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A FINITE DISPLACEMENT FORMULATION OF ELASTIC SHELLS

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1. INTRODUCTION

The theory of shells is formulated by reducing the relations in three dimensional continua under the so-called Kirchhoff-Love hypothesis to those in two dimensional continua which are expressed by the quantities of the middle surfaces of shells. This paper aims to present a general formulation for elastic shells under the Kirchhoff-Love hypothesis with particular emphasis on geometrical nonlinearity.

There are two ways to formulate shell theories; one directly using the energy principles for elastic continua[1)~6)], the other being the integration of equilibrium equations for three dimensional continua with respect to thickness component of shells[7)~9)]. The latter can be understood as a variation of static methods.

The formulation based on energy principles involves complicated mathematical manipulation, and further the boundary conditions are obtained only under the restrictions of small strains and small rotations[3),6)]. On the other hand, the mathematical procedure involved in the integration method is not so complex as the former, particularly when tensor expression is utilized. The physical meanings which are clear in the equilibrium equations will not be lost in integration, and hence the meanings of the resulting equations are also obvious.

This integration has been tried to derive the small displacement theory of shells[7),8)], and to discuss the effectiveness of Lagrangian expressions of stress resultant tensors and others in the finite displacement theory of shells[9)]. In those papers, however, attention has been paid only on field equations, and thus the boundary conditions compatible with the field equations have not been discussed.

Although Sanders[1)] and Koiter[6)] have derived the strain-displacement relations and equilibrium equations in finite displacements on the basis of the Kirchhoff-Love hypothesis, appropriate constitutive equations have not been found. The rigorous treatment for the elastic constitutive equations of shells has only been discussed by Seide[2)], Naghdi and Nordgren[3)], utilizing strain energy functions.

Here in this paper, by making use of tensor expressions[7)], an unified finite displacement theory for elastic thin shells is derived by integrating the well-established equilibrium equations for three dimensional continua with respect to the shell thickness. Particular interest is given to present the rigorous boundary conditions and constitutive equations consistent with the field equations for the finite displacement theory of shells which have remained to be resolved.

2. STRAIN TENSORS IN TERMS OF METRIC AND CURVATURE TENSORS

The right-hand curvilinear coordinates (x^1, x^2, x^3) are defined in the three dimensional space containing the body, in which the coordinates (x^1, x^2) are on the middle surface of a shell, and the coordinate x^3 is normal to the surface. Position vectors $\boldsymbol{s}$ and $\boldsymbol{r}$ are covariant base vectors

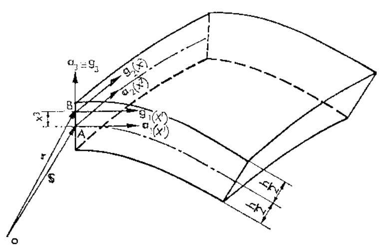

Fig. 1 A Shell Element before Deformation.

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$\boldsymbol{a}_i$ and $\boldsymbol{g}_i$ are defined before deformation of a shell with thickness h as shown in **Fig. 1**, in which $\boldsymbol{s}$ and $\boldsymbol{a}_i$ are given on the middle surface, while $\boldsymbol{r}$ and $\boldsymbol{g}_i$ are for an arbitrary point along the thickness. The quantities after deformation are hereinafter denoted using symbol (^) such as $\hat{\boldsymbol{s}}$ and $\hat{\boldsymbol{a}}_i$. In the descriptions to follow, Roman suffix $(i, j, k, \ldots)$ indicates three dimensional coordinates (1, 2, 3), while Greek suffix $(\alpha, \beta, \ldots)$ indicates two dimensional plane coordinates (1, 2). Further, the rule of summation convention is used throughout this paper.

By the definition of position vectors and base vectors, the following relation holds;

$$\boldsymbol{a}_\alpha(x^1, x^2, 0)=\boldsymbol{s}_{,\alpha} \quad \cdots\cdots (1)$$

in which (,) denotes differentiation as given by

$$(\ )_{,\alpha}=\partial(\ )/\partial x^\alpha \quad \cdots\cdots (2)$$

Since the coordinate x^3 is straight and normal to the middle surface, base vector $\boldsymbol{a}_3$ becomes the unit vector as given by

$$\boldsymbol{a}_3=(\boldsymbol{a}_1\times\boldsymbol{a}_2)/|\boldsymbol{a}_1\times\boldsymbol{a}_2| \quad \cdots\cdots (3)$$

As obvious from **Fig. 1**, position vector $\boldsymbol{r}$ is expressed by

$$\boldsymbol{r}(x^1, x^2, x^3)=\boldsymbol{s}(x^1, x^2, 0)+x^3\boldsymbol{a}_3(x^1, x^2, 0) \quad \cdots (4)$$

By the definition of base vectors,

$$\boldsymbol{g}_\alpha\equiv\boldsymbol{r}_{,\alpha}=\boldsymbol{s}_{,\alpha}+x^3\boldsymbol{a}_{3,\alpha} \quad \cdots\cdots (5\cdot a)$$

The base vector toward the normal direction is

$$\boldsymbol{g}_3\equiv\boldsymbol{a}_3 \quad \cdots\cdots (5\cdot b)$$

Since $\boldsymbol{a}_3$ is perpendicular to $\boldsymbol{a}_\alpha$, $\boldsymbol{a}_{3,\alpha}$ becomes a vector on the middle surface, and thus $\boldsymbol{a}_{3,\alpha}$ can be decomposed to

$$\boldsymbol{a}_{3,\alpha}\equiv -b_\alpha{}^\beta\boldsymbol{a}_\beta\equiv -b_{\alpha\beta}\boldsymbol{a}^\beta \quad \cdots\cdots (6)$$

Quantities $b_\beta{}^\alpha$, $b_{\alpha\beta}$ are called the curvature tensors of the middle surface, and $\boldsymbol{a}^\beta$ is contravariant base vector as defined by

$$\boldsymbol{a}_\alpha\boldsymbol{a}^\beta=\delta_\alpha{}^\beta \quad \cdots\cdots (7)$$

in which $\delta_\alpha{}^\beta$ is the Kronecker's delta as given by

$$\delta_\alpha{}^\beta=\begin{cases}1 & (\alpha=\beta)\\ 0 & (\alpha\neq\beta)\end{cases} \quad \cdots\cdots (8)$$

Substitution of Eq. (6) into Eq. (5·a) yields

$$\boldsymbol{g}_\alpha=\mu_\alpha{}^\beta\boldsymbol{a}_\beta \quad \cdots\cdots (9)$$

in which

$$\mu_\alpha{}^\beta=\delta_\alpha{}^\beta-x^3 b_\alpha{}^\beta \quad \cdots\cdots (10)$$

The Kirchhoff-Love hypothesis and the assumption of thickness unchanged are expressed mathematically by

$$\left.\begin{aligned}\hat{\boldsymbol{g}}_\alpha\hat{\boldsymbol{g}}_3\equiv\hat{g}_{3\alpha}=0\\ \hat{\boldsymbol{g}}_3\hat{\boldsymbol{g}}_3\equiv\hat{g}_{33}=1\end{aligned}\right\} \quad \cdots\cdots (11\cdot a, b)$$

using deformed base vectors respectively, in which $g_{ij}=\boldsymbol{g}_i\boldsymbol{g}_j$ is metric tensor at an arbitrary point of a shell. The assumptions of Eqs. (11) render the tensor formulation simple even for the deformed state. Using metric tensor $a_{\alpha\beta}=\boldsymbol{a}_\alpha\boldsymbol{a}_\beta$ and curvature tensor $b_{\alpha\beta}$ as defined by Eq. (6) on the middle surface both for undeformed and deformed states, Green's strain tensor η_{ij} can be expressed as

$$\left.\begin{aligned}\eta_{\alpha\beta}&=\{(\hat{a}_{\alpha\beta}-a_{\alpha\beta})-2x^3(\hat{b}_{\alpha\beta}-b_{\alpha\beta})\\ &\quad+(x^3)^2(\hat{b}_\alpha{}^\gamma\hat{b}_{\beta\gamma}-b_\alpha{}^\gamma b_{\beta\gamma})\}/2\\ \eta_{\alpha3}&=0\\ \eta_{33}&=0\end{aligned}\right\} \quad \cdots\cdots (12\cdot a\sim c)$$

3. EQUILIBRIUM EQUATIONS AND MECHANICAL BOUNDARY CONDITIONS

(1) Equilibrium Equations

The equilibrium equation for the deformed three dimensional infinitesimal element of a continuum is given by[7)]

$$\sigma^{ij}|_i\hat{\boldsymbol{g}}_j+X^k\boldsymbol{g}_k=0 \quad \cdots\cdots (13)$$

in which σ^{ij} is the $\hat{\boldsymbol{g}}_j$-component of Kirchhoff's stress tensor acting on the section normal to $\hat{\boldsymbol{g}}_i$, defined for undeformed unit area, and X^k is the $\boldsymbol{g}_k$-component of body force defined for undeformed unit volume. Symbol $(|_i)$ denotes three dimensional covariant derivative with respect to x^i in terms of $\hat{\boldsymbol{g}}_i$.

Deformed infinitesimal volume $d\hat{V}$ can be expressed by

$$d\hat{V}=\hat{\epsilon}_{12}\hat{\mu}dx^1dx^2dx^3 \quad \cdots\cdots (14)$$

in which $\hat{\epsilon}_{12}$ is a deformed component of permutation tensor as defined by

$$\epsilon_{\alpha\beta}=\epsilon_{\alpha\beta3} \quad \cdots\cdots (15)$$

Permutation tensor ϵ_{ijk} is given by $\epsilon_{ijk}=\sqrt{g}\,e_{ijk}$ in which e_{ijk} is permutation symbol and $g=\det|g_{ij}|$[7)]. Symbol $\hat{\mu}$ in Eq. (14) is deformed quantity of μ which is given by

$$\begin{aligned}\mu=\det|\mu_\beta{}^\alpha|&=\begin{vmatrix}\mu_1{}^1\mu_2{}^1\\ \mu_1{}^2\mu_2{}^2\end{vmatrix}\\ &=1-x^3b_\lambda{}^\lambda+(x^3)^2\det|b_\beta{}^\alpha| \quad \cdots\cdots (16)\end{aligned}$$

Multiplying Eq. (13) by $d\hat{V}$ of Eq. (14) and integrating with respect to x^3 from $-h/2$ to $h/2$ lead to the following equilibrium equations in terms of stress resultants;

$$\begin{aligned}&(\hat{N}^{\beta\alpha}||_\beta+\hat{Q}^{\beta3}\hat{b}_\beta{}^\alpha+\hat{P}^\alpha)\hat{\boldsymbol{a}}_\alpha+(\hat{N}^{\alpha\beta}\hat{b}_{\alpha\beta}-\hat{Q}^{\alpha3}||_\alpha\\ &\quad+\hat{P}^3)\hat{\boldsymbol{a}}_3+p^\alpha\boldsymbol{a}_\alpha+p^3\boldsymbol{a}_3=0 \quad \cdots\cdots (17)\end{aligned}$$

in which

$$\left.\begin{aligned}\hat{N}^{\alpha\beta}&=\int_{-h/2}^{h/2}\sigma^{\alpha\delta}\hat{\mu}\hat{\mu}_\delta{}^\beta dx^3\\ \hat{Q}^{\alpha3}&=-\int_{-h/2}^{h/2}\sigma^{\alpha3}\hat{\mu}dx^3\\ \hat{P}^\alpha&=[{}_s\sigma^{3\beta}\hat{\mu}_\beta{}^\alpha\hat{\mu}]_{-h/2}^{h/2}\\ \hat{P}^3&=[{}_s\sigma^{33}\hat{\mu}]_{-h/2}^{h/2}\end{aligned}\right\} \quad \cdots\cdots (18\cdot a\sim f)$$

$$p^{\alpha}=\int_{-h/2}^{h/2} X^{\beta}\mu_{\beta}{}^{\alpha}\hat{\mu}dx^3$$
$$p^{3}=\int_{-h/2}^{h/2} X^{3}\hat{\mu}dx^3$$

Symbol $(||_{\beta})$ indicates two dimensional covariant derivative with respect to x^{β}[7] in terms of $\hat{\boldsymbol{a}}_{\beta}$. Symbols $\hat{N}^{\alpha\beta}$ and $\hat{Q}^{\alpha 3}$ are understood as the stress resultants of the $\hat{\boldsymbol{a}}_{\beta}$- and $\hat{\boldsymbol{a}}_{3}$- directions per unit length of the middle surface respectively acting on the section normal to $\hat{\boldsymbol{a}}_{\alpha}$, and $\hat{P}^{i}$ and p^{i} are resultant components of surface force and body force respectively as defined above. Symbols ${}_{s}\sigma^{3\beta}$ and ${}_{s}\sigma^{33}$ that appear in Eqs. (18·c, d) can be expressed, using surface force $\boldsymbol{q}=q^{i}\boldsymbol{g}_{i}$ defined for undeformed unit area, by

$$\left.\begin{aligned} {}_{s}\sigma^{3\beta} &= \hat{\lambda}_{\delta}{}^{\beta}(q^{\alpha}\mu_{\alpha}{}^{\gamma}\beta_{\gamma}{}^{\hat{\delta}}+q^{3}\beta_{3}{}^{\hat{\delta}}) \\ {}_{s}\sigma^{33} &= q^{\alpha}\mu_{\alpha}{}^{\gamma}\beta_{\gamma}{}^{\hat{\delta}}+q^{3}\beta_{3}{}^{\hat{3}} \end{aligned}\right\} \cdots\cdots\cdots(19\cdot\text{a, b})$$

in which $\lambda_{\alpha}{}^{\beta}$ and $\beta_{j}{}^{i}$ are given by

$$\lambda_{\alpha}{}^{\beta}=\boldsymbol{g}^{\beta}\boldsymbol{a}_{\alpha}\,,\quad \beta_{j}{}^{i}=\boldsymbol{a}_{j}\boldsymbol{a}^{i} \cdots\cdots\cdots(20\cdot\text{a, b})$$

Without surface force $\boldsymbol{q}$, the resulting force $\hat{P}^{i}$ is always equal to zero.

Consider the equilibrium of moment, which can be expressed with respect to an arbitrary point of the middle surface for infinitesimal volume $d\hat{V}$ as

$$x^{3}\hat{\boldsymbol{g}}_{3}\times\{\sigma^{ij}|_{i}\hat{\boldsymbol{g}}_{j}+X^{k}\boldsymbol{g}_{k}\}+\sigma^{ij}\hat{\boldsymbol{g}}_{i}\times\hat{\boldsymbol{g}}_{j}=0 \cdots\cdots(21)$$

Substituting Eq. (13) into Eq. (21) renders the first term of the left hand side of Eq. (21) zero and thus the second term leads to

$$\sigma^{ij}=\sigma^{ji} \cdots\cdots\cdots(22)$$

which indicates that stress tensor σ^{ij} is symmetric.

Multiplying Eq. (21) by $d\hat{V}$ and intergrating with respect to x^3 as before, while noting Eq. (22), yield

$$(-\hat{M}^{\beta\alpha}||_{\beta}+\hat{Q}^{\alpha 3}+\hat{Q}_{\alpha})\hat{\boldsymbol{a}}^{\alpha} + \int_{\hat{V}}\sqrt{\hat{a}}\,\hat{\mu}(\sigma^{12}-\sigma^{21})d\hat{V}\hat{\boldsymbol{a}}^{3}=0 \cdots\cdots\cdots(23)$$

Additional consideration of Eq. (22) on Eq. (23) gives the following equilibrium equation of moment;

$$(-\hat{M}^{\beta\alpha}||_{\beta}+\hat{Q}^{\alpha 3}+\hat{Q}_{\alpha})\hat{\boldsymbol{a}}^{\alpha}=0 \cdots\cdots\cdots(24)$$

in which

$$\left.\begin{aligned} \hat{M}^{\alpha\beta} &= -\int_{-h/2}^{h/2}\sigma^{\alpha\delta}\hat{\mu}_{\delta}{}^{\beta}\hat{\mu}x^{3}dx^{3} \\ \hat{Q}_{\alpha} &= [{}_{s}\sigma^{3\beta}\hat{\mu}_{\beta}{}^{\alpha}\hat{\mu}x^{3}]_{-h/2}^{h/2} \\ &\quad +\int_{-h/2}^{h/2}x^{3}\hat{\mu}(X^{\beta}\mu_{\beta}{}^{\gamma}\beta_{\gamma}{}^{\hat{\delta}}+X^{3}\beta_{3}{}^{\hat{\delta}})\hat{\epsilon}_{\delta\alpha}dx^{3} \end{aligned}\right\} \cdots\cdots\cdots(25\cdot\text{a, b})$$

Symbol $\hat{M}^{\alpha\beta}$ is understood as the $\epsilon_{\beta\gamma}\hat{\boldsymbol{a}}^{\gamma}$- component of resulting moment per unit length of the middle surface acting on the section normal to $\hat{\boldsymbol{a}}_{\alpha}$, in which $e_{\alpha\beta}$ is the permutation symbol of second order, and $\hat{Q}_{\alpha}$ is resulting external force.

Equilibrium equations (17) and (24) resulting from Eq. (13) and Eq. (21) respectively have been obtained by a similar procedure as employed by Flügge[7] for the small displacements of shells.

As indicated by Eq. (23), the $\hat{\boldsymbol{a}}_{3}$-component of moment is always equal to zero. Therefore, the number of independent equilibrium equations becomes five in all, three from Eq. (17) and two from Eq. (24). Those equilibrium equations in finite displacements agree, though expressed differently, with the results given by Sanders[1].

(2) Mechanical Boundary Conditions

Consider a deformed infinitesimal triangular element which is surrounded by boundary $ds\hat{\bar{\boldsymbol{g}}}_{2}$ and coordinates $dx^{i}\hat{\boldsymbol{g}}_{i}$ as shown by the shaded portion in **Fig. 2**. The equilibrium of forces acting on sections dA_{i} and dA for this element is given by

$$\boldsymbol{T}dA-\boldsymbol{\sigma}_{\alpha}dA_{\alpha}-\boldsymbol{\sigma}_{3,3}dA_{3}dx^{3}=0 \cdots\cdots\cdots(26)$$

in which

$$\left.\begin{aligned} dA &= \mu\epsilon_{\bar{1}\bar{2}}dsdx^{3} \\ dA_{\alpha} &= \mu\epsilon_{\alpha\beta}dx^{\beta}dx^{3} \\ dA_{3} &= (1/2)\mu\epsilon_{12}dx^{1}dx^{2} \end{aligned}\right\} \cdots\cdots\cdots(27\cdot\text{a}\sim\text{c})$$

Symbol $\boldsymbol{T}$ is external force per underformed unit area acting on the boundary, and $\boldsymbol{\sigma}_{\alpha}$ is stress vector acting on section A_{α}, both of which are decomposed to

$$\boldsymbol{T}=T^{\bar{1}k}\boldsymbol{g}_{k}\,,\quad \boldsymbol{\sigma}_{\alpha}=\sigma^{\alpha i}\hat{\boldsymbol{g}}_{i} \cdots\cdots\cdots(28\cdot\text{a, b})$$

Symbol $(^{-})$ at the top of suffix indicates quantities regarding boundary coordinate $\bar{\boldsymbol{g}}_{i}$. For example, $\epsilon_{\bar{1}\bar{2}}$ is component of permutation tensor of Eq. (15) defined now for the boundary coordinates.

Substituting Eqs. (27) and (28) into Eq. (26) and integrating with respect to x^3 give

$$N^{\bar{1}\alpha}\boldsymbol{a}_{\alpha}-Q^{\bar{1}3}\boldsymbol{a}_{3}=\{\epsilon'_{\alpha\gamma}\nu^{\gamma}(\hat{N}^{\alpha\beta}\hat{\boldsymbol{a}}_{\beta}-\hat{Q}^{\alpha 3}\hat{\boldsymbol{a}})\} \cdots(29)$$

in which

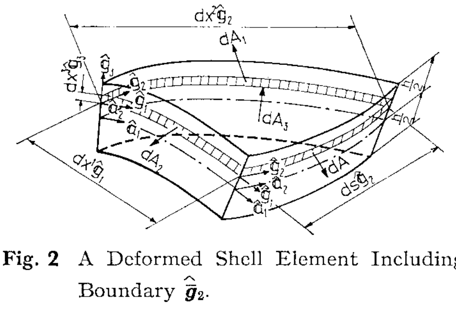

Fig. 2 A Deformed Shell Element Including Boundary $\hat{\bar{\boldsymbol{g}}}_{2}$.

$$\left.\begin{aligned} N^{\bar{1}\alpha} &= \int_{-h/2}^{h/2} T^{\bar{1}\delta}\hat{\mu}\mu_{\alpha}{}^{\delta} dx^3 \\ Q^{\bar{1}3} &= -\int_{-h/2}^{h/2} T^{\bar{1}3}\hat{\mu} dx^3 \end{aligned}\right\} \cdots\cdots\cdots\cdots(30\cdot\text{a, b})$$

Symbols $N^{\bar{1}\alpha}$ and $Q^{\bar{1}3}$ are components of external force $\boldsymbol{T}$ on the boundary, which correspond to stress resultants as defined by Eqs. (18·a, b). Symbols $\epsilon'_{\alpha\beta}$ and ν^{γ} that appear in Eq. (29) are defined by

$$\left.\begin{aligned} \epsilon'_{\alpha\beta} &= \epsilon_{\alpha\beta}/\epsilon_{\bar{1}\bar{2}} \\ \nu^{\gamma} &= dx^{\gamma}/ds \end{aligned}\right\} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(31\cdot\text{a, b})$$

In the derivation of Eq. (29), the integral of $\boldsymbol{\sigma}_{3,3}dA_3dx^3$ has been neglected because of higher order infinitesimal quantity and μ in Eqs. (27) has been replaced by $\hat{\mu}$ without detriment to the consequence, in order for the expression of Eq. (29) to be consistent with that of Eq. (17).

As for the derivation of Eq. (24), the equilibrium of moment for the triangular element can be obtained from Eq. (26) as

$$M^{\bar{1}i}\beta_i{}^{\hat{\beta}}\hat{\epsilon}_{\beta\delta}\hat{\boldsymbol{a}}^{\delta} = \hat{M}^{\alpha\beta}\nu^{\gamma}\epsilon'_{\alpha\gamma}\hat{\epsilon}_{\beta\delta}\hat{\boldsymbol{a}}^{\delta} \cdots\cdots\cdots\cdots\cdots\cdots(32)$$

in which

$$\left.\begin{aligned} M^{\bar{1}\alpha} &= -\int_{-h/2}^{h/2} T^{\bar{1}\delta}\mu_{\delta}{}^{\alpha}\hat{\mu}x^3 dx^3 \\ M^{\bar{1}3} &= -\int_{-h/2}^{h/2} T^{\bar{1}3}\hat{\mu}x^3 dx^3 \end{aligned}\right\} \cdots\cdots(33\cdot\text{a, b})$$

Symbol $M^{\bar{1}i}$ is component of external force $\boldsymbol{T}$ on the boundary, which corresponds to resulting moment as defined by Eq. (25·a).

Equations (29) and (32) give the mechanical boundary conditions, the number of which totals five. Mathematically, however, four boundary conditions are necessary and sufficient from the order of the corresponding differential equations. Thus as discussed by Kirchhoff[10], torsional moment in Eq. (32) must be replaced by the equivalent couple of forces and added to the boundary condition of forces of Eq. (29).

For the simplicity in writing, the left and right hand sides of Eq. (29) are denoted by $\boldsymbol{F}$ and $\tilde{\boldsymbol{F}}$, as given by

$$\left.\begin{aligned} \boldsymbol{F} &= N^{\bar{1}\alpha}\boldsymbol{a}_{\alpha} - Q^{\bar{1}3}\boldsymbol{a}_3 = F^{\bar{1}j}\boldsymbol{a}_j \\ \tilde{\boldsymbol{F}} &= \epsilon'_{\alpha\beta}\nu^{\gamma}(\hat{N}^{\alpha\beta}\hat{\boldsymbol{a}}_{\beta} - \hat{Q}^{\alpha3}\hat{\boldsymbol{a}}_3) = \tilde{F}^{1j}\hat{\boldsymbol{a}}_j \end{aligned}\right\}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(34\cdot\text{a, b})$$

which express external forces and stress resultants respectively. In a similar way, the left and right hand sides of Eq. (32) are denoted by

$$\left.\begin{aligned} \boldsymbol{L} &= M^{\bar{1}i}\beta_i{}^{\hat{\beta}}\hat{\epsilon}_{\beta\delta}\hat{\boldsymbol{a}}^{\delta} = L_{\bar{1}\bar{\delta}}\hat{\bar{\boldsymbol{a}}}^{\delta} \\ \tilde{\boldsymbol{L}} &= \hat{M}^{\alpha\beta}\nu^{\gamma}\epsilon'_{\alpha\gamma}\hat{\epsilon}_{\beta\delta}\hat{\boldsymbol{a}}^{\delta} = \tilde{L}_{\bar{1}\bar{\delta}}\hat{\bar{\boldsymbol{a}}}^{\delta} \end{aligned}\right\} \cdots\cdots\cdots(35\cdot\text{a, b})$$

respectively, in which

$$\left.\begin{aligned} L_{\bar{1}\bar{\delta}} &= M^{\bar{1}i}\beta_i{}^{\hat{\gamma}}\hat{\epsilon}_{\gamma\zeta}\beta_{\hat{\bar{\delta}}}{}^{\zeta} \\ \tilde{L}_{\bar{1}\bar{\delta}} &= \hat{M}^{\alpha\beta}\nu\epsilon'_{\alpha\gamma}\hat{\epsilon}_{\beta\zeta}\beta_{\hat{\bar{\delta}}}{}^{\zeta} \end{aligned}\right\} \cdots\cdots\cdots\cdots\cdots(36\cdot\text{a, b})$$

The $\hat{\bar{\boldsymbol{a}}}^1$-component of moment identified as torsional moment in Eq. (35) is then replaced by a couple of forces with the arm of $\hat{\bar{\boldsymbol{a}}}_2 ds$ on the boundary, which are equivalent forces denoted by $\boldsymbol{R}$ and $\tilde{\boldsymbol{R}}$ as

$$\left.\begin{aligned} \boldsymbol{R} &= R^{\bar{1}\bar{i}}\hat{\bar{\boldsymbol{a}}}_i \\ \tilde{\boldsymbol{R}} &= \tilde{R}^{\bar{1}\bar{i}}\hat{\bar{\boldsymbol{a}}}_i \end{aligned}\right\} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(37\cdot\text{a, b})$$

As detailed in the Appendix, $R^{\bar{1}\bar{i}}$ in Eqs. (37) can be expressed by

$$\begin{aligned} R^{\bar{1}\bar{i}}\hat{\bar{\boldsymbol{a}}}_i &= -(L_{\bar{1}\bar{1}}\hat{\boldsymbol{a}}_3)_{,\bar{2}}/\hat{\epsilon}_{\bar{1}\bar{2}} \\ &= \{L_{\bar{1}\bar{1}}\hat{b}_{\bar{2}}{}^{\bar{\alpha}}\hat{\bar{\boldsymbol{a}}}_{\alpha} - (L_{\bar{1}\bar{1},\bar{2}} + L_{\bar{1}\bar{1}}\hat{\Gamma}^{\bar{1}}{}_{\bar{2}\bar{1}})\hat{\boldsymbol{a}}_3\}/\hat{\epsilon}_{\bar{1}\bar{2}} \end{aligned}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(38)$$

in which $\Gamma^{\gamma}{}_{\alpha\beta}$ is Christoffel's symbol defined on the middle surface by

$$\Gamma^{\gamma}_{\alpha\beta} = \boldsymbol{a}_{\alpha,\beta}\boldsymbol{a}^{\gamma} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(39)$$

Proceeding as for Eq. (38) leads to a similar expression for $\tilde{R}^{\bar{1}\bar{i}}$, in which $L_{\bar{1}\bar{1}}$ is replaced by $\tilde{L}_{\bar{1}\bar{1}}$.

Adding Eqs. (37) to both sides of Eq. (29) finally yields the following mechanical boundary conditions of forces with the expressions of Eqs. (34);

$$(F^{\bar{1}j} + R^{\bar{1}\bar{i}}\beta_{\hat{\bar{i}}}{}^{j})\boldsymbol{a}_j = (\tilde{F}^{1j}\beta_{\hat{i}}{}^{j} + \tilde{R}^{\bar{1}\bar{i}}\beta_{\hat{\bar{i}}}{}^{j})\boldsymbol{a}_j \cdots\cdots(40)$$

As for the boundary condition of moment, only the $\hat{\bar{\boldsymbol{a}}}^2$-component of $\boldsymbol{L}$ in Eqs. (35) becomes independent, and thus the remaining mechanical boundary condition is given by

$$L_{\bar{1}\bar{2}}\hat{\bar{\boldsymbol{a}}}^2 = \tilde{L}_{\bar{1}\bar{2}}\hat{\bar{\boldsymbol{a}}}^2 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(41)$$

Eventually, the mechanical boundary conditions are expressed by four independent Eqs. (40) and (41), which are mathematically consistent with the corresponding differential equations.

4. GEOMETRICAL BOUNDARY CONDITIONS

The mechanical boundary conditions of Eqs. (40) and (41) are here inclusively expressed by

$$\boldsymbol{Q} = \tilde{\boldsymbol{Q}} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(42)$$

Denoting general displacement vector such as displacement and rotation by $\boldsymbol{w}$, and work done by external force $\boldsymbol{Q}$ by W, the following relation holds;

$$\boldsymbol{Q}d\boldsymbol{w} = dW \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(43)$$

From Eq. (43), the geometrical boundary conditions are generally given by

$$\boldsymbol{w} = \boldsymbol{w}_0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(44)$$

in which $\boldsymbol{w}_0$ is the prescribed displacement on the boundary.

Displacement $\boldsymbol{v}$ at an arbitrary point of a shell is expressed using displacement $\boldsymbol{u}$ on the middle surface by

$$\boldsymbol{v}=\boldsymbol{u}+x^3(\hat{\boldsymbol{a}}_3-\boldsymbol{a}_3) \quad \cdots\cdots(45\cdot\text{a})$$

therefore

$$d\boldsymbol{v}=d\boldsymbol{u}+x^3d\hat{\boldsymbol{a}}_3 \quad \cdots\cdots(45\cdot\text{b})$$

in which

$$d\boldsymbol{u}=du_i\boldsymbol{a}^i \quad \cdots\cdots(45\cdot\text{c})$$

Infinitesimal quantity $d\boldsymbol{\omega}$ representing rotation is now defined by

$$d\hat{\boldsymbol{a}}_3=d\boldsymbol{\omega}\times\hat{\boldsymbol{a}}^3 \quad \cdots\cdots(46\cdot\text{a})$$

in which

$$d\boldsymbol{\omega}=d\omega^i\hat{\boldsymbol{a}}_i \quad \cdots\cdots(46\cdot\text{b})$$

By the definition of Eqs. (46), $d\omega^{\bar{\alpha}}$ is given by

$$d\omega^{\bar{\alpha}}=\hat{\boldsymbol{a}}_\beta d\hat{\boldsymbol{a}}_3/\hat{\epsilon}_{\bar{\beta}\bar{\alpha}} \quad \cdots\cdots(47)$$

Meanwhile, the Kirchhoff-Love hypothesis of Eq. (11·a) leads to

$$\hat{\boldsymbol{a}}_\beta\hat{\boldsymbol{a}}_3=0 \quad \cdots\cdots(48)$$

Differentiation of Eq. (48) yields

$$d\hat{\boldsymbol{a}}_\beta\hat{\boldsymbol{a}}_3+\hat{\boldsymbol{a}}_\beta d\hat{\boldsymbol{a}}_3=0 \quad \cdots\cdots(49)$$

Substituting Eq. (49) into Eq. (47) leads to

$$d\omega^{\bar{\alpha}}=-\hat{\boldsymbol{a}}_3d\hat{\boldsymbol{a}}_\beta/\hat{\epsilon}_{\bar{\beta}\bar{\alpha}} \quad \cdots\cdots(50)$$

Using the above expressions regarding displacements and rotations, the meanings of Eq. (43) are examined hereinafter. The equilibrium equation of an infinitesimal triangular element with the boundary has been given by Eq. (26). With Eq. (26), a virtual work done by virtual displacement $d\boldsymbol{v}$ becomes zero, and thus the following equation holds;

$$\oint_C\int_{-h/2}^{h/2}(\boldsymbol{T}dA-\boldsymbol{\sigma}_\alpha dA_\alpha-\boldsymbol{\sigma}_{3,3}dx^3dA_3)d\boldsymbol{v}=0 \quad \cdots\cdots(51)$$

in which integration is performed along the closed boundary C and thickness h.

Substitution of Eqs. (45·b) and (46·a) into Eq. (51) leads to

$$\oint_C\int_{-h/2}^{h/2}\boldsymbol{\sigma}_\alpha dA_\alpha d\boldsymbol{u}+\oint_C\int_{-h/2}^{h/2}x^3\hat{\boldsymbol{a}}_3\times\boldsymbol{\sigma}_\alpha dA_\alpha d\boldsymbol{\omega}$$
$$=\oint_C\int_{-h/2}^{h/2}\boldsymbol{T}dAd\boldsymbol{u}+\oint_C\int_{-h/2}^{h/2}(x^3\hat{\boldsymbol{a}}_3\times\boldsymbol{T}dA)d\boldsymbol{\omega} \quad \cdots\cdots(52)$$

in which integration of term $\boldsymbol{\sigma}_{3,3}dx^3dA_3$ has been reduced to zero as explained before for Eqs. (29) and (32).

The terms of external forces in the right hand side of Eq. (52) are now examined. Use of Eqs. (34·a) and (35·a) transforms the right hand side of Eq. (52) to

$$\oint_C\int_{-h/2}^{h/2}\boldsymbol{T}dAd\boldsymbol{u}+\oint_C\int_{-h/2}^{h/2}(x^3\hat{\boldsymbol{a}}_3\times\boldsymbol{T}dA)d\boldsymbol{\omega}$$
$$=\oint_C F^{\bar{1}i}du_jds+\oint_C L_{\bar{1}\bar{\alpha}}d\omega^{\bar{\alpha}}ds \quad \cdots\cdots(53)$$

Torsional moment $L_{\bar{1}\bar{1}}$ in Eq. (53) needs to be replaced by equivalent force $\boldsymbol{R}$ of Eq. (38). Denoting displacement corresponding to the equivalent force $\boldsymbol{R}$ by $\boldsymbol{u}^*$ and with integration by parts for the expression of Eq. (38) for $\boldsymbol{R}$, equivalence of work done by $L_{\bar{1}\bar{1}}$ and that by $\boldsymbol{R}$ leads to

$$\oint_C L_{\bar{1}\bar{1}}d\omega^{\bar{1}}ds=\oint_C\boldsymbol{R}d\boldsymbol{u}^*ds$$
$$=\oint_C L_{\bar{1}\bar{1}}\hat{\boldsymbol{a}}_3d\boldsymbol{u}^*{}_{,\bar{2}}/\hat{\epsilon}_{\bar{1}\bar{2}}ds \quad \cdots\cdots(54)$$

Substituting Eq. (50) into $d\omega^{\bar{1}}$ in Eq. (54) and comparing the left and the right hand sides of Eq. (54), the following relation holds;

$$d\boldsymbol{u}^*=d\boldsymbol{u} \quad \cdots\cdots(55)$$

Thus, Eq. (54) is reduced to

$$\oint_C L_{\bar{1}\bar{1}}d\omega^{\bar{1}}ds=\oint_C\boldsymbol{R}d\boldsymbol{u}ds \quad \cdots\cdots(56)$$

Substitution of Eqs. (56) and (37·a) into Eq. (53) finally leads to the following expression for the right hand side of Eq. (52);

$$\oint_C\int_{-h/2}^{h/2}\boldsymbol{T}dAd\boldsymbol{u}+\oint_C\int_{-h/2}^{h/2}(x^3\hat{\boldsymbol{a}}_3\times\boldsymbol{T}dA)d\boldsymbol{\omega}$$
$$=\oint_C(F^{\bar{1}j}+R^{\bar{1}\bar{i}}\beta_{\hat{\bar{i}}}{}^j)du_jds+\oint_C L_{\bar{1}\bar{2}}d\omega^{\bar{2}}ds \quad \cdots\cdots(57)$$

A similar procedure that can be performed for internal forces of the left hand side of Eq. (52) leads to the expression with terms du_j and $d\omega^{\bar{2}}$ as in Eq. (57). From this operation, the geometrical boundary conditions which correspond to the mechanical boundary conditions of Eqs. (39) and (40) are eventually given by

$$\left.\begin{aligned}u_j&={}_0u_j\\ \omega^{\bar{2}}&={}_0\omega^{\bar{2}}\end{aligned}\right\} \quad \cdots\cdots(58\cdot\text{a, b})$$

in which ${}_0u_j$ and ${}_0\omega^{\bar{2}}$ are the prescribed displacement and rotation on the boundary.

The mathematical expression for rotation $\omega^{\bar{2}}$ that appears in Eq. (58·b) has been given only by the differential form of Eq. (47) or (50). From those relations, the explicit expression of $\omega^{\bar{2}}$ is not tried to be found. Displacement vector $\boldsymbol{u}$ on the boundary is expressed by

$$\boldsymbol{u}=u^{\bar{i}}\bar{\boldsymbol{a}}_i \quad \cdots\cdots(59)$$

Using Eq. (59), deformed base vector $\hat{\boldsymbol{a}}_\alpha$ on the boundary is given by

$$\hat{\boldsymbol{a}}_\alpha=(\delta_\alpha{}^i+u^{\bar{i}}|_{\bar{\alpha}})\bar{\boldsymbol{a}}_i \quad \cdots\cdots(60)$$

By the assumptions of the Kirchhoff-Love hypothesis and unchanged thickness of Eqs. (11), the deformed normal base vector is

$$\hat{\boldsymbol{a}}_3=(\hat{\bar{\boldsymbol{a}}}_1\times\hat{\bar{\boldsymbol{a}}}_2)/|\hat{\bar{\boldsymbol{a}}}_1\times\hat{\bar{\boldsymbol{a}}}_2| \quad\cdots\cdots(61)$$

Base vector $\hat{\boldsymbol{a}}_3$ of Eq. (61) is decomposed to

$$\hat{\boldsymbol{a}}_3=\beta_{\hat{3}\bar{i}}\bar{\boldsymbol{a}}^i \quad\cdots\cdots(62)$$

Using Eqs. (60) and (61), $\beta_{\hat{3}\bar{1}}$ in Eq. (62) is expressed in terms of displacements as

$$\left.\begin{aligned}\beta_{\hat{3}\bar{1}}&=\sqrt{\bar{a}/\hat{\bar{a}}}\,(-u^3|_{\bar{1}}-u^3|_{\bar{1}}u^{\bar{2}}|_{\bar{2}}+u^{\bar{2}}|_{\bar{1}}u^3|_{\bar{2}})\\ \beta_{\hat{3}\bar{2}}&=\sqrt{\bar{a}/\hat{\bar{a}}}\,(-u^3|_{\bar{2}}-u^3|_{\bar{2}}u^{\bar{1}}|_{\bar{1}}+u^3|_{\bar{1}}u^{\bar{1}}|_{\bar{2}})\\ \beta_{\hat{3}\bar{3}}&=\sqrt{\bar{a}|\hat{\bar{a}}}\,(1+u^{\bar{1}}|_{\bar{1}}+u^{\bar{2}}|_{\bar{2}}+u^{\bar{1}}|_{\bar{1}}u^{\bar{2}}|_{\bar{2}}-u^{\bar{2}}|_{\bar{1}}u^{\bar{1}}|_{\bar{2}})\end{aligned}\right\} \quad\cdots\cdots(63\cdot\mathrm{a}\sim\mathrm{c})$$

in which

$$a=(\boldsymbol{a}_1\times\boldsymbol{a}_2)^2=\det|a_{\alpha\beta}| \quad\cdots\cdots(64)$$

Meanwhile, using Eqs. (15) and (50), $d\omega^{\bar{2}}$ is transformed to

$$d\omega^{\bar{2}}=-\hat{\boldsymbol{a}}_3 d\hat{\bar{\boldsymbol{a}}}_1/\sqrt{\hat{\bar{a}}} \quad\cdots\cdots(65)$$

Substitution of Eq. (62) into Eq. (65) yields

$$d\omega^{\bar{2}}=-\beta_{\hat{3}\bar{i}}du^{\bar{i}}|_{\bar{1}}/\sqrt{\hat{\bar{a}}} \quad\cdots\cdots(66)$$

Since it seems difficult from Eq. (66) to express $\omega^{\bar{2}}$ as a function of $u^{\bar{i}}$, the following artificial condition is introduced;

$$\beta_{\hat{3}\bar{i}}/\sqrt{\hat{\bar{a}}}=\text{constant} \quad\cdots\cdots(67)$$

Use of Eq. (67) for Eq. (66) leads to

$$d\omega^{\bar{2}}=d(-\beta_{\hat{3}\bar{i}}u^{\bar{i}}|_{\bar{1}}/\sqrt{\hat{\bar{a}}}) \quad\cdots\cdots(68)$$

With the help of Eqs. (63), the differential form of Eq. (68) finally results in

$$\omega^{\bar{2}}=\beta_{\hat{3}\bar{1}}/\sqrt{\hat{\bar{a}}} \quad\cdots\cdots(69)$$

in which integration constant is trivial and thus ignored. Equation (69) is the substantial form of rotation $\omega^{\bar{2}}$ to be used for the expression of the geometrical boundary condition of Eq. (58·b).

As obvious from Eq. (62), the artificial condition of Eq. (67) physically means that the value of $\hat{\boldsymbol{a}}^3/\sqrt{\hat{\bar{a}}}$ be given on the boundary. Meanwhile, as indicated by Eqs. (62) and (69), $\omega^{\bar{2}}$ is the $\bar{\boldsymbol{a}}^1$-component of $\hat{\boldsymbol{a}}^3/\sqrt{\hat{\bar{a}}}$ irrespective of the $\bar{\boldsymbol{a}}^2$- and $\bar{\boldsymbol{a}}^3$- components as given by $\beta_{\hat{3}\bar{2}}/\sqrt{\hat{\bar{a}}}$ and $\beta_{\hat{3}\bar{3}}/\sqrt{\hat{\bar{a}}}$ respectively. Thus, the geometrical boundary condition given by $\omega^{\bar{2}}$ is equivalent to the value only of $\beta_{\hat{3}\bar{1}}/\sqrt{\hat{\bar{a}}}$ being given. Saying differently, the condition of Eq. (67) is automatically satisfied when Eq. (69) is used for the geometrical boundary condition for bending moment and the values of $\beta_{\hat{3}\bar{2}}/\sqrt{\hat{\bar{a}}}$ and $\beta_{\hat{3}\bar{3}}/\sqrt{\hat{\bar{a}}}$ are understood as arbitrary constants. By this reasoning, the values of $\beta_{\hat{3}\bar{2}}/\sqrt{\hat{\bar{a}}}$ and $\beta_{\hat{3}\bar{3}}/\sqrt{\hat{\bar{a}}}$ can be interpreted as the resulting $\bar{\boldsymbol{a}}^2$- and $\bar{\boldsymbol{a}}^3$- components of $\hat{\boldsymbol{a}}^3/\sqrt{\hat{\bar{a}}}$ respectively, calculated from the prescribed rotation of ${}_0\omega^{\bar{2}}$.

In the publications available, the variational principle has often been used for the derivation of the boundary conditions of shells. Sanders[1] used the principle of virtual work which was obtained from the equilibrium equations multiplied by corresponding virtual displacements and rotations. The boundary condition obtained thus constituted of five equations both for mechanical and geometrical boundary conditions. Four boundary equations necessary and sufficient to solve the field equation were only acquired with the help of small strains and small rotations, that is, the boundary conditions obtained thus are valid only for small strains and small rotations.

Washizu[5] made a similar derivation as above, and thus suggested that the geometrical boundary conditions for small displacements are applicable also for finite displacements. Noting, however, that the geometrical boundary condition regarding rotation includes the second order term of $u^{\bar{i}}|_\alpha$ in $\beta_{\hat{3}\bar{1}}$ as seen in Eq. (63·a), the geometrical boundary conditions differ for small and finite displacements.

The use of the finite displacement field equations and the small displacement boundary conditions is inconsistent and thus the solution may not be satisfactory for finite displacement problems.

Nishimura[4] made a formulation of shells based on the principle of minimum potential energy, but the number of the resulting mechanical and geometrical boundary conditions remained six for both of them.

The application of the variational principle as reported in literature[1),5)] seems to have failed in presenting the complete form of the boundary conditions for finite displacements. The reason may come from tedious and cumbersome manipulations involved. The formulation presented in this paper has first made it possible to derive the accurate boundary conditions for finite displacements.

5. CONSTITUTIVE EQUATIONS

The stress-strain relationship for isotropic homogeneous elastic continua is given by[7]

$$E\eta_j{}^i=(1+\nu)\sigma_j{}^i-\nu\sigma_m{}^m\delta_j{}^i \quad\cdots\cdots(70)$$

in which E and ν are Young's modulus and Poisson's ratio respectively. For elastic thin shells, the following plane stress state is postulated

$$\sigma_3{}^\alpha=\sigma_\alpha{}^3=\sigma_3{}^3=0 \quad\cdots\cdots(71)$$

Substitution of Eq. (71) into Eq. (70) leads to

$$\sigma_\gamma{}^\alpha=\bar{E}\{(1-\nu)\eta_\gamma{}^\alpha+\nu\delta_\gamma{}^\alpha\eta_\zeta{}^\zeta\} \quad \cdots\cdots(72)$$

in which

$$\bar{E}=E/(1-\nu^2) \quad \cdots\cdots(73)$$

Contravariant component $\sigma^{\alpha\delta}$ of stress tensor can be transformed from mixed-variance component $\sigma_\delta{}^\alpha$ to

$$\sigma^{\alpha\delta}=\sigma_\gamma{}^\alpha\hat{\lambda}_\rho{}^\gamma\hat{\lambda}_\sigma{}^\delta\hat{a}^{\rho\sigma} \quad \cdots\cdots(74)$$

in which $\lambda_\beta{}^\alpha$ that has been defined by Eq. (20·a) is expressed here by

$$\hat{\lambda}_\beta{}^\alpha=\delta_\beta{}^\alpha+\hat{b}_\beta{}^\alpha x^3+\hat{b}_\gamma{}^\alpha\hat{b}_\beta{}^\gamma(x^3)^2+\cdots \quad \cdots\cdots(75)$$

Substituting Eq. (74) into Eqs. (18·a) and (25·a), stress resultants are expressed in terms of metric tensor $a^{\alpha\beta}$, strain tensor $\eta_{\alpha\beta}$ and tensor $\lambda_\beta{}^\alpha$ as

$$\begin{Bmatrix}\hat{N}^{\alpha\beta}\\ \hat{M}^{\alpha\beta}\end{Bmatrix}=\bar{E}\hat{a}^{\rho\beta}\hat{a}^{\mu\nu}\int_{-h/2}^{h/2}\{(1-\nu)\delta_\gamma{}^\zeta\hat{\lambda}_\mu{}^\alpha+\nu\delta_\gamma{}^\alpha\hat{\lambda}_\mu{}^\zeta\}\eta_{\zeta\delta}\hat{\lambda}_\nu{}^\delta\hat{\lambda}_\rho{}^\gamma\hat{\mu}\begin{Bmatrix}1\\ -x^3\end{Bmatrix}dx^3 \quad \cdots\cdots(76)$$

Substitution of Eq. (75) into (76) and integration with respect to x^3 give the explicit form of the stress-strain relationship called constitutive equations.

Symbol $\hat{b}_\beta{}^\alpha$ that appears in the right hand side of Eq. (75) is the curvature tensor of the deformed middle surface as defined by Eq. (6). The n-th term of the right hand side of Eq. (75) is small quantity of the (n-1)th order of (thickness/radius of curvature). By this reason, the higher order terms may be neglected without any significant loss of accuracy in the expression of the constitutive Eq. (76). Retaining the first two terms, the constitutive equation becomes

$$\begin{Bmatrix}\hat{N}^{\alpha\beta}\\ \hat{M}^{\alpha\beta}\end{Bmatrix}=\bar{E}\hat{a}^{\rho\beta}\hat{a}^{\mu\nu}\begin{Bmatrix}hI_1+(h^3/12)I_3\\ -(h^3/12)I_2\end{Bmatrix} \quad \cdots\cdots(77)$$

in which symbols I_1, I_2 and I_3 are given by

$$I_i=(1-\nu)D^1_{i-1}+\nu D^2_{i-1} \quad (i=1\sim3) \quad \cdots\cdots(78)$$

with

$$\left.\begin{aligned}
&D_0{}^1=\eta_{0\rho\nu}\delta_\mu{}^\alpha\\
&D_0{}^2=\eta_{0\mu\nu}\delta_\rho{}^\alpha\\
&D_1{}^1=\eta_{0\rho\nu}\hat{b}_\mu{}^\alpha+\delta_\mu{}^\alpha(\eta_{0\rho\delta}\hat{b}_\nu{}^\delta+\eta_{0\gamma\nu}\hat{b}_\rho{}^\gamma-\eta_{0\rho\nu}\hat{b}_\lambda{}^\lambda+\kappa_{1\rho\nu})\\
&D_1{}^2=\eta_{0\mu\nu}\hat{b}_\rho{}^\alpha+\delta_\rho{}^\alpha(\eta_{0\mu\delta}\hat{b}_\nu{}^\delta+\eta_{0\gamma\nu}\hat{b}_\mu{}^\gamma-\eta_{0\mu\nu}\hat{b}_\lambda{}^\lambda+\kappa_{1\mu\nu})\\
&D_2{}^1=\kappa_{2\rho\nu}\delta_\mu{}^\alpha\\
&D_2{}^2=\kappa_{2\mu\nu}\delta_\rho{}^\alpha
\end{aligned}\right\} \cdots\cdots(79\cdot a\sim f)$$

in which

$$\left.\begin{aligned}
&\eta_{0\gamma\delta}=(\hat{a}_{\gamma\delta}-a_{\gamma\delta})/2\\
&\kappa_{1\gamma\delta}=-(\hat{b}_{\gamma\delta}-b_{\gamma\delta})\\
&\kappa_{2\rho\nu}=(\hat{b}_\rho{}^\varepsilon\hat{b}_{\nu\varepsilon}-b_\rho{}^\varepsilon b_{\nu\varepsilon})/2
\end{aligned}\right\} \cdots\cdots(80\cdot a\sim c)$$

The constitutive equations have been obtained by substituting the elastic stress-strain relation of the material into the definitions of the stress resultants as given by Eqs. (18·a) for $\hat{N}^{\alpha\beta}$ and Eq. (25·a) for $\hat{M}^{\alpha\beta}$. Seide[2] used similar definitions as above, but has expressed the resulting constitutive equations only in a vague form. The derivation of the constitutive equations above seems similar to that of Tosaka and Tsuboi[11]. However, it should be noted that Tosaka et al. used Cauchy's (Euler's) stress tensors for expressions, while this paper has presented the constitutive equations on the basis of Kirchhoff's stress tensors.

6. DISCUSSIONS

Throughout the procedure presented in this paper, the tensor quantities for the original state and for the deformed state are defined in an identical way. This definition at the deformed state has greatly facilitated the mathematical formulations of shells. The use of these tensor quantities together with the direct integration of equilibrium equations for three dimensional continua with respect to thickness has simplified the derivation of the finite displacement shell theory, compared with the use of the energy principles[2],[3].

Stress tensor σ^{ij} has been defined for the undeformed unit area. On the other hand, if it is defined for the deformed unit area, quantity $\sqrt{a/\hat{a}}\,(\mu/\hat{\mu})X^k$ must be used in place of body force X^k in Eq. (13), resulting in somewhat complicated expressions.

Integration for the equilibrium equations has been performed over the deformed domain of a shell. This is because the equilibrium Eqs. (17) and (24) are expressed by deformed tensor quantities such as $\hat{N}^{\alpha\beta}$ and $\hat{M}^{\alpha\beta}$. If integration is performed over the undeformed domain[5], the derivations of Eqs. (17) and (24) are much more complex.

In this paper, the metric tensors and curvature tensors of the middle surface at the deformed state have been chosen as unknowns. Position vector $\hat{\boldsymbol{s}}$ of the deformed middle surface is expressed as

$$\hat{\boldsymbol{s}}=\boldsymbol{s}+\boldsymbol{u} \quad \cdots\cdots(81\cdot a)$$

with

$$\boldsymbol{u}=u^i\boldsymbol{a}_i \quad \cdots\cdots(81\cdot b)$$

using the undeformed position vector $\boldsymbol{s}$ and the displacement vector $\boldsymbol{u}$. Applying Eqs. (1) and (3) for Eq. (81) yields

$$\hat{\boldsymbol{a}}_\alpha=\hat{\boldsymbol{s}}_{,\alpha}=\boldsymbol{s}_{,\alpha}+\boldsymbol{u}_{,\alpha}=\boldsymbol{a}_\alpha+u^i|_\alpha\boldsymbol{a}_i \quad \cdots\cdots(82\cdot a)$$

$$\hat{\boldsymbol{a}}_3=\hat{\boldsymbol{a}}_1\times\hat{\boldsymbol{a}}_2/|\hat{\boldsymbol{a}}_1\times\hat{\boldsymbol{a}}_2| \quad \cdots\cdots(82\cdot b)$$

With the help of Eq. (82), the metric tensor $\hat{a}_{\alpha\beta}$

$=\hat{\boldsymbol{a}}_\alpha \hat{\boldsymbol{a}}_\beta$ and the curvature tensor $\hat{b}_\alpha{}^\beta$, $\hat{b}_{\alpha\beta}$ of Eq. (6) at the deformed state can easily be expressed in terms of the undeformed base vectors $\boldsymbol{a}_i$ and the displacement vector $\boldsymbol{u}$.

The deformed component normal to the middle surface in the equilibrium of moment of Eq. (23) is always equal to zero because of the symmetry of stress tensor. This is consistent with the fact that the number of unknowns is equal to that of the governing equations. This proof has first been systematically ascertained here for finite displacements, although it has widely been recognized for small displacements[7].

The exact boundary conditions including rotations have first been presented for finite displacements as given by Eqs. (40), (41), (58) and (69), in which the geometrical boundary conditions have been obtained with the aid of the principle of virtual work.

The constitutive equations have already been obtained by Seide[2], Naghdi and Nordgren[3], using the potential functions with some vague aspects for the assessment of the potential. On the other hand, the constitutive equations presented by Eq. (76) are rigorous under the Kirchhoff-Love hypothesis and the assumption of plane stress, when the series of Eq. (75) are considered up to infinitive.

A tensor analysis for elastic shells which this paper has described is an extention of that of Reference 7) by Flügge to the finite displacements of shells. There are, however, some differences. Flügge used displacements as unknowns, while, metric and curvature tensors have been chosen as unknowns in this paper. This facilitates to express the formulation in a simpler manner, and the physical meanings of the equations and quantities involved become obvious. The boundary conditions are not presented in Reference 7). As discussed before, the expressions of the boundary conditions of shells have been confused particularly for finite displacements. Thus the boundary conditions presented in this paper deserve to be noted.

7. CONCLUDING REMARKS

An unified finite displacement theory of elastic thin shells has been formulated under the Kirchhoff-Love hypothesis. Special attention has been given to the precise presentation of the boundary conditions and constitutive equations for which some questions remained in literature.

The field equations and the associated mechanical boundary conditions have been obtained by integrating the equilibrium equations of three dimensional continua with respect to thickness component of a shell. This method has been proved tractable and effective, compared with the use of the energy principles. The equilibrium of moment in the thickness direction after deformation has been found to be always satisfied.

The metric and curvature tensors on the middle surface of a shell have been used as unknowns, with which the derivation of the finite displacement shell theory becomes simple and similar to that for small displacements.

The geometrical boundary conditions consistent with the field equations and mechanical boundary conditions for finite displacements have been presented with the aid of the principle of virtual work.

Under the condition of plane stress, the constitutive equations for elastic shells have been obtained in the form of the relations of stress resultants and strains on the middle surface. The constitutive equations so obtained are expressed in the form of infinite series in a rigorous sense, but the use only of a few terms may guarantee a good accuracy because of thin thickness of shells.

8. APPENDIX: DERIVATION OF EQUIVALENT FORCE

Eq. (38) which expresses equivalent force $R^{\bar{1}\bar{1}}$ in terms of torsional moment $L_{\bar{1}\bar{1}}$ can be obtained as follows.

Among the moments acting on the boundary, the torsional moment $\boldsymbol{L}_{\bar{1}}$ of the $\hat{\bar{\boldsymbol{a}}}^1$-direction is considered as shown in **Fig. A·1** with the boundary coordinates $\hat{\bar{\boldsymbol{a}}}_\alpha$ and $\hat{\boldsymbol{a}}_3$. It is supposed that torsional moment $\boldsymbol{L}_{\bar{1}} d\bar{x}^2$ acts on point A on the boundary, while $(\boldsymbol{L}_{\bar{1}}+d\boldsymbol{L}_{\bar{1}})d\bar{x}^2$ acts on point B infinitesimally apart from point A, where $d\boldsymbol{L}_{\bar{1}}$ indicates the increment of $\boldsymbol{L}_{\bar{1}}$ toward the $\hat{\bar{\boldsymbol{a}}}_2$ -direction.

Torsional moment $\boldsymbol{L}_{\bar{1}} d\bar{x}^2$ is transformed to a couple of forces $\boldsymbol{Z}$ and $\bar{\boldsymbol{Z}}$ acting on the points apart from point A by $(1/2)\hat{\bar{\boldsymbol{a}}}_2 d\bar{x}^2$ in the negative and positive directions respectively, with additional force $\boldsymbol{S}$ acting on point A which results

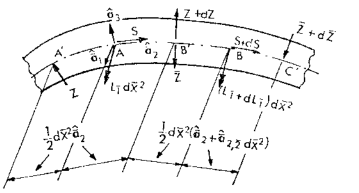

Fig. A·1 Boundary Coordinates and Equivalent Forces.

from the curvature of the middle surface of a shell. When the direction of forces $\boldsymbol{Z}$ and $\bar{\boldsymbol{Z}}$ is selected normal to the middle surface at point A, $\boldsymbol{Z}$ and $\bar{\boldsymbol{Z}}$ are expressed by

$$\left.\begin{aligned}\boldsymbol{Z}&=\tilde{Z}^{\bar{1}3}\hat{\boldsymbol{a}}_3-(1/2)(\tilde{Z}^{\bar{1}3}\hat{\boldsymbol{a}}_3)_{,\bar{2}}d\bar{x}^2\\ \bar{\boldsymbol{Z}}&=-\{\tilde{Z}^{\bar{1}3}\hat{\boldsymbol{a}}_3+(1/2)(\tilde{Z}^{\bar{1}3}\hat{\boldsymbol{a}}_3)_{,\bar{2}}d\bar{x}^2\}\end{aligned}\right\} \quad \cdots\cdots(\mathrm{A\cdot 1\ a,\ b})$$

Equality of torsional moment $\boldsymbol{L}_{\bar{1}}d\bar{x}^2$ with moment induced by a couple of forces $\boldsymbol{Z}$ and $\bar{\boldsymbol{Z}}$ leads to

$$\begin{aligned}\boldsymbol{L}_{\bar{1}}d\bar{x}^2&=L_{\bar{1}\bar{1}}\hat{\bar{\boldsymbol{a}}}^1d\bar{x}^2\\ &=-(1/2)(\hat{\bar{\boldsymbol{a}}}_2d\bar{x}^2)\times\boldsymbol{Z}+(1/2)(\hat{\bar{\boldsymbol{a}}}_2d\bar{x}^2)\times\bar{\boldsymbol{Z}}\\ &=-\hat{\epsilon}_{\bar{1}\bar{2}}\tilde{Z}^{\bar{1}3}d\bar{x}^2\hat{\bar{\boldsymbol{a}}}^1 \quad \cdots\cdots(\mathrm{A\cdot 2})\end{aligned}$$

When new symbol $Z^{\bar{1}3}$ is introduced by

$$Z^{\bar{1}3}=\tilde{Z}^{\bar{1}3}\hat{\epsilon}_{\bar{1}\bar{2}} \quad \cdots\cdots(\mathrm{A\cdot 3})$$

the following relation is obtained from Eq. (A·2);

$$Z^{\bar{1}3}=-L_{\bar{1}\bar{1}} \quad \cdots\cdots(\mathrm{A\cdot 4})$$

The sum of forces $\boldsymbol{Z}$, $\bar{\boldsymbol{Z}}$ and $\boldsymbol{S}$ should be equal to zero, as given by

$$\boldsymbol{Z}+\bar{\boldsymbol{Z}}+\boldsymbol{S}=0 \quad \cdots\cdots(\mathrm{A\cdot 5})$$

from the equivalence to torsional moment $\boldsymbol{L}_{\bar{1}}d\bar{x}^2$. Torsional moment $(\boldsymbol{L}_{\bar{1}}+d\boldsymbol{L}_{\bar{1}})d\bar{x}^2$ acting on point B can be transformed to a couple of forces $\boldsymbol{Z}+d\boldsymbol{Z}$ and $\bar{\boldsymbol{Z}}+d\bar{\boldsymbol{Z}}$, and additional force $\boldsymbol{S}+d\boldsymbol{S}$ in a similar way as above. Proceeding as for Eq. (A·5) leads to

$$d\boldsymbol{Z}+d\boldsymbol{S}+d\bar{\boldsymbol{Z}}=0 \quad \cdots\cdots(\mathrm{A\cdot 6})$$

with the use of Eq. (A·5).

In order to replace torsional moment $L_{\bar{1}}$ by equivalent force $\boldsymbol{R}$ on the boundary with the use of forces $\boldsymbol{Z}$, $\bar{\boldsymbol{Z}}$ and $\boldsymbol{S}$, resultant force acting on the small interval between points A and B is calculated as given by

$$\boldsymbol{R}d\bar{x}^2\equiv R^{\bar{1}i}\hat{\bar{\boldsymbol{a}}}_id\bar{x}^2=\boldsymbol{S}/2+\bar{\boldsymbol{Z}}+\boldsymbol{Z}+d\boldsymbol{Z}+(\boldsymbol{S}+d\boldsymbol{S})/2 \quad \cdots\cdots(\mathrm{A\cdot 7})$$

in which $\boldsymbol{R}$ is defined per unit length of the boundary. Forces $\boldsymbol{S}$ and $\boldsymbol{S}+d\boldsymbol{S}$ are halved because duplicate counting for adjacent intervals must be avoided.

Substitution of Eqs. (A·5) and (A·6) into Eq. (A·7) leads to

$$R^{\bar{1}i}\hat{\bar{\boldsymbol{a}}}_id\bar{x}^2=(d\boldsymbol{Z}-d\bar{\boldsymbol{Z}})/2 \quad \cdots\cdots(\mathrm{A\cdot 8})$$

Substitution of Eqs. (A·1) into Eq. (A·8), while noting Eq. (A·3), yields

$$R^{\bar{1}i}\hat{\bar{\boldsymbol{a}}}_i=(\tilde{Z}^{\bar{1}3}\hat{\boldsymbol{a}}_3)_{,\bar{2}} \quad \cdots\cdots(\mathrm{A\cdot 9})$$

Differentiation of the right hand side of Eq. (A·9) leads to

$$(\tilde{Z}^{\bar{1}3}\hat{\boldsymbol{a}}_3)_{,\bar{2}}=\tilde{Z}^{\bar{1}3}\hat{\Gamma}_{\bar{2}3}^{\bar{\alpha}}\hat{\bar{\boldsymbol{a}}}_\alpha+(\tilde{Z}_{,}{}^{\bar{1}3}{}_{\bar{2}}+\tilde{Z}^{\bar{1}3}\hat{\Gamma}^{\bar{1}}{}_{\bar{2}\bar{1}})\hat{\boldsymbol{a}}_3 \quad \cdots\cdots(\mathrm{A\cdot 10})$$

in which $\hat{\Gamma}^{\bar{\alpha}}{}_{\bar{2}3}$ is given by the definition of Chistoffel's symbol as

$$\hat{\Gamma}_{\bar{2}3}^{\bar{\alpha}}=-\hat{b}_{\bar{2}}{}^{\bar{\alpha}} \quad \cdots\cdots(\mathrm{A\cdot 11})$$

Expressing $\tilde{Z}^{\bar{1}3}$ of Eq. (A·10) by component $L_{\bar{1}\bar{1}}$ of torsional moment with the use of Eqs. (A·3) and (A·4), while noting Eq. (A·11), equivalent force $\boldsymbol{R}$ is finally obtained as follows.

$$R^{\bar{1}i}\hat{\bar{\boldsymbol{a}}}_i=\{L_{\bar{1}\bar{1}}\hat{b}_{\bar{2}}{}^{\bar{\alpha}}\hat{\bar{\boldsymbol{a}}}_\alpha-(L_{\bar{1}\bar{1},\bar{2}}+L_{\bar{1}\bar{1}}\hat{\Gamma}_{\bar{2}\bar{1}}{}^{\bar{1}})\hat{\boldsymbol{a}}_3\}/\hat{\epsilon}_{\bar{1}\bar{2}} \quad \cdots\cdots(\mathrm{A\cdot 12})$$

REFERENCES

1) Sanders, J. L.: Nonlinear Theories for Thin Shells, Quarterly of Applied Mathematics, Vol. 21, pp. 21–36, 1963.
2) Seide, P.: Some Implications of Strain Energy Expressions for the Love-Kirchhoff Nonlinear Theory of Plates and Shells, International Journal of Non-Linear Mechanics, Vol. 6, pp. 361–376, 1971.
3) Naghdi, P. M. and R. P. Nordgren: On the Nonlinear Theory of Elastic Shells under the Kirchhoff Hypothesis, Quarterly of Applied Mathematics, Vol. 21, pp. 49–59, 1963.
4) Nishimura, T.: Tensors and Shell Theories, Syokokusha Publishing Inc., Tokyo, 1977 (in Japanese).
5) Washizu, K.: Variational Methods in Elasticity and Plasticity, Pergamon Press Ltd., Oxford, 1968.
6) Koiter, W. T.: On the Nonlinear Theory of Thin Elastic Shells, Parts I, II and III, proceedings of the Koninkliyke Nederlandse Akademie van Wetenschappen, B69, pp. 1–54, 1966.
7) Flügge, W.: Tensor Analysis and Continuum Mechanics, Springer-Verlarg, Berlin, 1972.
8) Abe, H.: On Bending Terms in the Linear Thin Shell Equations in Terms of the Displacement Components, International Journal of Engineering Science, Vol. 13, pp. 1055–1065, 1975.
9) Shrivastava, J. P. and P. G. Glockner: Lagrangian Formulation of Static of Shells, Proceedings of the American Society of Civil Engineers, Vol. 96, No. EM5, pp. 547–563, 1970.
10) Timoshenko, S. P. and S. W. Krieger: Theory of Plates and Shells, 2nd Edition, McGraw-Hill, New York, 1959.
11) Tosaka, N. and Y. Tsuboi: The Nonlinear Theory of Thin Elastic Shells under the Kirchhoff-Love Hypotheses., Transactions of the Architectural Institute of Japan, No. 235, pp. 27–38, September, 1975.

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