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ANALYSIS OF CIRCULAR PLATE RESTING ON NON-TENSION FOUNDATION

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1. INTRODUCTION

Many works have been done on beams supported by Winkler's foundation, which is symbolised by a row of isolated elastic springs[1]. This foundation is sometimes not the case for real applications. The ground for example does not move upward, therefore the Winkler's idea is not applicable to it if strictly speaking. The nontension foundation has been introduced to overcome this inaccuracy. It induces upward reaction force when it sinks, while it does not deflect upward together with beams on it. That is there is a separation between the foundation and the beams when they displace upwards.

Nien-chien Tsai and R. A. Westmann[2], and Y. Weitsman[3] gave solutions for beams and circular plate of infinite radius on the nontension foundation. N. Kamiya derived a solution for a clamped circular plate[4].

In this report an analytical procedure will be carried out on a weightless circular plate subjected to a point load at the center and supported on non-tension Winkler's foundation.

2. METHOD OF ANALYSIS

(1) Governing Equations

Denote the displacement of the middle surface of the plate by w (w positive downward) and the radial distance by r. The radius of contact is denoted by r_0 (unknown) where the plate separates from the elastic foundation.

The governing equations of axisymmetric plates on the nontension foundation are[5]:

at $0<r<r_0$ and $w>0$;

$$D\nabla_r^2\nabla_r^2 w_1 + k w_1 = 0 \quad \cdots\cdots (1)$$

at $r>r_0$ and $w<0$;

$$\nabla_r^2\nabla_r^2 w_2 = 0 \quad \cdots\cdots (2)$$

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In Eqs. (1) and (2) ∇_r^2 denotes Laplacian operator for 1-dimension and

$$\nabla_r^2 = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) \quad \cdots\cdots (3)$$

D is the flexural rigidity of the plate, k is the modulus of the foundation and the subscripts 1 and 2 refer to the contact and the no-contact regions respectively.

Denote

$$l^4 = \frac{D}{k} \quad \cdots\cdots (4)$$

Introduce the dimensionless quantities x, x_0 and γ defined by

$$x = r/l, \quad x_0 = r_0/l, \quad \gamma = a/l \quad \cdots\cdots (5\cdot a, b, c)$$

where a denotes radius of the plate.

The general solutions of Eqs. (1) and (2) are respectively[5),6)]:

$$w_1(x) = C_0 \operatorname{ber} x + E_0 \operatorname{bei} x + F_0 \ker x + G_0 \operatorname{kei} \quad \cdots\cdots (6)$$

$$w_2(x) = A + Bx^2 + C\ln x + Ex^2 \ln x \quad \cdots\cdots (7)$$

where ber x, bei x, ker x and kei x are Kelvin functions of order 0.

Because the deflection must be finite at the origin, F_0 is set to equal zero since ker x becomes singular at the origin[7]. Thus Eq. (6) becomes

$$w_1(x) = C_0 \operatorname{ber} x + E_0 \operatorname{bei} x + G_0 \operatorname{kei} x \quad \cdots\cdots (6\cdot a)$$

(2) Boundary and Continuity Conditions

For the contact and no-contact domains of the plate, the boundary conditions and continuity conditions are respectively:

$$(M_{r2})_{x=\gamma} = 0 \quad \cdots\cdots (8)$$
$$(Q_{r2})_{x=\gamma} = 0 \quad \cdots\cdots (9)$$
$$(W_1)_{x=x_0} = 0 \quad \cdots\cdots (10)$$
$$(w_2)_{x=x_0} = 0 \quad \cdots\cdots (11)$$
$$(Q_{r1})_{x=x_0} = 0 \quad \cdots\cdots (12)$$
$$(w_1')_{x=x_0} = (w_2')_{x=x_0} \quad \cdots\cdots (13)$$
$$(M_{r1})_{x=x_0} = (M_{r2})_{x=x_0} \quad \cdots\cdots (14)$$

The equilibrium of transverse forces requires that

$$\left[\int_0^{2\pi} Q_r r d\theta\right]_{r=\epsilon} + P = 0 \quad \cdots\cdots (15)$$

where ϵ is an infinitesimal positive value.

M_r and Q_r are respectively the radial bending moment and the radial shearing force per unit width of the section and are

$$M_r=-\frac{D}{l^2}\left[\frac{d^2w}{dx^2}+\nu\frac{1}{x}\frac{dw}{dx}\right] \quad\cdots\cdots(16)$$

$$Q_r=-\frac{d}{l^3}\frac{d}{dx}(\nabla_r{}^2w) \quad\cdots\cdots(17)$$

Other stress resultant besides those defined in Eqs. (16) and (17) is

$$M_\theta=-\frac{D}{l^2}\left(\frac{1}{x}\frac{dw}{dx}+\nu\frac{d^2w}{dx^2}\right) \quad\cdots\cdots(18)$$

The stress resultants as defined in Eqs. (16) to (18) can be evaluated with the help of the following expressions for the derivatives of Kelvin functions[7] which appear in Eq. (6).

$$\frac{d}{dx}f_m=\frac{m}{x}f_m+\frac{1}{\sqrt{2}}(f_{m+1}+g_{m+1}),\quad x>0 \quad\cdots\cdots(19)$$

$$\frac{d^2}{dx^2}f_m=\frac{m}{x^2}(m-1)f_m-\frac{1}{x\sqrt{2}}(f_{m+1}+g_{m+1})-g_m,\quad x>0 \quad\cdots\cdots(20)$$

$$\frac{d^3}{dx^3}f_m=\frac{m(m-1)(m-2)}{x^3}f_m+\frac{1}{x^2\sqrt{2}}(m^2+2+x^2)f_{m+1}+\frac{1}{x^2\sqrt{2}}(m^2+2-x^2)g_{m+1}-\frac{m-1}{x}g_m,\quad x>0 \quad\cdots\cdots(21)$$

where f_m and g_m stand for one of the following four sets of Kelvin functions of mth order

$$\left.\begin{array}{l} f_m=\ \mathrm{ber}_m x \\ g_m=\ \mathrm{bei}_m x \end{array}\right\} \qquad \left.\begin{array}{l} f_m=\ \mathrm{bei}_m x \\ g_m=-\mathrm{ber}_m x \end{array}\right\} \quad\cdots\cdots(22\cdot\text{a, b})$$

$$\left.\begin{array}{l} f_m=\ \mathrm{ker}_m x \\ g_m=\ \mathrm{kei}_m x \end{array}\right\} \qquad \left.\begin{array}{l} f_m=\ \mathrm{kei}_m x \\ g_m=-\mathrm{ker}_m x \end{array}\right\} \quad\cdots\cdots(22\cdot\text{c, d})$$

The behaviour of each Kelvin function and its contribution to the stress resultants are shown in Appendix.

3. SOLUTIONS

Substituting Eq. (17) into Eq. (15) and knowing that as x tends to zero, kei x is the only term in Eq. (6·a) which contributes to the values of Q_r and this leads to

$$G_0=-\frac{Pl^3}{D}\frac{1}{\int_0^{2\pi}\left[-\frac{1}{\sqrt{2}}(\mathrm{ker}_1 x+\mathrm{kei}_1 x)\right]r d\theta} \quad\cdots\cdots(23)$$

after introducing a integrated denominator,

$$G_0=-\frac{Pl^2}{2\pi D} \quad\cdots\cdots(24)$$

At $x=\gamma$, the radial shear vanishes, thus the Eq. (9) leads to

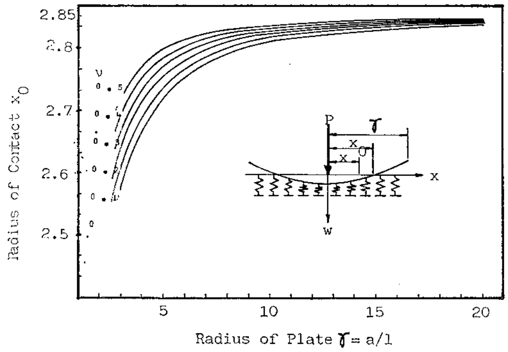

Fig. 1 Radius of Contact x_0(Solutions of Eq. (36)).

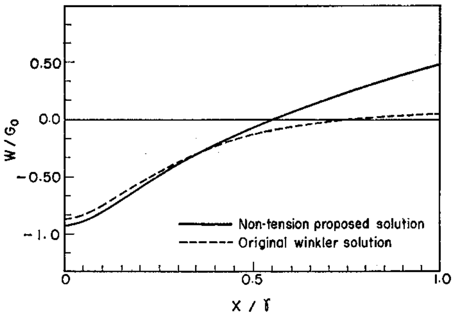

Fig. 2 Deflections of plates under a concentrated load at the center of plates. a=radius of plate, D=flexural rigidity of plate, k=modulus of foundation, P=concentrated load, r=radial coordinate, w=deflection, $l=\sqrt[4]{D/k}$, $G_0=-Pl^2/2\pi D$, $x=r/l$, $x/\gamma=r/a$. ($\gamma=5$, $\nu=0.2$)

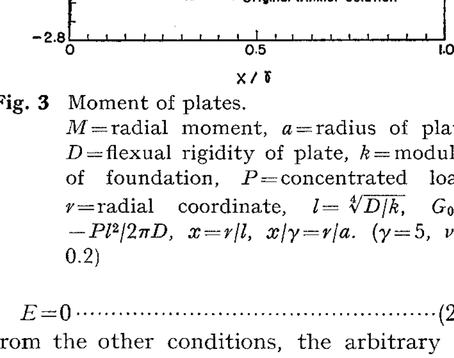

Fig. 3 Moment of plates. M=radial moment, a=radius of plate, D=flexual rigidity of plate, k=modulus of foundation, P=concentrated load, r=radial coordinate, $l=\sqrt[4]{D/k}$, $G_0=-Pl^2/2\pi D$, $x=r/l$, $x/\gamma=r/a$. ($\gamma=5$, $\nu=0.2$)

$$E=0 \quad\cdots\cdots(25)$$

From the other conditions, the arbitrary constants C_0, E_0, A, B and C are respectively,

$$C_0=-\frac{G_0}{\mathrm{ber}\,x_0}[\mathrm{kei}\,x_0-\zeta_1(x_0)\,\mathrm{bei}\,x_0] \quad\cdots\cdots(26)$$

$$E_0=-G_0\zeta_1(x_0) \quad\cdots\cdots(27)$$

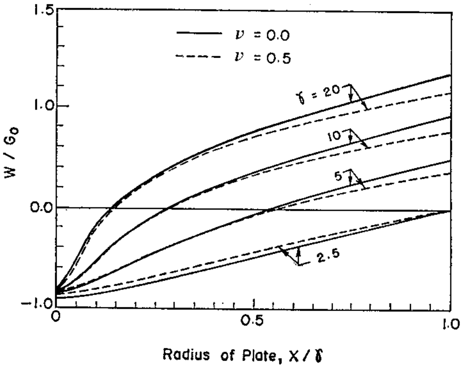

Fig. 4 Deflection of plate
ν=Poisson's ratio, a=radius of plate, D=flexural rigidity of plate, k=modulus of foundation, P=concentrated load, r= radial coordinate, w=deflection, $l=\sqrt[4]{D/k}$, $G_0=-Pl^2/2\pi D$, $x=r/l$, $x/\gamma=r/a$.

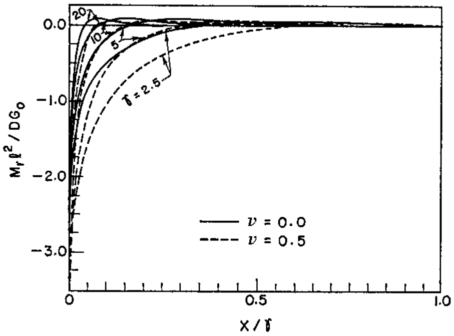

Fig. 5 Distribution of radial bending moment. Refer to notations of Fig. 3.

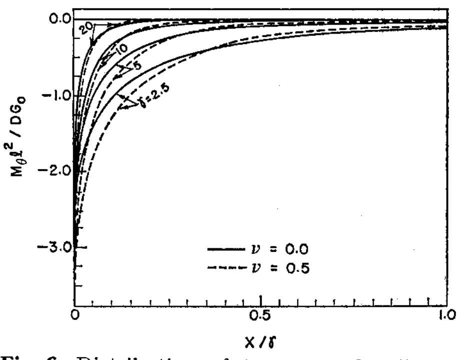

Fig. 6 Distribution of transverse bending moment M_θ.
Refer to notations of **Fig. 3**.

$$C=-\frac{G_0x_0}{2}\frac{\text{kei}\,x_0-\zeta_1(x_0)\,\text{bei}\,x_0}{\text{ber}\,x_0}[\{\sqrt{2}\,R_1(x_0)+x_0\,\text{bei}\,x_0\}+\zeta_1(x_0)\{\sqrt{2}\,S_1(x_0)-x_0\,\text{ber}\,x_0\}+\{\sqrt{2}\,U_1(x_0)-x_0\,\text{ker}\,x_0\}] \qquad \cdots\cdots(28)$$

$$B=\frac{-G_0}{2x_0}\left[\frac{R_1(x_0)}{\sqrt{2}}\frac{\text{kei}\,x_0-\zeta_1(x_0)\,\text{bei}\,x_0}{\text{ber}\,x_0}\right.$$

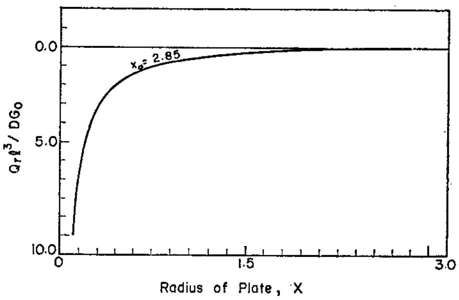

Fig. 7 Distribution of radial shering force Q_r. Refer to notations of **Fig. 3**. $X_0=2.85$ for a ideal case of $\nu=0.5$ and $\gamma=\infty$.

$$\left.+\zeta_1(x_0)\frac{S_1(x_0)}{\sqrt{2}}+\frac{U_1(x_0)}{\sqrt{2}}\frac{C}{x_0}\right] \qquad \cdots\cdots(29)$$

$$A=-Bx_0^2-C\ln x_0 \qquad \cdots\cdots(30)$$

where

$$R_1(x_0)=\text{ber}_1\,x_0+\text{bei}_1\,x_0 \qquad \cdots\cdots(31)$$

$$S_1(x_0)=\text{bei}_1\,x_0-\text{ber}_1\,x_0 \qquad \cdots\cdots(32)$$

$$T_1(x_0)=\text{ker}_1\,x_0+\text{kei}_1\,x_0 \qquad \cdots\cdots(33)$$

$$U_1(x_0)=\text{kei}_1\,x_0-\text{ker}_1\,x_0 \qquad \cdots\cdots(34)$$

$$\zeta_1(x_0)=\frac{\text{kei}\,x_0S_1(x_0)+\text{ber}\,x_0T_1(x_0)}{\text{bei}\,x_0S_1(x_0)+\text{ber}\,x_0R_1(x_0)} \qquad \cdots\cdots(35)$$

The condition for unknown x_0 is derived,

$$-\frac{1}{\text{ber}\,x_0}[\text{kei}\,x_0-\zeta_1(x_0)\,\text{bei}\,x_0]\cdot\left[\text{bei}\,x_0+\frac{1}{\gamma^2}\frac{(1-\nu)}{(1+\nu)}x_0\{\sqrt{2}\,R_1(x_0)+x_0\,\text{bei}\,x_0\}\right]-\zeta_1(x_0)\left[\text{ber}\,x_0-\frac{1}{\gamma^2}\frac{(1-\nu)}{(1+\nu)}x_0\cdot\{\sqrt{2}\,S_1(x_0)-x_0\,\text{ber}\,x_0\}\right]+\left[\text{ker}\,x_0-\frac{1}{\gamma^2}\frac{(1-\nu)}{(1+\nu)}x_0\{\sqrt{2}\,U_1(x_0)-x_0\,\text{ker}\,x_0\}\right]=0 \qquad \cdots\cdots(36)$$

From Eq. (36), the weightless plate lifts up at a distance which is independent of the magnitude of P. The unknown X_0 is derived numerically by a trial and error method for given γ and ν values. The contact radius X_0 increases as γ or ν increases.

Substituting Eqs. (28), (29) and Eq. (30) into Eq. (7) leads to

$$\frac{w_2}{G_0}=-\frac{x_0}{2}\left[\frac{(x^2-x_0^2)}{2\gamma^2}\frac{(1-\nu)}{(1+\nu)}+\ln\frac{x}{x_0}\right]\cdot\left[\frac{\text{kei}\,x_0-\zeta_1(x_0)\,\text{bei}\,x_0}{\text{ber}\,x_0}\{\sqrt{2}\,R_1(x_0)+x_0\,\text{bei}\,x_0\}+\zeta_1(x_0)\{\sqrt{2}\,S_1(x_0)-x_0\,\text{ber}\,x_0\}+\{\sqrt{2}\,U_1(x_0)-x_0\,\text{ker}\,x_0\}\right] \qquad \cdots\cdots(37)$$

Substituting Eqs. (26) and (27) into Eq. (6a) leads to

$$\frac{w_1}{G_0}=-\frac{1}{\text{ber}\, x_0}[\text{kei}\, x_0-\zeta_1(x_0)\,\text{bei}\, x_0]\,\text{ber}\, x$$
$$-\zeta_1(x_0)\,\text{bei}\, x+\text{kei}\, x \cdots\cdots\cdots\cdots\cdots\cdots\cdots(38)$$

Finally substituting Eqs. (37) and (38) into Eqs. (16) to (18) leads to the equation of stress resultants as follows:

$$M_{r1}\frac{l^2}{DG_0}=-\frac{1}{\text{ber}\, x_0}(\text{kei}\, x_0-\zeta_1(x_0)\,\text{bei}\, x_0)$$
$$\cdot\left\{\frac{1-\nu}{x\sqrt{2}}R_1(x_0)+\text{bei}\, x\right\}$$
$$-\zeta_1(x_0)\left\{\frac{(1-\nu)}{x\sqrt{2}}S_1(x_0)-\text{ber}\, x\right\}$$
$$+\left\{\frac{1-\nu}{x\sqrt{2}}U_1(x)-\text{ker}\, x\right\} \cdots\cdots(39)$$

$$M_{r2}\frac{l^2}{DG_0}=-\frac{x_0}{2}\left[\frac{1-\nu}{1+\nu}\frac{1}{\gamma^2}-\frac{1}{x^2}\right.$$
$$\left.+\nu\left\{\frac{1-\nu}{1+\nu}\frac{1}{\gamma^2}+\frac{1}{x^2}\right\}\right]\left[-\frac{1}{\text{ber}\, x_0}\right.$$
$$\cdot\{\text{kei}\, x_0-\zeta_1(x_0)\,\text{bei}\, x_0\}\{\sqrt{2}R_1(x_0)$$
$$+x_0\,\text{bei}\, x_0\}-\zeta_1(x_0)\{\sqrt{2}S_1(x_0)$$
$$-x_0\,\text{ber}\, x_0\}+\{\sqrt{2}U_1(x_0)$$
$$\left.-x_0\,\text{ker}\, x_0\}\right]\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(40)$$

$$M_{\theta 1}\frac{l^2}{DG_0}=-\frac{1}{\text{ber}\, x_0}(\text{kei}\, x_0-\zeta_1(x_0)\,\text{bei}\, x_0)$$
$$\cdot\left[\nu\,\text{bei}\, x-\frac{1-\nu}{x\sqrt{2}}R_1(x)\right]$$
$$-\zeta_1(x_0)\left[-\nu\,\text{ber}\, x-\frac{1-\nu}{x\sqrt{2}}S_1(x)\right]$$
$$-\nu\,\text{ker}\, x-\frac{1-\nu}{x\sqrt{2}}U_1(x) \cdots\cdots(41)$$

$$M_{\theta 2}\frac{l^2}{DG_0}=-\frac{x_0}{2}\left[\frac{1-\nu}{1+\nu}\frac{1}{\gamma^2}+\frac{1}{x^2}\right.$$
$$\left.+\nu\left\{\frac{1-\nu}{1+\nu}\frac{1}{\gamma^2}-\frac{1}{x^2}\right\}\right]\left[-\frac{1}{\text{ber}\, x_0}\right.$$
$$\cdot\{\text{kei}\, x_0-\zeta_1(x_0)\,\text{bei}\, x_0\}$$
$$\cdot\{\sqrt{2}R_1(x_0)+x_0\,\text{bei}\, x_0\}$$
$$-\zeta_1(x_0)\{\sqrt{2}S_1(x_0)-x_0\,\text{ber}\, x_0\}$$
$$\left.+\{\sqrt{2}U_1(x_0)-x_0\,\text{ker}\, x_0\}\right]\cdots(42)$$

$$\frac{Q_{r1}}{G_0}=-\frac{S_1(x)}{\sqrt{2}\,\text{ber}\, x_0}\{\text{kei}\, x_0-\zeta_1(x_0)\,\text{bei}\, x_0\}$$
$$+\frac{\zeta_1(x_0)}{\sqrt{2}}R_1(x)-\frac{T_1(x)}{\sqrt{2}} \cdots\cdots\cdots\cdots\cdots(43)$$

Some numerical results are shown in figures.

4. ACKNOWLEDGEMENTS

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APPENDIX

The contribution of the Kelvin functions to the deflection and the stress results are given below:

Item	Expression		
$w(x)$	$\text{ber}\, x$	$\text{bei}\, x$	$\text{kei}\, x$
$\frac{l^2}{D}M_r$	$\frac{1-\nu}{x\sqrt{2}}(\text{ber}_1\, x+\text{bei}_1\, x)-\text{ber}\, x$	$\frac{1-\nu}{x\sqrt{2}}(\text{bei}_1\, x-\text{ber}_1\, x)-\text{ber}\, x$	$\frac{1-\nu}{x\sqrt{2}}(\text{kei}_1\, x-\text{ker}_1\, x)-\text{ker}\, x$
$\frac{l^2}{D}M_\theta$	$\nu\,\text{bei}\, x-\frac{1-\nu}{x\sqrt{2}}(\text{ber}_1\, x+\text{bei}_1\, x)$	$-\nu\,\text{ber}\, x-\frac{1-\nu}{x\sqrt{2}}(\text{bei}_1\, x-\text{ber}_1\, x)$	$-\nu\,\text{kei}\, x-\frac{1-\nu}{x\sqrt{2}}(\text{kei}_1\, x-\text{ker}_1\, x)$
$\frac{l^3}{D}Q_r$	$\frac{1}{\sqrt{2}}(\text{bei}_1\, x-\text{ber}_1\, x)$	$-\frac{1}{\sqrt{2}}(\text{bei}_1\, x+\text{ber}_1\, x)$	$-\frac{1}{\sqrt{2}}(\text{ker}_1\, x+\text{kei}_1\, x)$

Subscripts 1 to bei and ber functions denote the functions of order 1

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