.

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線形粘弾性基礎上の矩形厚板の解析

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1. まえがき

弾性基礎上のせん断変形を考慮した矩形板の解析に対しては，若干の報告がある．すなわち，4辺単純支持板に対する級数解，4辺自由板に対するFEM，差分，そして級数解である．しかしながら，粘弾性基礎上の同種の解析に対する報告は見当たらない．著者は先に，各種の境界をもつ薄板に対し，自由振動問題におけるモード関数(固有関数)を用いて弾性基礎問題に対する解を求めた後に，弾性-粘弾性問題における〈対応原理〉により，その解を粘弾性問題の解へと拡張する方法を示した．本論文は，この手法をせん断変形を考慮した矩形板の解析に適用し，相対2辺が単純支持された矩形板の一般解を誘導したものである．

平板の古典理論（薄板理論）に対し，せん断変形を含めるための修正理論がいくつか提案されているが，本論文ではMindlin理論を用いた．この理論は，せん断修正係数を5/6と採ることにより，変位および断面力が，板厚方向の直応力 σ_z の効果を無視したReissner理論，そしてPancの修正理論と一致する．

取り扱った基礎の粘弾性モデルは，Kelvin (Voigt)，Maxwell，そしてクリープや応力緩和をよりよく表わし得るStandard linear solidであり，荷重は時間に依存しないものとした．

2. 粘弾性解

弾性基礎上のMindlin板の支配方程式は，w を板中立面のたわみ，ϕ_x, ϕ_y を曲げによる板の回転とすると，次式で与えられる．

$$\frac{D}{2}\left[(1-\nu)\nabla^2\phi_x+(1+\nu)\frac{\partial\phi}{\partial x}\right]+\kappa Gh\left(\frac{\partial w}{\partial x}-\phi_x\right)=0$$
$$\frac{D}{2}\left[(1-\nu)\nabla^2\phi_y+(1+\nu)\frac{\partial\phi}{\partial y}\right]+\kappa Gh\left(\frac{\partial w}{\partial y}-\phi_y\right)=0$$
$$\kappa Gh(\nabla^2 w-\phi)=kw-q \quad \cdots\cdots(1)$$

ここで，$\phi=\partial\phi_x/\partial x+\partial\phi_y/\partial y$；$D=Eh^3/12(1-\nu^2)$ は曲げ剛性；E, G はそれぞれヤング率，せん断弾性係数；h は板厚；ν はポアソン比；k は基礎係数；q は荷重；κ はせん断修正係数；$\nabla^2=\partial^2/\partial x^2+\partial^2/\partial y^2$．

$W_{mn}(x,y)$, $\Psi_{x,mn}(x,y)$, $\Psi_{y,mn}(x,y)$ を境界条件を満足するMindlin板の m, n 次の固有関数とすれば，弾性基礎問題，式(1)の解は

$$\begin{bmatrix} w(x,y) \\ \phi_x(x,y) \\ \phi_y(x,y) \end{bmatrix}=\frac{a^4}{D}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{q_{nn}}{\lambda_{mn}^4+K^4}\begin{bmatrix} W_{mn}(x,y) \\ \Psi_{x,mn}(x,y) \\ \Psi_{y,mn}(x,y) \end{bmatrix} \quad \cdots\cdots(2)$$

となる．ここで，$K^4=ka^4/D$；a はスパン長；q_{mn} は荷重展開係数；λ_{mn} は固有値．基礎反力は $p=kw$ として求められる．

式(2)において，基礎係数が固有関数に含まれていないので，〈対応原理〉を適用すればラプラスの逆変換が容易にとれ，粘弾性基礎問題の解式を得ることができる．

階段荷重 $q(x,y)H(t)$ に対するたわみの粘弾性解のラプラス変換式 $\bar{W}$ とその逆変換式(粘弾性解) W，および基礎反力 P の粘弾性解は，

$$\bar{W}(x,y,s)=a^4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\bar{q}_{mn}(s)}{D\lambda_{mn}^4+\bar{k}(s)a^4}W_{mn}(x,y)$$
$$W(x,y,t)=w(x,y)-\frac{a^4}{D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{q_{mn}T_1(t)}{\lambda_{mn}^4+K^4}W_{mn}(x,y)$$
$$P(x,y,t)=p(x,y)+K^4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{q_{mn}T_2(t)}{\lambda_{mn}^4+K^4}W_{mn}(x,y) \quad \cdots\cdots(3)$$

となる．各種粘弾性モデルに対する粘弾性オペレーター $\bar{k}(s)$ と時間関数 $T_1(t)$, $T_2(t)$ を表に示した．