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# A NEW APPROACH TO PREDICT THE STRENGTH OF COMPRESSED STEEL PLATES

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## 1. INTRODUCTION

The stability and the load-carrying capacity of plates is still a matter of immense diversity and ever-growing research interest in the field of civil engineering[1),2)].

In many cases where slender plate elements are used, they are known to buckle locally and the member may fail prematurely before reaching the yielding point. However, they do not generally fail by the elastic buckling but have significant reservation of the postbuckling strength. This is known as one of the major differences between the plates and the bars.

The early attempts of the elasto-plastic buckling analysis of plates have been seemingly made by Bijlaard, Ilyushin, Stowell[3)], and Bleich[4)]. They have derived the fundamental differential equations of the plates, and obtained explicitly the closed-form solutions under various edge conditions and various loading conditions. These classical methods of approach are summarized by Okumura et al[5)].

Recently, the prime research interests are shifting to the evaluation of the elasto-plastic strengths of imperfect plates with different width-thickness ratios by means of discretization methods such as finite differences and finite element methods and the procedures of solving sets of nonlinear simultaneous equations[6)~14)].

In this paper, however, attempts are made to evaluate the load-carrying capacity of plates in the elasto-plastic range in a simple manner making use of a knowledge of the catastrophe theory[15)].

Here, the above-mentioned nonlinear procedures are not required, but the effects of the imperfections and the width-thickness ratios can be designated explicitly in a closed form expression.

The great emphasis in the presentation will be placed on the evaluations of the elasto-plastic buckling loads, the postbuckling equilibrium paths, the plastic unloading curve or the mechanism curve, and the imperfection sensitivity curve[16)].

## 2. BASIC CONCEPTS

### (1) Ideal Elasto-Plastic Buckling Load

The elasto-plastic buckling stress of the initially flat plates with a proper residual stress distribution in the cross section is firstly determined. As a basic model, a rectangular plate with four edges simply supported under uniaxial compression as shown in **Fig. 1** is considered subsequently.

The distribution of the residual stresses of the simply supported plates is assumed in either a parabolic, a triangular or a trapezoidal form uniform in the axial direction as shown in **Fig. 1**(a)~(c). Let $E_t$, $\bar{\varepsilon}$, and $\bar{\sigma}$ designate the tangent modulus, the average axial strain, and the average axial stress of the plate during the loading, respectively, then, their relations can be easily derived as:

$$E_t \equiv \frac{d\bar{\sigma}}{d\bar{\varepsilon}} = kE,$$

and

$$\left.\begin{array}{l}\bar{\sigma}=\sigma_Y-(3-2k)k^2\sigma_r \\ \bar{\varepsilon}=[\sigma_Y+3(1-k)^2\sigma_r-\sigma_r]/E\end{array}\right\}\text{ for parabolic residual stress}$$

$$\left.\begin{array}{l}\bar{\sigma}=\sigma_Y-k^2\sigma_r \\ \bar{\varepsilon}=[\sigma_Y-(2k-1)\sigma_r]/E\end{array}\right\}\text{ for triangular residual stress}\quad 0\le k\le 1$$

$$\left.\begin{array}{l}\bar{\sigma}=\sigma_Y-k^2(\sigma_r+\sigma_Y)^2/(4\sigma_r) \\ \bar{\varepsilon}=[2\sigma_Y-k(\sigma_r+\sigma_Y)/(2\sigma_r)]/E\end{array}\right\}\text{ for trapezoidal residual stress}\quad 0\le k\le \frac{2\sigma_r}{\sigma_r+\sigma_Y} \quad \cdots\cdots(1)$$

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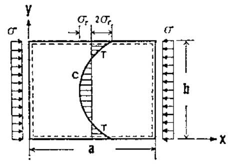
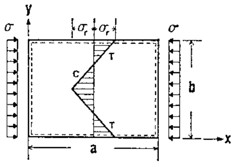
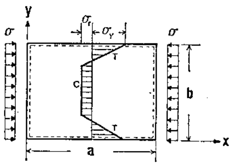



(a) Parabolic Distribution




(b) Triangular Distribution




(c) Trapezoidal Distribution

**Fig. 1** Distributions of residual stresses.

where $E$, $\sigma_Y$ and $\sigma_r$ refers to the Young's elastic modulus, the yielding strength and the magnitude of the maximum longitudinal compressive residual stress, respectively. Moreover, $k$ refers to the ratio of the elastic portion of the cross section to the total section, and implies the global tangent modulus factor of the plate section.

From the boundary condition, the following buckling mode and the initial imperfection are adopted for both the elastic and elasto-plastic buckling referring to the coordinate system as shown in **Fig. 1**:

$$\left.\begin{aligned} &W = wY(y)\sin\frac{m\pi x}{a} \\ &W_0 = w_0 Y(y)\sin\frac{m\pi x}{a} \\ &Y(y) = \sin\frac{n\pi y}{b} \quad : \quad \text{all edges simply supported} \end{aligned}\right\} \cdots(2)$$

where $w$, $w_0$, $Y(y)$, $m$ and $n$ designates the magnitude of the total out-of-plane deflection and the initial deflection of the plate, the mode of the deflection in $y$-direction, the number of half waves in $x$ and $y$ direction, respectively.

There are certainly a number of possible expressions for the moment-curvature relationship for the elasto-plastic buckling of plates based on the classical method using partial differential equations[3)~5)]. For example, reference 5) classifies such classical methods into three groups as shown in **Appendix A.**

Based on these classical methods of approach, the basic non-dimensionalized equations of equilibrium can be written as[5)]:

$$\left.\begin{aligned} &\frac{D_e(1-\nu^2)}{\sigma_Y t}\tilde{\nabla}_p^4 W + \tilde{\sigma}\frac{\partial^2 W}{\partial x^2} = 0\,, \\ &\text{where} \\ &D_e \equiv \frac{Et^3}{12(1-\nu^2)}\,, \qquad \tilde{\sigma} \equiv \frac{\sigma}{\sigma_Y} = \frac{P}{\sigma_Y bt} \\ &\tilde{\nabla}_p^4 \equiv \frac{\partial}{\partial x^2}\left(k_1\frac{\partial^2}{\partial x^2} + k_2\frac{\partial^2}{\partial y^2}\right) \\ &\qquad + 4\frac{\partial^2}{\partial x\partial y}\left(k_4\frac{\partial^2}{\partial x\partial y}\right) \\ &\qquad + \frac{\partial}{\partial y^2}\left(k_2\frac{\partial^2}{\partial x^2} + k_3\frac{\partial^2}{\partial y^2}\right) \end{aligned}\right\} \cdots(3)$$

in which $k_j\,(j=1, \cdots, 4)$, $b$, $D_e$, $P$, $t$, and $\nu$ denotes some constants, the width, the elastic flexural rigidity of isotropic plates, the total compressive load, the thickness, and the Poisson's ratio, respectively. Superscript symbol "$\sim$" will be used hereafter to designate either the non-dimensionalized stresses or deflections as divided by the yielding stress or the thickness, respectively, unless otherwise referred to.

Let $W$, in Eq. (2) be substituted into Eq. (3) adopting the Galerkin's method, and let $f$ be defined by:

$$\left.\begin{aligned} &f \equiv \frac{D_e(1-\nu^2)\int_0^b YY_1\,dy}{\sigma_E t\int_0^b\left(\frac{m\pi}{a}\right)^2 Y^2 dy}\,, \qquad \sigma_E \equiv \frac{K_E\pi^2 D_e}{b^2 t} \\ &\text{where} \\ &Y_1(y) \equiv k_1\left(\frac{m\pi}{a}\right)^4 Y - 2(k_2+2k_4)\left(\frac{m\pi}{a}\right)^2\frac{d^2Y}{dy^2} \\ &\qquad + k_3\frac{d^4Y}{dy^4} \\ &K_E = n^2\left(\frac{mb}{na} + \frac{na}{mb}\right)^2 \end{aligned}\right\} \cdots\cdots(4)$$

furthermore, $K_E$ and $\sigma_E$ refers to the Euler buckling coefficient and the corresponding buckling stress, respectively. Then, the non-dimensionalized equation of equilibrium can be obtained in the following form:

$$\left.\begin{aligned} &f\tilde{\sigma}_E w - \tilde{\sigma}w = 0 \\ &\text{where} \\ &\tilde{\sigma}_E \equiv \frac{\sigma_E}{\sigma_Y} = \frac{1}{R^2}\,; \qquad R \equiv \frac{b}{\pi t}\sqrt{\frac{12(1-\nu^2)\sigma_Y}{K_E E}} \end{aligned}\right\} \cdots\cdots(5)$$

whereas, $R$ refers to the generalized width-thickness ratio.

In this way, a pseudo-potential, $U(w, \tilde{\sigma})$ may be constructed so that the equilibrium equation is given by Eq. (5), that is,

$$U' \equiv \frac{\partial U}{\partial w} = f \bar{\sigma}_E w - \bar{\sigma} w = 0 \quad \cdots\cdots\cdots\cdots\cdots\cdots (6)$$

The value of $f$ can be evaluated from Eq. (4) once the coefficients of elasto-plastic plates are known. **Appendix A** provides several possible classical expressions for $f$, based on the methods by Bleich, Chwalla, Stowell, Bijlaard, and Pearson. In this paper, numerical examples are demonstrated using only the Bleich's method to evaluate the buckling stress. Since this method does not explicitly consider the effect of the residual stress, a modification is made taking that into account using the tangent modulus defined by Eq. (1) and through Eq. (4). Then, it can be shown that the value of $f$ can be determined by:

$$f^c \equiv f\Big|_{\bar{\sigma}=\bar{\sigma}_{cr}} = \frac{\sqrt{k_c}\, n^2}{K_E}\left(\frac{\sqrt[4]{k_c}\, mb}{na} + \frac{na}{\sqrt[4]{k_c}\, mb}\right)^2 \quad \cdots\cdots\cdots\cdots\cdots\cdots (7)$$

where $k_c$ refers to the critical value of the ratio, $k$, when the elasto-plastic bifurcation occurs.

Upon substitution of Eq. (7) into Eq. (6), the elasto-plastic buckling stress, $\bar{\sigma}_{cr}$, of the perfect flat plate, that is, for $w_0=0$ can be obtained by:*

$$\bar{\sigma}_{cr} \equiv \frac{\sigma_{cr}}{\sigma_Y} = \frac{4\sqrt{k_c}}{K_E}\bar{\sigma}_E \quad \text{for } n=1 \text{ and } \frac{a}{mb} = \sqrt[4]{k_c} \quad \cdots\cdots\cdots\cdots\cdots\cdots (8)$$

This relationship implies that the elasto-plastic buckling stress can be expressed in the form similar to that of columns.

### (2) Postbuckling Path

In general, it is well known that the plate components possess the significant postbuckling strength in the elastic range. Such postbuckling equilibrium path may be obtained from von Kármán's nonlinear equations, which have been solved by many researchers[17),18)]. Thus, for simply supported plates, the following postbuckling path can be obtained using the deflection mode prescribed by Eq. (2):

$$\left.\begin{aligned} \bar{\sigma} &= \sigma_E + C_E w^2 \\ C_E &= \frac{n^2\pi^2 E}{16b}\left[\left(\frac{mb}{na}\right)^2 + \left(\frac{na}{mb}\right)^2\right] \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (9)$$

The rigorous determination of the postbuckling behavior in the closed form, however, is extremely difficult in the elasto-plastic range. Therefore, a modification on the von Kármán's equations may be accomplished in the subsequent manner:

$$\left.\begin{aligned} &\nabla^4 F = E_s\left[\left(\frac{\partial^2 W}{\partial x \partial y}\right)^2 - \frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}\right] \\ &\text{and} \\ &D_e(1-\nu^2)\int_0^b \tilde{\nabla}_p^4 W Y(y)\,dy \\ &\quad = t\int_0^b \left[\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 W}{\partial x^2}Y(y) + \frac{\partial F^2}{\partial x^2}\frac{\partial^2 W}{\partial y^2}Y(y) - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 W}{\partial x \partial y}Y(y)\right]dy \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots (10)$$

where $F$ is the Airy's stress function, and $E_s$ designates the secant modulus of the total plate width defined by $E_s \equiv \bar{\sigma}/\bar{\varepsilon}$. Also, the second equation of Eq. (10) takes the similar form to Eq. (3).

The average stress on the postbuckling equilibrium path in the elasto-plastic range, may be modified and can be shown to take the form similar to Eq. (9):

$$\left.\begin{aligned} &\bar{\sigma} = \sigma_{cr} + C_p w^2 \\ &\text{where} \\ &C_p = \frac{n^2\pi^2 E_s}{16b^2}\left[\left(\frac{mb}{na}\right)^2 + \left(\frac{na}{mb}\right)^2\right] \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (11)$$

where $\sigma_{cr}$ designates the average elasto-plastic buclking stress determined by Eq. (8). Upon nondimensionalization of Eq. (11) by $\sigma_Y$, it is easy to show that:

$$\left.\begin{aligned} &\bar{\sigma} \equiv \frac{\bar{\sigma}}{\sigma_Y} = \bar{\sigma}_{cr} + \tilde{C}_p \bar{w}^2 \quad \text{(curve AC in \textbf{Fig. 3(a)})} \\ &\text{where} \\ &\tilde{C}_p \equiv \frac{3(1-\nu^2)E_s(1+k_c)}{4EK_E R^2\sqrt{k_c}} \\ &\bar{w} \equiv \frac{w}{t} \quad [n=1 \;\&\; a/(mb) = \sqrt[4]{k_c}]^* \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots (12)$$

### (3) Ultimate Strength and Imperfection Sensitivity

Ultimate strength of the actual plates can not be determined only from the critical elasto-plastic buckling stress with the residual stresses and the elasto-plastic postbuckling path. It is further affected by the initial lateral deflection, and by the plastic unloading curve corresponding to the failure mechanism of the plate[19)~22)]. Similarly to the case of the columns, the concept of the "equivalent bifurcation point" may also be introduced here[16)].

The equivalent bifurcation point for the plate can be defined by the intersection of the elasto-

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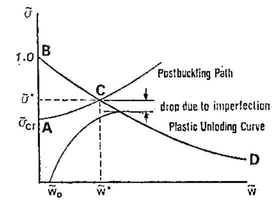
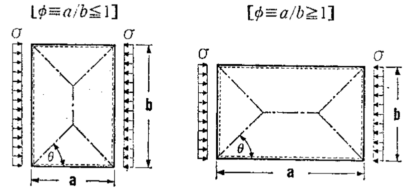
* In case of simply supported plates in the elastic range, $K_E=4$, thus, $\bar{\sigma}_{cr} = \sqrt{k_c}\,\bar{\sigma}_E$.

* In the elastic range, $C_p$ becomes $C_E$: $k_c=1$. and $E_s=E$.




(a) Postbuckling Path and Plastic Unloading Curve




(b) Plastic Fold Lines for Simply Supported Plates

**Fig. 2** Plastic failure mechanisms, post-buckling path and plastic unloading curve.

plastic postbuckling equilibrium path, Eq. (12), with the plastic unloading curve as shown in **Fig.** 2 (a). The latter curve can be determined by assuming the fold lines of the failure mechanism of the plate in the subsequent manner.

The failure mechanism of the rectangular plate under pure compression is shown graphically in **Fig.** 2 (b). Detailed discussions on the plastic unloading curve have been made by Sherbourne[19], Murray[20], Fujita[21], and Davies[22], but here in this study, the following interaction formula is assumed for simplicity, between the in-plane axial load and the bending moment:*

$$\left.\begin{aligned} &\bar{\sigma}^2+\bar{m}=1 \\ &\text{where} \\ &\bar{m}\equiv\frac{M}{M_p} \end{aligned}\right\} \quad \cdots\cdots(13)$$

in which $M$ and $M_p$ refers to the bending moment acting perpendicular to the fold line, and the plastic moment, respectively. Then, the plastic unloading curve can be obtained approximately by:

$$\left.\begin{aligned} &w=A\frac{1-\bar{\sigma}^2}{\bar{\sigma}}: \\ &A=\frac{1}{2}\,\frac{1+\phi\cot\theta}{2-\phi\tan\theta} \quad \text{for } \phi\leq 1 \\ &A=\frac{1}{2}(1+\phi\cot\theta) \quad \text{for } \phi>1 \end{aligned}\right\} \quad \cdots\cdots(14)$$

where $\theta$ and $\phi$ indicates the angle of the yielding fold line and the aspect ratio, $a/b$, respectively, as shown in this figure.

Now, let us define the equivalent bifurcation point as the intersection of Eq. (12) with Eq. (14), that is, as the solution of the following quartic polynomial equation:

$$\tilde{C}_pA^2\bar{\sigma}^4-\bar{\sigma}^3-(2\tilde{C}_pA^2-\bar{\sigma}_{cr})\bar{\sigma}^2+\tilde{C}_pA^2=0 \quad \cdots(15)$$

Let $\bar{\sigma}^*$ and $\bar{w}^*$ designates the real proper root of this equation and the corresponding deflection, respectively, then, the equivalent bifurcation point can be given by point C ($\bar{w}^*$, $\bar{\sigma}^*$) as shown in **Fig.** 2 (a).

For evaluation of the ultimate strength of the imperfect plate, another pseudo-potential, $V$, is also assumed to exist near point C in the form of $V=V(\bar{w}, \bar{w}_0, \bar{\sigma})$ similar to Eq. (6). In which $w_0$ refers to the magnitude of the initial lateral deflection with the same mode as the buckling mode of Eq. (2). Furthermore, $V$ is assumed so that the equilibrium equation near the point C is expressed with reference to Eq. (5):

$$\left.\begin{aligned} &V'\equiv\frac{\partial V}{\partial\bar{w}}=f\bar{\sigma}_E(\bar{w}-\bar{w}_0-\bar{w}^*)-\bar{\sigma}(\bar{w}-\bar{w}^*)=0 \\ &\text{where} \\ &f(w_d^*)\equiv f^c+\frac{1}{2}f_t'^c\bar{w}_d^*; \quad \bar{w}_d^*\equiv\bar{w}_0-\bar{w}-\bar{w}^* \end{aligned}\right\} \quad \cdots\cdots(16)$$

$f^c$ and $f_t'^c$ designates the secant or the tangent modulus factor of the plate and the coefficient for the 1st order term of the Taylor expansion of the secant modulus factor in terms of $\bar{w}_d^*$, respectively, but the both being evaluated at the equivalent bifurcation point. This equation defines an equilibrium surface, $M_V$, comprising the following set of points ($\bar{w}-\bar{w}^*$, $\bar{w}_0$, $\bar{\sigma}$):

$$M_V=\{(\bar{w}-\bar{w}^*, \bar{w}_0, \bar{\sigma})\,|\,V'(\bar{w}-\bar{w}^*, \bar{w}_0, \bar{\sigma})=0\}\subset\boldsymbol{R}^3 \quad \cdots\cdots(17)$$

**Fig. 3** illustrates the equibrium surface $M_V$ in 3-dimensional space ($\bar{w}-\bar{w}^*$, $\bar{w}_0$, $\bar{\sigma}$), and its projection to three orthogonal planes.

Therefore, the ultimate strength $\sigma_m$ of the imperfect plate can be evaluated in terms of the bifurcation set through the catastrophe theory in the manner similar to the case of the columns.[16] The imperfection sensitivity curve can be defined graphically by the curve $a_2c_2$ in the $\bar{w}_0\sim\bar{\sigma}$ space projecting vertically the singular fold line AC on the equilibrium surface. It takes the following form:*

* See Appendix B for the comparison among these theories in the case of compressed square plates.

* Considering a mapping from $(\bar{w}-\bar{w}^*, \bar{w}_0)\rightarrow(\bar{\sigma}, \bar{w}_0)$, and let the Jacobian vanish, then it can be shown that $\partial\bar{\sigma}/(\partial\bar{w}-\bar{w}^*)=0$. This may be identical to say that $V''=0$. Furthermore, $V'''=f_t'^c\bar{\sigma}_E\neq 0$.

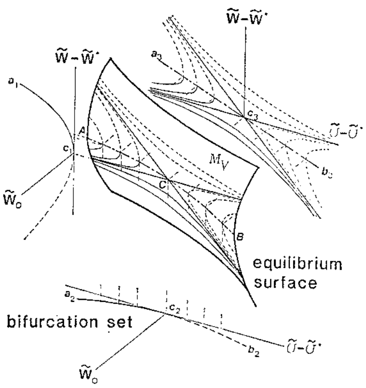



**Fig. 3** Equilibrium surface $M_V$ in 3-dimensional space.

$$\frac{\sigma_m}{\sigma^*}=1+\alpha^*\tilde{w}_0-\sqrt{2\alpha^*\tilde{w}_0\left(1+\frac{1}{2}\alpha^*\tilde{w}_0\right)}$$

where

$$\alpha^*\equiv-\frac{f_t'^c}{f^c} \qquad \cdots\cdots(18)$$

Since $f_t'^c$ is quite difficult to be determined, an attempt is made to approximate it by the slope of the plastic unloading curve at point C $(0, 0, \tilde{\sigma}^*)$ as in the case of the columns, that is,

$$\alpha^*\equiv-\frac{1}{\tilde{\sigma}^*}\frac{d\tilde{\sigma}}{d\tilde{w}}\bigg|_{\tilde{\sigma}=\tilde{\sigma}^*}=\frac{\tilde{\sigma}^*}{A(1+\tilde{\sigma}^{*2})} \qquad \cdots\cdots(19)$$

On the other hand, the standard form of the bifurcation set of "FOLD CATASTROPHE" according to the catastrophe theory,[15] can be given by

$$\frac{\sigma_m}{\sigma^*}=1-\frac{1}{\sigma^*}\sqrt{\frac{2V_{111}^c\dot{V}_1^c}{\overset{\circ}{V}{}_{11}^{c2}}}\tilde{w}_0=1-\sqrt{2\alpha^*\tilde{w}_0}$$

where

$$\dot{V}_1^c\equiv\frac{\partial V'}{\partial\tilde{w}_0}\bigg|_{\tilde{w}=\tilde{w}^*,\tilde{\sigma}=\tilde{\sigma}^*,\tilde{w}_0=0}=-\tilde{\sigma}_*$$

$$\overset{\circ}{V}{}_{11}^c\equiv\frac{\partial V''}{\partial\tilde{\sigma}}\bigg|_{\tilde{w}=\tilde{w}^*,\tilde{\sigma}=\tilde{\sigma}^*,\tilde{w}_0=0}=-1. \qquad \cdots\cdots(20)$$

It is obvious that Eqs. (18) and (20) are identical provided that $\alpha^*\tilde{w}_0\ll 1$.

### (4) Modification for Imperfection

The deflection at the point of the maximum load can be obtained using Eq. (16) and its further differentiation. Hence, it will be quite easy to show that

$$\tilde{w}=\tilde{w}^*+\sqrt{\frac{2\tilde{w}_0}{\alpha^*}+\tilde{w}_0^2} \qquad \cdots\cdots(21)$$

Now, if the initial deflection $\tilde{w}_0$ is to be non-dimensionalized through the width $b$ of the plate, then Eq. (21) can be rewritten as follows using slenderness $R$:

$$\frac{w}{b}=\frac{w^*}{b}+\sqrt{\frac{2}{\alpha^*\pi R}\sqrt{\frac{12(1-\nu^2)\sigma_Y}{K_E E}}\frac{w_0}{b}+\left(\frac{w_0}{b}\right)^2} \qquad \cdots\cdots(22)$$

Let us consider a particular case of very stocky plate, that is, the case of $R=0$, then it can be easily shown that $\tilde{w}^*\to 0$ & $\alpha^*\neq 0$ in view of Eqs. (14) and (19). However, deflection $w/b$ remains indefinite since the first term in the square root of Eq. (25) tends to become singular as $R\to 0$. Now, it may be quite natural and reasonable in engineering sense to expect that extremely stocky plate will fail plastically without any significant increase of deflection $w$ from the value of $w_0$. In order to take this expection into account, it will be necessary to pose the limit: $w\to w_0$ as $R\to 0$. Consequently, the equivalent imperfection, $w_0^*/b$, may be introduced through some function:

$$\frac{w_0^*}{b}=\mu(R)\frac{w_0}{b}$$

where

$$\mu(R)=\mu_c\left(\frac{R}{R_p}\right)^\beta \qquad \cdots\cdots(23)$$

and $R_p$ refers to the value of $R$ at which the buckling point changes from elasto-plastic to purely elastic. Here $\mu(R)$ must be determined so that the following limit holds as $R\to 0$:

$$\lim_{R\to 0}\frac{1}{R}\frac{w_0^*}{b}=0 \qquad \cdots\cdots(24)$$

Besides, taking into account the numerical results by Crisfield, Little, Harding, Dawson and Horne et al., the form of $\mu(R)$ can be finally approximated by

$$\mu_c=\frac{1}{8},\quad \beta=2\left(1-\frac{R}{R_p}\right) \qquad \cdots\cdots(25)$$

The determination of the form of $\beta$ reflects another observation that the effect of the imperfection will diminish as the plates become more slender in the elastic region.

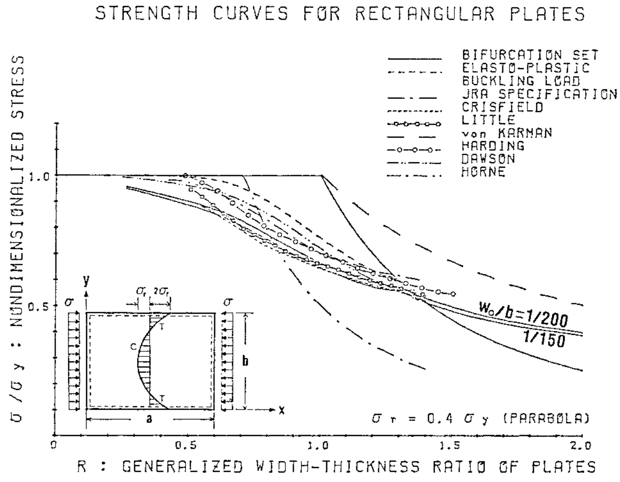
Finally, the imperfection sensitivity can be obtained by Eq. (18); however, with the equivalent imperfection of Eq. (23).

## 3. NUMERICAL EXAMPLES AND DISCUSSIONS

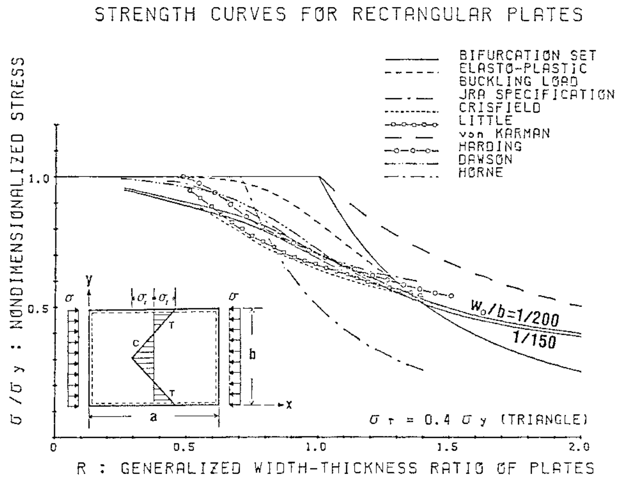
Now, as the numerical illustration, let us examine the simply supported plates under uniaxial compression. The type of the distribution of the residual stress considered herein is either of a parabola, a triangle, or a trapezoidal as shown in **Fig. 1** (a)~(c), respectively. The magnitude of the maximum compressive residual stress is restricted for practical reason to $0.4\sigma_Y$ for all types. Also, the magnitude of the initial deflections of the plates considered herein is taken to be either $b/200$ or $b/150$, considering the tolerance of $b/150$ as allowed by the JRA Specifications for highway bridges[23], where $b$ represents the width of the loaded edge of the plate.

The numerical results of plates are illustrated in **Fig. 4** (a)~(c) for the residual stress distribution of the parabola, the triangle, and the trapezoidal, respectively. The abscissa indicates the generalized width-thickness ratio, $R$, and the ordinate indicates the non-dimensionalized ultimate strength with respect to the yield stress.

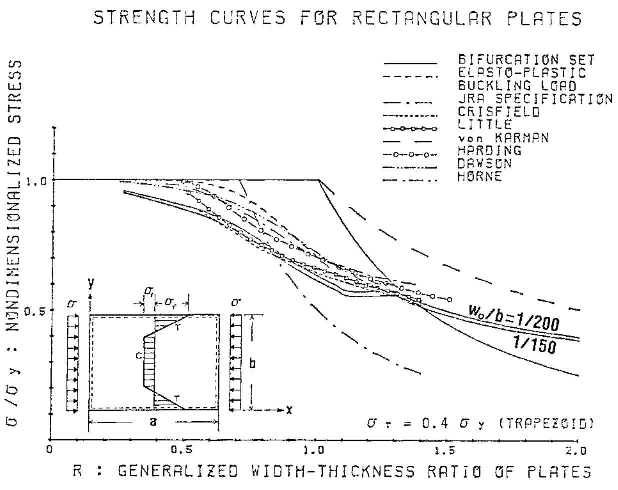
The proposed bifurcation sets are compared with six ultimate strength curves herein: by von Kármán, Crisfield's finite element large deflection elasto-plastic buckling analysis[11], Little's energy minimization[12], Harding's finite difference analysis with a dynamic relaxation[9], Dawson's simplified elastic large deflection perturbation analysis[13], and by Horne's effective width approach[10]. These strength curves were drawn for simply supported rectangular plates with $\phi=0.7\sim1$, $w_0/b=1/200$ and the rectangular band width at $3t$ of the residual yield tensile stress.




(a)




(b)




(c)

**Fig. 4** Ultimate strength curves for compressed rectangular plates with trapezoidal residual stress.

The ultimate strength curves in the elastic range for the slender plates are throughout the same regardless of the residual stress types. But, in the elasto-plastic range for intermediate values of $R$, the ultimate strengths in the case of the trapezoidal distribution are found to be the lowest, and those of the triangular distribution are the highest, independent of the magnitude of the imperfections.

Furthermore, it must be mentioned that all of the bifurcation sets for the rectangular plates are obtained for such aspect ratio, $\phi$, that the least buckling coefficients are obtained both in the elastic and in the elasto-plastic range, that is, for $n=1$ and $\phi/m=\sqrt[4]{k_c}$.

## 4. CONCLUSIONS

The main conclusions are summarized as follows:

**(1)** The inelastic strength of the plates may be explicitly evaluated in terms of the imperfection sensitivity characterized by the 1/2-power of the imperfection.

**(2)** The imperfection sensitivity can be defined explicitly in a closed form near the equivalent bifurcation point being the intersection of the elasto-plastic postbuckling path with a plastic unloading curve of the failure mechanism of the plate.

**(3)** The actual imperfections are modified and replaced by the equivalent imperfections so that the strength curves are in good correlation with those by many previous researchers.

**(4)** The validity of the assumption of the pseudo-potential is investigated from the viewpoint of the singularity mapping in the catastrophe theory, and it is found reasonable.

**(5)** The general philosophy adopted herein may also be applicable to other types of engineering structures including columns, beams, and shells.

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## 5. APPENDIX

### A) BUCKLING COEFFICIENT OF ELASTO-PLASTIC PLATES

The buckling coefficient evaluated through the so-called classical methods in the case of simply supported plates can be tabulated in **Table A·1**:

**Table A·1** Buckling Coefficient

| Type | Theory | Buckling Coecient: $K$ <br> $K=fK_E$ | Comments |
|---|---|---|---|
| Elastic | Hooke | 4 | |
| Plastic | Bleich | $4\sqrt{\tau}$ * | analogous to elastic |
| | Chwalla | $4\tau_r$**** | |
| | Stowell | $4\tau_s\left(\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{4}+\frac{3}{4}\tau}\right)$** | Deformation Theory |
| | Bijlaard | $2(1-\nu^2)(\sqrt{k_1k_3}+k_2+2k_4)$*** | |
| | Pearson | $2\left[1+\frac{1}{2}\sqrt{1+3\tau}\right]$. | Flow Theory |

* $\tau\equiv E_t/E$: tangent modulus factor
** $\tau_s\equiv E_s/E$: secant modulus factor
*** $k_1=[1+3(1+e)\tau]/d$, $k_2=[2-(2-4\nu)\tau]/d$, $k_3=4/d$, $k_4=1/d'$; $d\equiv5-4\nu-(1-2\nu)^2\tau+3e$, $d'\equiv2+2\nu+3e$, $e\equiv1/\tau_s-1$, $\nu$: Poisson's ratio
**** $\tau_r\equiv4\tau/(1+\sqrt{\tau})^2$: reduced modulus factor

### B) COMPARISON OF UNLOADING CURVES

Several theories have been proposed so far regarding the plastic failure mechanisms of compressed rectangular plates by different researchers[19)~22)]. From the comparison among these theories, nevertheless, it can be seen that there exists no truly exact solution and one

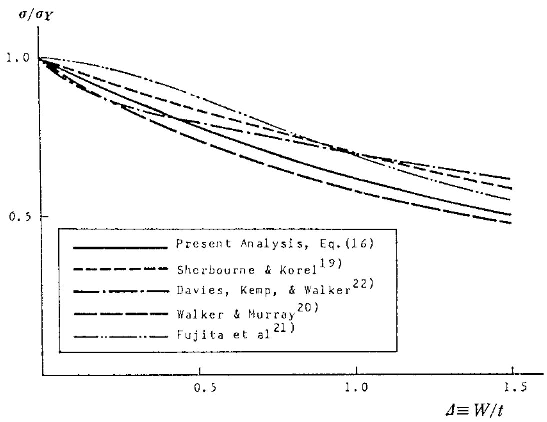


**Fig. B** Load-deflection curves in plastic failure mechanism. Compressed square plates: $\phi=1$; $\theta=45°$.

theory differes from another considerably. Thus, an example is given in **Fig. B** to show the load-deflection curves obtained by using different theories associated with the plastic failure mechanism of simply supported compressed square plates.

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# 鋼圧縮板の強度推定のための新しいアプローチ

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今日，プレートガーダー，箱桁，塔，橋脚などの土木構造物の圧縮部材として，板パネルが幅広く使用されている．圧縮板要素の安定性と耐荷力を評価することは，非常に重要な問題である．

圧縮板の弾塑性耐荷力を求めるためには，材料的非線形性および幾何学的非線形性の両方を考慮した弾塑性大変形解析が，一般に採用されてきている．このとき，得られた耐荷力は，縦横比，残留応力等はもとより，幅厚比や幾何学的初期不整を含む多くのパラメーターの空間内で，単に離散的な孤立点として求められるにすぎない．また，その算定には膨大な非線形計算が必要となる．

本研究は，このような非線形弾塑性大変形解析を経由せずに，カタストロフィー理論の基礎知識を適用して，幅厚比と幾何学的初期不整が弾塑性域における圧縮板の耐荷力に及ぼす影響を陽な形で評価しようとするものである．本論文で提案する圧縮矩形板に対する弾塑性耐荷力の簡易計算法は，以下のとおりである．

（1）　材料の弾塑性挙動や残留応力を考慮して，弾塑性分岐点を求める．この分岐点は，弾塑性座屈荷重を与えるとともに，弾性域と同様な安定対称分岐としての弾塑性後座屈曲線を決定できる．

（2）　圧縮矩形板の極限荷重を示す塑性崩壊機構曲線を求める．本論文では，簡易化し中心圧縮柱と類似の塑性相関関係が採用される．

（3）　（1）と（2）の交点を算定し，これを「等価分岐点」とよぶ．等価分岐点を特異点にもつように「擬似ポテンシャル」を，この点の近傍で定義する．

（4）　幾何学的初期不整は，実際の圧縮板の挙動を考慮し修正して「等価初期不整」に置き換えられる．

（5）　（3）の擬似ポテンシャルを用いて，等価分岐点近傍で圧縮板の弾塑性耐荷力を，（4）の等価初期不整に対する敏感性の形で評価する．

以上の弾塑性耐荷力簡易評価法を圧縮矩形板に適用し，得られた結果は次のとおりである．

（1）　矩形板の弾塑性圧縮強度は，「折り目」のカタストロフィーの分岐集合で表現され，その形は近似的に1/2乗法則に従う．

（2）　この分岐集合は，圧縮矩形板の塑性崩壊機構曲線と弾塑性後座屈曲線の交点である等価分岐点の近傍において定義できる．

（3）　周辺単純支持矩形圧縮板に対する結果より，既往の解析的研究や実験的研究により得られた耐荷力曲線を参考にして，本論文で定義した等価初期不整を，幾何学的初期不整として採用するのが妥当であることが示された．

（4）　カタストロフィー理論における特異点写像という観点から，擬似ポテンシャル導入の妥当性が検証された．

（5）　本簡易評価法は一般的な概念として，柱はもちろんはりやシェルの安定性と耐荷力の問題にも適用できるものと思われる．