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PROPOSAL OF A YIELD FUNCTION AND DESCRIPTION OF PLASTIC BEHAVIOR OF SOFT ROCKS

By Hiroyoshi HIRAI and Masao SATAKE***

1. INTRODUCTION

So far various criteria on fracture of rocks have been proposed by many investigators. One can find the ambiguous usage of terms in different contexts in the literature dealing with the fracture criterion. In order to study the fracture of rocks on the basis of precise concept, the difinition of terms proposed by Bieniawski[1] will be accepted in what follows.

When a rock is subjected to a compressive stress in the axial direction under low confining pressure in triaxial compression test, the stress at strength failure tends to take the value close to that at yield, namely, brittle fracture occurs. For this reason, we have often failed to discriminate between the criterion on strength failure and that on yield strictly when rocks are subjected to low confining pressure. On the other hand, when a rock is subjected to a compressive stress in the axial direction under high confining pressure, the stress at strength failure shows the value apparently different from that at yield; and the behavior of the rock becomes to exhibit ductile fracture in general. Accordingly, a precise distinction must be made concerning yield and fracture criteria in multiaxial stress conditions.

Mohr[2] introduced a hypothesis for failure of soils and rocks. Drucker and Prager[3] generalized the Mohr's criterion in terms of invariants of stress tensor. Griffith[4] postulated a fracture criterion based on the presence of small cracks in a material. According to studies of failure of rocks, it has been found that failure criteria applicable to predicting the strength of materials may be those by Mohr and Griffith. Hu and Pae[5] took the influence of hydrostatic pressure on the plastic behavior into account to propose a yield criterion. However, if these criteria are applied to the plastic potential in flow rule, it is found that the equations of plastic strain rate derived from foregoing criteria do not adequately describe the plastic deformation of some rocks. This may be because the above-mentioned criteria are not functions of cap's type which means closed form in stress space. Thus, it may be suggested that the four criteria are not relevant to the yield criterion of rocks.

The objective of the present paper is to deduce a yield criterion in order to describe the plastic behavior of soft rocks precisely. For this purpose, a yield function is proposed on the basis of invariants of tensors and experimental evidences. Particularly the form of work-hardening parameter is expressed by taking account of plastic work divided into two parts related respectively with the change in volume and with the change in shape. Furthermore the appropriateness of the proposed yield function is investigated through experimental data[6],[7].

2. PROPOSAL OF A YIELD FUNCTION OF SOFT ROCKS

In the theory of plasticity, many assumptions have been made in order to offer a mathematical expression of the mechanical behavior of materials. It may be acceptable assumption that there exists a scalar function, called a yield function, and denoted by $f(T^{ij}, E_{ij}, \kappa)$, in which T^{ij} are components of stress tensor; E_{ij} are components of strain tensor; κ represents a work-hardening parameter which depends on the plastic deformation history.

If a material is isotropic, a yield function depends only on invariants of stress and strain, and variables related to the plastic deformation history. Further, if the yield function depends on stress and work-hardening parameter and the work-hardening is isotropic, the yield function can be expressed in the form

$$f=f(I_1, J_2, J_3, \kappa) \quad \cdots\cdots(1)$$

* Member of JSCE, Dr. Eng., Research Associate of Civil Engineering, Tohoku University.

** Member of JSCE, Dr. Eng., Professor of Civil Engineering, Tohoku University.

where I_1 is the first invariant of stress defined as $I_1=T_k^k$ and J_2 and J_3 are the second and third invariants of deviatoric stress T'^i_j defined as $J_2=T'^i_jT'^j_i/2$ and $J_3=T'^i_jT'^j_kT'^k_i/3$ respectively.

Assuming that J_3 does not appear in the yield function, Mróz et al.[8] propounded a yield function which comprises the terms of the first and second powers of invariants I_1 and $J_2^{1/2}$ in the mechanical analysis of soils. This yield function is considered to be appropriate for soft rocks as well as soils because it has necessary terms of invariants to represent the plastic behavior of soft rocks, as may be mentioned later. Arranging the terms of invariants and the work-hardening parameter in the yield function given by Mróz et al, the following yield function may be applied to soft rocks in the form

$$f=J_2+\alpha J_2^{1/2}+\beta I_1^2+\gamma I_1-\kappa=0 \quad\cdots\cdots(2)$$

where α, β and γ are material constants. It is noted from Eq. (2) that yielding of soft rocks occurs for hydrostatic pressure which corresponds to preconsolidation pressure of clays.

Let us consider a form of the work-hardening parameter κ. It may be reasonable to consider that the work-hardening parameter κ is a function of the internal energy which is a state variable in the thermodynamics[9]. However, it is assumed usually that κ may be regarded as a function of the plastic work[10], i.e.,

$$\kappa=\kappa\left(\int_0^t T^{ij}\dot{E}_{ij}^{(p)}dt\right) \quad\cdots\cdots(3)$$

where t denotes time and $\dot{E}_{ij}^{(p)}$ are the plastic strain rates. The plastic work rate $T^{ij}\dot{E}_{ij}^{(p)}$ can be divided into two parts related respectively with the change in volume and with the change in shape. Then the plastic work rate per unit volume is expressed as

$$T^{ij}\dot{E}_{ij}^{(p)}=T_i^i\dot{E}_j^{j(p)}/3+T'^{ij}\dot{E}_{ij}^{(p)} \quad\cdots\cdots(4)$$

where the former and the latter of the right-hand side are concerned with the change in volume and with the change in shape respectively.

In experimental results of soft rocks[6),7)], it has been reported that the plastic volumetric strain rate $\dot{E}_j^{j(p)}$ does not vanish. This may imply that the plastic deformation of soft rocks will depend not only on the plastic work related with the change in shape, but also on that with the change in volume. Thus, for the rate of the work-hardening parameter $\dot{\kappa}$, a linear equation which comprises two variables of the first and second terms in the right-hand side of Eq. (4) may be assumed as follows

$$\dot{\kappa}=\phi_1T_i^i\dot{E}_j^{j(p)}/3+\phi_2T'^{ij}\dot{E}_{ij}^{(p)} \quad\cdots\cdots(5)$$

where ϕ_1 and ϕ_2 are material constants to be

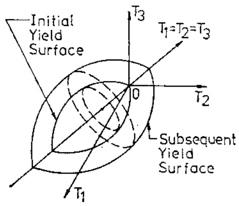

Fig. 1 Yield loci in the coordinates of principal stresses.

determined from experimental results. From Eqs. (2) and (5) is assumed isotropic hardening which expands uniformly from the initial yield surface and the initial and subsequent yield functions are illustrated schematically in **Fig. 1**.

Hence, the yield function of soft rocks proposed here is given by Eqs. (2) and (5).

3. STRESS-STRAIN RELATIONSHIP DERIVED FROM THE PROPOSED YIELD FUNCTION

For the stress-strain relationship in the plastic state, the flow rule proposed by Prager[11] is expressed in the form

$$\dot{E}_{ij}^{(p)}=\frac{1}{h}\frac{\partial f}{\partial T^{mn}}\dot{T}^{mn}\frac{\partial f}{\partial T^{ij}} \quad\cdots\cdots(6)$$

where

$$h=-\left[\frac{\partial f}{\partial E_{kl}^{(p)}}+\frac{\partial f}{\partial \kappa}\frac{\partial \kappa}{\partial E_{kl}^{(p)}}\right]\frac{\partial f}{\partial T^{kl}} \quad\cdots\cdots(7)$$

This rule is known as the normality condition by the reason that Eq. (6) requires the normality of the plastic strain rate vector to the yield surface in the stress space.

The plastic stress-strain relationship can be neatly formulated through some assumptions. If the yield function Eq. (2) and the work-hardening parameter Eq. (5) are adopted, Eq. (6) is written as

$$\dot{E}_{ij}^{(p)}=\left[\left(1+\frac{\alpha}{2J_2^{1/2}}\right)T'_{mn}+(2\beta I_1+\gamma)g_{mn}\right]\dot{T}^{mn}$$
$$\times\left[\left(1+\frac{\alpha}{2J_2^{1/2}}\right)T'_{ij}+(2\beta I_1+\gamma)g_{ij}\right]$$
$$/[\phi_1I_1(2\beta I_1+\gamma)+\phi_2J_2^{1/2}(2J_2^{1/2}+\alpha)] \quad\cdots(8)$$

where g_{ij} are covariant components of the metric tensor.

Let $E_{ij}^{(e)}$ be components of elastic strain. As the strain rate is divisible into elastic and plastic parts in the case of infinitesimal strain, we have

$$\dot{E}_{ij}=\dot{E}_{ij}^{(e)}+\dot{E}_{ij}^{(p)} \quad\cdots\cdots(9)$$

The stress tensor is related to the elastic strain tensor by the tensor of elastic constant, C^{ijkl}, of the material. Thus

$$T^{ij}=C^{ijkl}E_{kl}^{(e)} \quad \cdots\cdots(10)$$

Assuming Eqs. (2) and (5) and using Eqs. (6), (9) and (10), we obtain

$$\dot{T}^{ij}=D^{ijkl}\dot{E}_{kl} \quad \cdots\cdots(11)$$

where

$$D^{ijkl}=C^{ijkl}-\frac{C^{ijpq}\dfrac{\partial f}{\partial T^{pq}}\dfrac{\partial f}{\partial T^{mn}}C^{mnkl}}{\dfrac{\partial \kappa}{\partial E_{st}^{(p)}}\dfrac{\partial f}{\partial T^{st}}+C^{stuv}\dfrac{\partial f}{\partial T^{st}}\dfrac{\partial f}{\partial T^{uv}}} \quad \cdots\cdots(12)$$

Substituting Eqs. (2) and (5) and the tensor C^{ijkl} in Eq. (10) into Eq. (12), we can express the total stress-strain relationship Eq. (11) in terms of material constants.

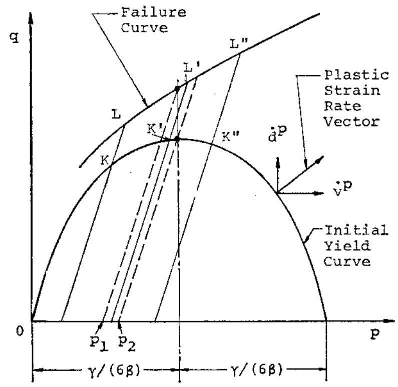

Fig. 2 Initial yield and failure curves and plastic strain rate vector.

4. DETERMINATION OF MATERIAL CONSTANTS IN THE PROPOSED YIELD FUNCTION

To determine the value of material constants in the proposed yield function given by Eqs. (2) and (5), it is necessary to carry out adequate experiments. Current methods of testing rocks have been, as yet, almost always restricted to the conventional triaxial tests, in particular triaxial compression tests. Two stress parameters used in triaxial compression tests such that $\sigma_1'=\sigma_2'<\sigma_3'$ are the mean principal stress

$$p=(\sigma_1'+\sigma_2'+\sigma_3')/3=(2\sigma_1'+\sigma_3')/3 \quad \cdots\cdots(13)$$

and the deviatoric stress

$$q=\sigma_3'-\sigma_1' \quad \cdots\cdots(14)$$

where σ_1' and σ_2' are the principal effective radial stresses, which are called the confining pressures; σ_3' is the principal effective axial stress; the compressive stresses are taken to be positive. Considering the condition such that $e_1=e_2<e_3$ and taking compressive strains as positive, we shall use two strain parameters, i.e., the volumetric strain

$$v=e_1+e_2+e_3=2e_1+e_3 \quad \cdots\cdots(15)$$

and the deviatoric strain

$$d=2/3(e_3-e_1) \quad \cdots\cdots(16)$$

where e_1 and e_2 are the strains in the radial direction, and e_3 is the strain in the axial direction.

In triaxial compression tests, the yield function Eq. (2) is expressed as

$$f=q^2/3+\alpha/3^{1/2}q+9\beta p^2-3\gamma p-\kappa=0 \quad \cdots\cdots(17)$$

It may be suitable for soft rocks to assume that yielding will occur under the state that $p<0$. Therefore, it is postulated that the work-hardening parameter κ in Eq. (17) is nought at the initial yielding. The yield function given by Eq. (17) with $\kappa=0$ is illustrated in **Fig. 2.**

For soft rocks possessing elastoplastic properties, assuming the normality condition of Eq. (6), we get

$$\dot{p}/\dot{q}=-\dot{v}^p/\dot{d}^p \quad \cdots\cdots(18)$$

where $\dot{p}$ and $\dot{q}$ are the rates of stresses and $\dot{v}^p$ and $\dot{d}^p$ are the plastic volumetric strain rate and the plastic deviatoric strain rate respectively. The stress-strain relationship at the initial yielding is derived from Eqs. (17) and (18) in the following form

$$x=3\frac{y^2-18\xi\beta y-27\beta}{\xi y^2+6y-27\xi\beta} \quad \cdots\cdots(19)$$

where

$$x=-\dot{v}^p/\dot{d}^p \quad \cdots\cdots(20)$$

$$y=q/p \quad \cdots\cdots(21)$$

$$\xi=-\alpha/(3^{1/2}\gamma) \quad \cdots\cdots(22)$$

The material constants ξ and β are determined from the experimental data plotted in (x, y) plane by using Eq. (19).

In **Fig. 2**, $\dot{v}^p$ is drawn parallel to the p-axis and $\dot{d}^p$ is parallel to the q-axis. If we assume that the work-hardening up to strength failure is isotropic, the characteristic of the plastic volumetric strain which depends on the confining pressure is classified as follows:

(i) When the confining pressure is less than p_1, the plastic volume expansion occurs; e.g., K→L in **Fig. 2.**

(ii) When the confining pressure is between p_1 and p_2, the plastic volumetric strain up to strength failure is small in magnitude; e.g., K'→L' in **Fig. 2.**

(iii) When the confining pressure is greater

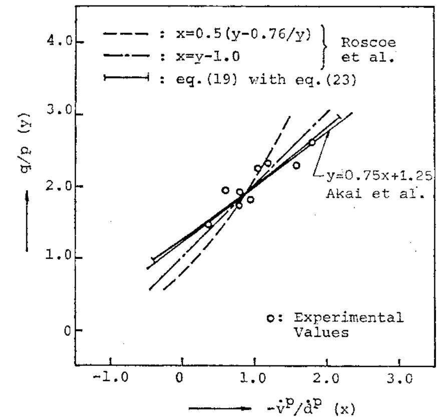

Fig. 3 Relationships between q/p and $\dot{v}^p/\dot{d}^p$ at initial yielding (from Akai et al. 6)).

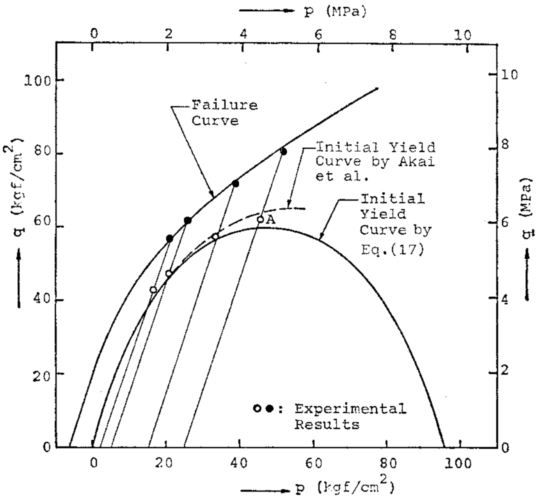

Fig. 4 Initial yield and failure curves for soft rock (from 6)).

than p_2, the plastic volume contraction occurs: e.g., K″→L″ in **Fig. 2**. Following this classification, we determine the material constants in the proposed yield function.

Experimental data of triaxial compression tests by Akai et al.[6] on a soft sedimentary rock (porous tuff) are shown in **Figs. 3 to 7**. Using the data as an example, we determine the material constants here. **Fig. 3** shows the relationship between the ratio of stresses q/p and that of plastic strain rates $\dot{v}^p/\dot{d}^p$ at the initial yielding. Applying the data in **Fig. 3** to Eq. (19), we obtain

$$\xi=-0.734,\quad \beta=0.149 \quad \cdots\cdots(23)$$

Roscoe and Burland[12] investigated constitutive equations and yield functions for "wet" (i.e., normally and lightly overconsolidated saturated) clays. In **Fig. 3**, the stress-strain curves provided by Roscoe et al. and the curve given by Eqs. (19) and (23) are shown. It is found that the equations by Roscoe et al. can not sufficiently describe the experimental data, on the other hand, Eq. (19) with Eq. (23) is able to express the stress-strain relationship of experimental results adequately and is much the same as the experimental equation given by Akai et al.

From the relationship between the deviatoric stress and the volumetric strain under various confining pressures concerning the soft rock tested by Akai et al., it is found that the plastic volume expansion occurs when the confining pressure is less than $p=25$ kgf/cm² (2.45 MPa) and the plastic volume contraction tends to occur when the confining pressure is $p=25$ kgf/cm² (2.45 MPa). In **Fig. 4**, this may imply that $p=\gamma/(6\beta)$, which is a half length of the intercept of line $q=0$ with the initial curve (see **Fig. 2**), is the neighborhood of the initial yield point A [$p=46$ kgf/cm² (4.51 MPa)] under the confining pressure $p=25$ kgf/cm² (2.45 MPa). Then, it is assumed approximately that

$$\gamma/(6\beta)=48.0\ \text{kgf/cm}^2\ (4.70\ \text{MPa}) \quad \cdots\cdots(24)$$

From Eqs. (22) to (24), we obtain

$$\left.\begin{array}{l}\alpha=54.6\ \text{kgf/cm}^2\ (5.35\ \text{MPa})\\ \beta=0.149\\ \gamma=42.9\ \text{kgf/cm}^2\ (4.20\ \text{MPa})\end{array}\right\} \quad \cdots\cdots(25)$$

Applying Eq. (8) to triaxial compression tests, we have the relationship between the deviatoric stress q and the plastic deviatoric strain d^p in the form

$$\begin{aligned}\dot{d}^p/\dot{q}=&(2/3q+\alpha/3^{1/2})\{(2/3+2\beta)q+\alpha/3^{1/2}\\&+6\beta\sigma_1-\gamma\}/[(2\beta\phi_1+2/3\phi_2)q^2\\&+\{3\phi_1(4\beta\sigma_1-\gamma/3)+\alpha\phi_2/3^{1/2}\}q\\&+3\phi_1\sigma_1(6\beta\sigma_1-\gamma)]\end{aligned} \quad \cdots\cdots(26)$$

The stress-strain relationships obtained from experiments of soft rocks depend on the confining pressure. Then, it was found from Eq. (26) that the material constants ϕ_1 and ϕ_2 would rather not take unique values for every state of stresses p and q in order to describe the experimental results more adequately. For this reason, it may be convenient to consider the material constants ϕ_1 and ϕ_2 seperately for two different ranges where the plastic volume expansion and contraction occur.

As the experimental data given by Akai et al.

are almost concerned with the case where the plastic volume expansion appears, let us consider the material constants ϕ_1 and ϕ_2 in this case. By use of Eq. (26), the material constants ϕ_1, ϕ_2 are obtained from the values of gradients at two points on a experimental curve between the plastic deviatoric strain d^p and the deviatoric stress q. Then, for two confining pressures of 5 and 15 kgf/cm² (0.49 and 1.47 MPa), the material constants ϕ_1 and ϕ_2 in Eq. (26) may be assumed to take the following values

$$\left.\begin{array}{l}\phi_1=-1.12\times10^4\,\mathrm{kgf/cm^2}\ (-1.10\times10^3\,\mathrm{MPa})\\ \phi_2=\ \ 3.73\times10^3\,\mathrm{kgf/cm^2}\ (\ \ 3.66\times10^2\,\mathrm{MPa})\end{array}\right\} \qquad \cdots(27)$$

Substituting Eqs. (25) and (27) into Eq. (26) and integrating, we obtain the stress-strain relationship in the plastic range. It may be assumed that the stress-strain relationship in the elastic state is linear. Therefore, combining the elastic and plastic strains, we have the relationships between the deviatoric stress and the total deviatoric strain, as shown in **Fig. 5**.

From Eqs. (6) and (13) to (15), we get the plastic volumetric strain in the form

$$\begin{aligned}\dot{v}^p/\dot{q}=&3(2\beta q-\gamma+6\beta\sigma_1)\{(2/3+2\beta)q+\alpha/3^{1/2}\\&+6\beta\sigma_1-\gamma\}/[(2\beta\phi_1+2/3\phi_2)q^2\\&+\{3\phi_1(4\beta\sigma_1-\gamma/3)+\alpha\phi_2/3^{1/2}\}q\\&+3\phi_1\sigma_1(6\beta\sigma_1-\gamma)] \qquad \cdots(28)\end{aligned}$$

Substituting Eqs. (25) and (27) into Eq. (28) and integrating, we have the relationship between the deviatoric stress and the plastic volumetric strain. Therefore, combining the elastic and plastic volumetric strains, we obtain the relationships between the deviatoric stress and the total volumetric strain, as shown in **Fig. 6**.

Hence, it is found from **Figs. 5** and **6** that the stress-strain relationships of Eqs. (26) and (28) with Eqs. (25) and (27) can describe the experimental results of soft rock fairly well.

Griffith's[4] criterion may be assumed to be predominent for fracture criterion of soft rocks. This criterion under biaxial stresses such that $\sigma_1<\sigma_3$ is expressed in the form

$$(\sigma_1-\sigma_3)^2=8S_t(\sigma_1+\sigma_3) \qquad \cdots(29)$$

if

$$3\sigma_1+\sigma_3>0 \qquad \cdots(30)$$

and

$$\sigma_1=-S_t \qquad \cdots(31)$$

if

$$3\sigma_1+\sigma_3<0 \qquad \cdots(32)$$

where S_t is the uniaxial tensile strength of a material. According to Murrel[13], the envelope to various Mohr's circles which satisfy Eq. (29)

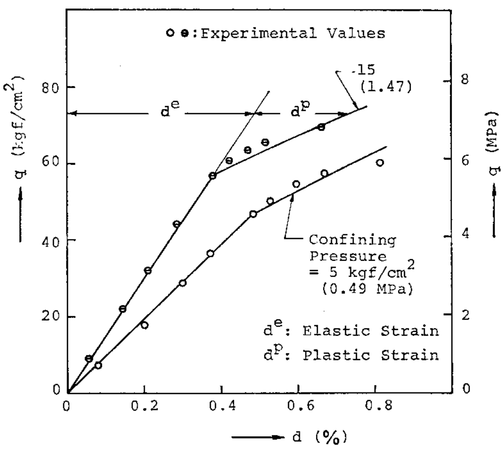

Fig. 5 Relationships between deviatoric stress q and deviatoric strain d (from 6)).

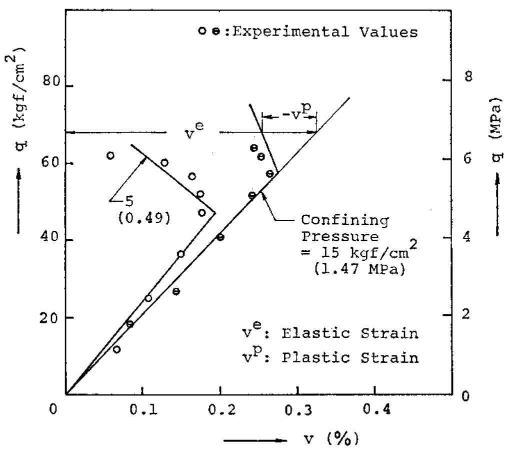

Fig. 6 Relationships between deviatoric stress q and volumetric strain v (from 6)).

is written in (σ, τ) plane in the form

$$\tau^2=4S_t(\sigma+S_t) \qquad \cdots(33)$$

Fig. 7 shows the Mohr's circles at strength failure in triaxial compression tests of a soft rock performed by Akai et al. If Griffith's criterion is suitable to the state of strength failure, applying Eq. (33) to the results showed in **Fig. 7** leads to

$$S_t=6.45\,\mathrm{kgf/cm^2}\ (0.632\,\mathrm{MPa}) \qquad \cdots(34)$$

Using the stresses p and q given by Eqs. (13) and (14), we can express Griffith's criterion for the case of triaxial compressive condition from Eqs. (29) to (32) in the form

$$48S_tp=q(3q-8S_t) \qquad \cdots(35)$$

if

$$12p>q \qquad \cdots(36)$$

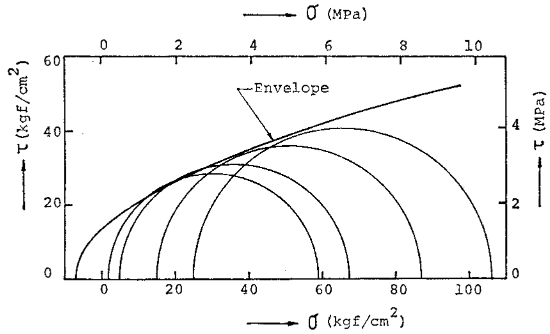

Fig. 7 Envelope of Mohr's circles at strength failure (from 6)).

and

$$3p = q - 3S_t \quad \cdots\cdots (37)$$

if

$$12p < q \quad \cdots\cdots (38)$$

The curve at strength failure presented by Eqs. (35) to (38) with Eq. (34) is illustrated in **Fig. 4.**

5. CONCLUDING REMARKS

On the basis of invariants of tensors and experimental evidence, a simple possible form of yield function of soft rocks was proposed. It was found that the stress-strain relationship derived from this yield function is able to describe properly the plastic behavior of a soft rock in the range where the plastic volume expansion appears. However, as for the range where the plastic volume contraction occurs, the material constants in the proposed yield function should be newly determined through the experimental data of the soft rock in this range. As the proposed yield function is applied to a soft rock (porous tuff), it is necessary in future to investigate the usefulness of application of it to the general problems of soft rocks. The constitutive equation Eq. (11) will be helpful for numerical calculations by use of finite element methods etc.

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