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NUMERICAL ANALYSIS OF VARIATIONAL INEQUALITIES FOR UNILATERAL PROBLEMS OF A BEAM

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1. INTRODUCTION

The present study is concerned with a development of numerical analysis for unilateral problems of bending of a linearly elastic beam governed by the variational inequality which had been obtained in author's previous work[1]. Two methods; namely the penalty method and the projectional relaxation method are specially considered to solve unilateral problems of a beam.

Rather much consideration is required for numerical treatments of variational inequalities related to fourth order differential operations, whereas the well-known Lagrangian multiplier method may be sufficient for variational inequalities related to the second order differential operator[2]. The speed of convergence of the Lagrangian multiplier method with Uzawa's algorithm to obtain a saddle point is very slow for unilateral problems of a beam as shown in the previous paper[1]. In order to obtain much faster algorithms than Uzawa's iterative method, we here apply the penalty and projectional relaxation methods, which are used for plane unilateral problems of linearly elastic bodies[3]. The mixed and reciprocal variational formulations are crucial to apply the relaxation method for a solver of variational inequalities, although the primal formulation can be solved effectively by the penalty method.

Among recent literatures of finite element methods, the penalty method is widely used to solve constraint problems such as the problem of incompressible linearly elastic bodies[4]. The origin of the penalty method is believed to be the address of Courant[5] in 1943 for the Direchlet boundary condition and this technique has been extensively applied in the field of optimization, see for example Luenberger[6]. The penalty method is recently applied for unilateral problems of plane elasticity and nonlinear plates[3],[7]. Moreover, equivalence of the penalty method to the method of the film or bond element[8] has been proved for a class of contact problems[9].

The projectional relaxation method is developed by Cea and Glowinski[10] in order to solve elliptic variational inequalities. This is based on the iterative algorithm such as the S.O.R. (super over relaxation) method and conjugate gradient method. The method is, however, not proper to apply to the problem defined by the fourth order differential operator. It is necessary for the relaxation method to obtain faster convergence that the matrix should be dominated by its diagonal. If we discretize the beam or plate by C^1-continuous finite elements, the stiffness matrix may not posses the required property for the relaxation method. To overcome this limitation we need to reformulate unilateral problems of a beam by other methods, for example by mixed and reciprocal method.

In this paper, we first define the unilateral problem of a beam which is classified as the signorini problem, following the previous paper[1]. Next a penalty formulation of the Signorini problem is given together with convergence of the penalized solutions w_ϵ to the solution w of the original variational inequality as the penalty parameter ϵ goes to zero. Applying techniques of numerical integration, the discretization of the penalized problem is obtained, and is solved by the successive iterative method. We examine choice of formula of numerical integration, relationship of the penalty parameter and the mesh size h of the finite element model, and the contact pressure by the penalized solution, using two numerical examples. A discretization of the mixed variational formulation is given by using linear finite elements. We here provide an algorithm of the projectional relaxation method to solve the system of inequalities as well as the penalty method. The third formulation of the Signorini problem is achieved by the reciprocal method which can be resolved by the relaxation algorithm. The seeking variable is the contact

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pressure instead of the deflection of the beam in this case. Special discretization is required for the *functional* of the contact pressure. Such a manner can be also applicable for approximations of integral equations.

The methods described in this paper are applicable to unilateral problems of not only for beams but also for plates, elastic bodies, and others. Main purposes of this paper are an introductions of the penalty method for inequality constraints, the mixed variational formulation with the projectional relaxation method and the reciprocal formulation with the special discretization. This paper is also aimed to be a supplementary article to the previous paper[1] in view of numerical analysis.

2. SIGNORINI'S PROBLEM

Suppose that a beam is spanned over a rigid foundation. Let L and EI be the length and stiffness of the beam, let f be the applied force, and let s be the distance between the beam and foundation. As shown in Kikuchi[1], such a problem can be formulated by the variational inequality

$$w \in K:\ \int_0^L EIw''(v-w)''dx \geq \int_0^L f(v-w)dx\,,\quad \forall v \in K \qquad (1)$$

where

$$K=\{v \in V:\ (v+s)(x)\geq 0 \text{ in } (0,L)\} \qquad (2)$$

$$V=\{v \in H^2(0,L):\ v(0)=v(L)=0\} \qquad (3)$$

and the prime $'$ means the derivative with respect to x. Here $H^m(0,L)$, $m>0$, is the m-th order Sobolev space[11] defined by

$$H^m(0,L)=\{v:\ v^{(j)} \in L^2(0,L),\ 0\leq j\leq m\} \qquad (4)$$

with the inner product and norm:

$$(w,v)_m=\sum_{j=0}^m \int_0^L w^{(j)}v^{(j)}dx\,,\quad \|v\|_m=(v,v)_m^{1/2}$$

where $v^{(j)}$ is the j-th derivative of the function v in the sence of distribution[11]. The inequality (1) is the *principle of virtual work* of the beam under the constraint due to the rigid foundation, and is called the *primal variational inequality* for unilateral bending problems of the elastic beam. The constraint

$$(w+s)(x)\geq 0 \text{ in } (0,L) \qquad (5)$$

physically means that the beam locates over/on the foundation after deformations. Since $w(0)=w(L)=0$,

$$s(x)\geq 0 \text{ on } x=0 \text{ and } x=L \qquad (6)$$

is a natural assumption.

If a contact problem for two beams is considered, a simple transformation of the variables implies an equivalent Signorini's problem to the original two-beam problem. Indeed by the third variable defined by

$$w_3(x)=w_1(x)-w_2(x) \qquad (7)$$

the two-beam contact problem

$$(w_1,w_2) \in K:\ \sum_{i=1}^2 \int_0^L EI_i w_i''(v_i-w_i)''dx \geq \sum_{i=1}^2 \int_0^L f_i(v_i-w_i)dx \text{ for every } (v_1,v_2) \in K \qquad (8)$$

can be transferred to the Signorini problem

$$(w_1,w_3) \in \hat{K}:\ \int_0^L EI_1 w_1''(v_1-w_1)''dx + \int_0^L EI_2(w_1-w_3)''(v_1-v_3-w_1+w_3)''dx \geq \int_0^L f_1(v_1-w_1)dx + \int_0^L f_2(v_1-v_3-w_1+w_3)dx\,,\quad (v_1,v_3) \in \hat{K} \qquad (9)$$

Here

$$K\{(v_1,v_2) \in V\times V:\ v_1(x)+s(x)\geq v_2(x)\} \qquad (10)$$

$$\hat{K}=\{(v_1,v_3) \in V\times V:\ v_3(x)+s(x)\geq 0\} \qquad (11)$$

The function $s(x)$ is the distance of two beams and we assume that both beams are simply supported in this case. Thus, it suffices to consider details of Signorini's problem for general two-beam contact problems.

3. PENALTY METHOD FOR THE PRIMAL VARIATIONAL INEQUALITY

A method commonly applied to resolve the constrained problem such as (1) is the Lagrangian multiplier method combining Uzawa's interative algorithm to find a saddle point which satisfies the Kuhn-Tucker sufficient condition[6]. This method had been applied to solve the variational inequality (1) in author's previous work[1]. However, the speed of convergence of Uzawa's iterative method was quite slow because of the characteristic of the stiffness matrix obtained by the Hermite interpolation in finite element analysis. While the Lagrangian multiplier method with Uzawa's algorithm is very powerfull for plane contact problems of linearly elastic bodies[3], it does not provide efficiency to the problem governed by the fourth order differential equation. To overcome this, we here applied the penalty method which is widely used in many fields of engineering for constrained problems, recently.

For the unilateral contact condition

$$(w+s)(x)\geq 0 \text{ in } (0,L) \qquad (12)$$

the *penalty virtual work* is defined by

$$-\epsilon^{-1}\int_0^L (w+s)(x)^- v(x)dx \qquad (13)$$

with the parameter of penalty ϵ such that $\epsilon\to 0$. Here

$$g(x)^- = \text{Max}(0., -g(x)) \quad \cdots\cdots(14)$$

for a continuous function g defined on $(0, L)$. The physical meaning of the penalty virtual work is that fairly large amount of virtual work is added to the system when the deflection w violates the contact constraint on a portion of $(0, L)$. In order to obtain zero penalty, the constrained condition

$$(w+s)(x)^- = 0 \text{ i.e., } (w+s)(x)\geq 0$$

must be satisfied on $(0, L)$. The penalty (13) is called the *exterior penalty* for the inequality constraint (12)[6].

Adding the penalty to the virtual work principle, we can replace the primal variational inequality (1) by the penalized principle of virtual work

$$w_\epsilon \in V:\ \int_0^L EIw_\epsilon''v''dx - \epsilon^{-1}\int_0^L (w_\epsilon+s)(x)^- \cdot v(x)dx = \int_0^L fvdx\,, \quad \text{for every } v\in V \quad \cdots(15)$$

For simplicity, we set

$$A(w)(v) = \int_0^L EIw''v''dx\,, \quad f(v)=\int_0^L fvdx \quad \cdots\cdots(16)$$

$$B(w)(v) = -\int_0^L (w+s)(x)^- v(x)dx \quad \cdots\cdots(17)$$

Forms $A(w)$, $B(w)$, and f are continuous linear functionals on the space V, and operators A and B map the product space $V\times V$ into R. Then (15) can be written as

$$w_\epsilon \in V:\ A(w_\epsilon)(v) + \epsilon^{-1}B(w_\epsilon)(v) = f(v)\,, \quad v\in V \quad \cdots\cdots(18)$$

We shall show that w_ϵ converge to the solution $w\in K$ of the variational inequality (1) as $\epsilon\to 0$. To this end the following inequalities on R are usefull:

$$\left.\begin{array}{l} (-a^-)a = (-a^-)^2 \quad \text{because of } a=a^+-a^- \\ (-a^-+b^-)(a-b)\geq(a^--b^-)^2 \\ (-a^-)b\leq a^-b^- \\ (-a^-+b^-)c\leq|a-b||c| \end{array}\right\} \quad \cdots\cdots(19)$$

where $a^+=\text{Max}(0., a)$ and $a^-=\text{Max}(0., -a)$. Applying (19),

$$(B(v)-B(w))(v-w)\geq||(v+s)^- - (w+s)^-||_0 \quad \cdots\cdots(20)$$

for every $v, w\in V$. On the other hand, the bilinear for $A(\,.\,)(\,.\,)$ satisfies the inequalities

$$c_1||v||_2{}^2 \leq A(v)(v) \leq c_2||v||_2{}^2 \quad \cdots\cdots(21)$$

for proper positive constants c_1 and c_2, and for every $v\in V$.

(1) Convergence of w_ϵ to w as $\epsilon\to 0$

The sequence w_ϵ of the solution of penalized problem (18) converges to the solution $w\in K$ of the variational inequality (1) as $\epsilon\to 0$.

(Uniform Boundedness of w_ϵ) Applying inequalities (20) and (21), for every $v\in K$ we have

$$\begin{aligned} f(v-w_\epsilon) &= A(w_\epsilon)(v-w_\epsilon) - 1/\epsilon(B(v)-B(w_\epsilon)) \\ &\cdot(v-w_\epsilon)\leq A(w_\epsilon)(v-w_\epsilon) - 1/\epsilon \\ &\cdot||(w_\epsilon+s)^-||_0{}^2 \leq c_2||w_\epsilon||_2||v||_2 \\ &- c_1||w_\epsilon||_2{}^2 - 1/\epsilon||(w_\epsilon+s)^-||_0{}^2 \quad \cdots(22) \end{aligned}$$

from equation (18). Then

$$||w_\epsilon||_2 \leq 2(c_2/c_1+1/2)||v||_2 + 2/c_1||f||_{-2} \quad \cdots\cdots(23)$$

and

$$||(w_\epsilon+s)^-||_0 \leq (c_2||w_\epsilon||_2||v||_2 + ||f||_{-2}(||v||_2+||w_\epsilon||_2)) \quad \cdots\cdots(24)$$

for any $v\in K$, where $||f||_{-2}$ is the dual norm to $||\,.\,||_2$.

The estimate (23) means that w_ϵ is uniformly bounded in ϵ, and then implies the existence of a subsequence of w_ϵ, denoted by w_ϵ again, which converges to an element w weakly in V. Moreover, (22) yields

$$\begin{aligned} f(v-w_\epsilon) &\leq A(w_\epsilon)(v-w_\epsilon) \\ &\leq A(w)(v-w_\epsilon) + (A(w_\epsilon)-A(w))(v-w) \end{aligned}$$

Passing to the limit $\epsilon\to 0$, we have

$$A(w)(v-w)\geq f(v-w)\,, \quad \forall v\in K\,,$$

since linearlity of A implies that $A(w_\epsilon)$ converges to $A(w)$ weakly. That is, the weak limit w of w_ϵ is a solution of the variational inequality (1) *if* $w\in K$.

($w\in K$) From the estimate (24),

$$||(w+s)^-||_0 \leq \liminf_{\epsilon\to 0}||(w_\epsilon+s)^-||_0 = 0$$

since the norm is convex and continuous, i.e., weakly lower simi-continuous. Thus

$$(w+s)^- = 0 \text{ i.e. } w+s\geq 0 \text{ i.e. } w\in K\,.$$

Therefore the weak limit of w is a solution of (1). Since the solution of (1) is unique, every convergent subsequence converges to the unique limit $w\in K$ as $\epsilon\to 0$. In turn, the original sequence w_ϵ converges to the solution $w\in K$.

(Strong Convergence) From (18) and (1),

$$\begin{aligned} c_1||w-w_\epsilon||_2{}^2 &\leq A(w-w_\epsilon)(w-w_\epsilon) \\ &\leq (A(w)-f)(w-w_\epsilon)\,. \end{aligned}$$

Since w_ϵ converges to w weakly in V,

$$\lim_{\epsilon\to 0}||w-w_\epsilon||_2{}^2 = 0 \quad \cdots\cdots(25)$$

that is, w must converge to $w\in K$ *strongly*.

Thus we have shown that the solution of the penalized problem (18) provides an approximation to the variational inequality (1). We next consider a discretization of (18) by finite element methods.

The first order Hermite interpolation is applied to keep C^1-continuity to the discrete model. We shall consider the penalized equation (18) as the summation over all elements;

$$\sum_{e=1}^{E}(A^e(w_\epsilon)+\epsilon^{-1}B^e(w_\epsilon))(v)=\sum_{e=1}^{E}f^e(v) \quad \forall v\in V \qquad \cdots\cdots(26)$$

where E_m is the number of total elements. Using shape functions

$$\left.\begin{array}{ll} N_1^e(\xi)=1-3\xi^2+2\xi^3 & M_1^e(\xi)=h\xi(\xi-1)^2 \\ N_2^e(\xi)=3\xi^2-2\xi^3 & M_2^e(\xi)=h\xi^2(\xi-1) \end{array}\right\} \qquad \cdots\cdots(27)$$

the deflection w is interpolated by

$$w(\xi)=\sum_{i=1}^{2}w^iN_i^e(\xi)+\theta^iM_i^e(\xi) \qquad \cdots\cdots(28)$$

Here h is the length of an element, $\xi\in(0,1)$ is the local coordinate attached to each finite element, and $\xi=0$ and $\xi=1$ correspond with nodal points of the element. Then

$$A^e(w)(v)=V^T\underset{\sim}{K}^eW \text{ and } f^e(v)=V^TF^e \qquad \cdots(29)$$

where

$$\left.\begin{array}{l} W^T=\{w^i,\theta^i\}\,, \quad (F^e)^T=\{f_j, m_j\}\,, \\ K^2_{2i-1,2j-1}=h^3\int_0^1 EIN_i^{e\prime\prime}N_j^{e\prime\prime}d\xi\,, \\ f_j=h\int_0^1 fN_j^ed\xi\,, \quad \text{etc.} \end{array}\right\} \qquad \cdots\cdots(30)$$

The penalty term $B^e(w_\epsilon)(v)$ is discretized by the rule of numerical integration such as the trapezoidal quadrature rule. Suppose that the rule of numerical integration

$$\int_{-1}^{1}f(\eta)d\eta=\sum_{i=1}^{N_g}g_if(\eta_i) \qquad \cdots\cdots(31)$$

is applied, where N_g is the number of integration points and g_i is its weight. Then the penalty term becomes

$$\left.\begin{array}{l} B^e(w)(v)=h\sum_{i}^{N_g}-g_i(w+s)(\xi_i)^-v(\xi_i) \\ \xi_i=(\eta_i+1)/2 \end{array}\right\} \qquad \cdots\cdots(32)$$

Applying the relation

$$-v^-=H(-v)v \text{ with } H(v)=\begin{cases}1 \text{ if } v>0\\ 0 \text{ if } v<0\end{cases} \qquad \cdots\cdots(33)$$

we have

$$B^e(w)(v)=h\sum_{i=1}^{N_g}H(-(w+s)(\xi_i)) \cdot g_i((w+s)(\xi_i))v(\xi_i)$$

i.e., in the vector matrix form,

$$B^e(w)(v)=V^T\underset{\sim}{K}_p^e(W)W+V^TS^e(W) \qquad \cdots\cdots(34)$$

Substituting (29) and (34) into (26), and summing up elementwise terms (i.e., $\underset{\sim}{K}=\sum_{e=1}^{E}\underset{\sim}{K}^e$, $\underset{\sim}{K}_p(W)=\sum_{e=1}^{E}\underset{\sim}{K}_p^e(W)$ etc.), we can meet the following global form

$$(\underset{\sim}{K}+\epsilon^{-1}\underset{\sim}{K}_p(W_\epsilon))W_\epsilon=F-\epsilon^{-1}S(W_\epsilon) \qquad \cdots\cdots(35)$$

since v is arbitrary in (26). The equation (35) is certainly nonlinear because of the penalty stiffness $\underset{\sim}{K}_p(W_\epsilon)$ and the penalty load vector $S(W_\epsilon)$. One of methods to solve (35) is the following *successive iteration* scheme

$$(\underset{\sim}{K}+\epsilon^{-1}\underset{\sim}{K}_p(W_\epsilon^n))W_\epsilon^{n+1}=F-\epsilon^{-1}S(W_\epsilon^n)\,, \quad n=1,2,\ldots \qquad \cdots\cdots(36)$$

(2) Choice of Integration Points

Natural choice of numerical integration is either the two-point trapezoidal rule ($N_g=2$, $g_i=1$, $i=1$ and 2, $\eta_1=-1$, $\eta_2=1$) or the four-point trapezoidal rule ($N_g=4$, $g_1=g_4=1/3$, $g_2=g_3=2/3$, $\eta_1=-1$, $\eta_2=-1/3$, $\eta_3=1/3$, $\eta_4=1$). If the two-point rule is applied, then the unilateral constraint $w+s\geq 0$ is controlled at the nodal point of the finite element model. For the four-point rule, we control the condition both inside and end points of each element. Since the first order Hermite interpolation is represented by the cubic polynomial which has four independent coefficients, $w+s$ is zero every-where in the element if $w+s$ is zero at four different points in the element. It is also possible to apply the four-point Gaussian rule instead of the trapezoidal rule. Numerical experiments show no essential differences between these choices as long as deflection is concerned. We will show this in the following example.

(EXAMPLE 1) Let a simply supported beam be spanned over a flat rigid foundation, and let the distance between the beam and foundation be $s=0.2$. Suppose that the stiffness EI and the length L of the beam are 1) and 2), respectively. Using symmetry of the problem, we just consider only the left half portion of beam which is divided into four finite elements. Suppose that the uniform load $f=-10$ is applied on the beam. Then the left edge of contact is given by

$$a=\sqrt[4]{24EIs/(-f)}=0.832\,4$$

and the deflection is obtained as

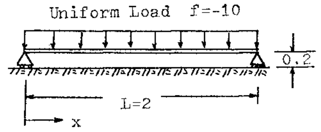
$$w(x)=\begin{cases}(f(a-x)^3/EI)(-(a-x)/24 \\ \quad +a/12)-s \quad \text{if } x\leq a \\ -s \quad \text{if } x\geq a\end{cases}$$

Moreover, the reaction force at the edge $x=a$ of contact is given by $R=-fa/2$.

Numerical results are given in **Table 1**. While more than ten iterations are necessary to obtain convergence of the successive iteration (36) for

Table 1 Deflection of the Beam (EXAMPLE 1).

x	2-Point Trapezoid	4-Point Trapezoid	4-Point Gaussian	Exact Solution
0.00	0.0000	0.0000	0.0000	0.0000
0.25	−0.1140	−0.1149	−0.1149	−0.1109
0.50	−0.1811	−0.1822	−0.1822	−0.1796
0.75	−0.2000	−0.1997	−0.1997	−0.1996
1.00	−0.2000	−0.2000	−0.2000	−0.2000
Number of Iterations	3	15	17	



the four-point rule, only three iterations are enough for the two-point rule and its quality of numerical solution is not bad as long as the deflection of the beam is concerned. However, at the point $x=0.75$, where contact does not occur theoretically, the beam contacts the foundation in the case of the two-point rule, whereas contact does not occur for the four-point rule.

(3) Relationship of ϵ and h

If the problem is well-posed, the penalty parameter ϵ should not depend on the mesh size h of the finite element model. The parameter h affects only on the "size" of the admissible set of the finite element model which is a subspace of the Sobolev space $H^2(0, L)$. On the other hand the convergence of the penalty formulation as $\epsilon \to 0$ is dominated by the topology of $H^2(0, L)$, but is not dominated by the "size" of the admissible set. However, if we adopt the successive iteration (36) to solve the nonlinear penalized equation (35), the penalty parameter ϵ seems to depend on the mesh size h. This is observed when we solve the contact problem described in EXAMPLE 2 below. If ϵ is small enough, say $\epsilon=10^{-3}$, and if $h=0.05$, then the numerical solution obtained by (36) is the one which is locked on the foundation. If $h=0.25$ for $\epsilon=10^{-3}$, we can get a reasonable solution as well as the case that $h=0.1$ for $\epsilon=10^{-1}$. It is possible to avoid this kind of dependence of ϵ upon h by changing the method of solver (36) of (35). Indeed, the following *two-step* iteration

$$(\underset{\sim}{K}+\epsilon^{-1}(t)\underset{\sim}{K}_p(W^n_{\epsilon(t)}))W^{n+1}_{\epsilon(t)}=F-\epsilon^{-1}(t)S(W^n_{\epsilon(t)}) \qquad \cdots\cdots(37)$$

leads independence of ϵ and h. Iteration is taken by n and t. First, for fixed t, we iterate (37) with respect to n until convergence is obtained, then $(t+1)$-th step is performed by taking $\epsilon(t)\to 0$ as $t\to 0$, for example $\epsilon(t)=10^{-t}$. The two-step successive iteration thus starts from mild penalty to severe one gradually as incremental method for nonlinear equations.

(EXAMPLE 2) Let a cantilever be supported on the rigid foundation, the shape of which is given by $y=-x^3$. Suppose that the stiffness EI is 1/12 and the length of the beam is $L=1$. Let a point load $P=-0.5$ be applied at the end of beam, and let the beam is divided into ten finite elements. In this case the length a and force P_c of contact are given by

$$a=-PL/(6EI-P) \text{ and } P_c=6EI-P \text{ at } x=a$$

The deflection of the cantilever is obtained as

$$w(x)=\begin{cases}-x^3 & \text{if } x\leq a\\ (P(3Lx^2-x^3)+P_c(3a^2x-a^3))/6EI\end{cases}$$

In the present example, $a=0.5$, $P_c=1.$ at $x=0.5$, and

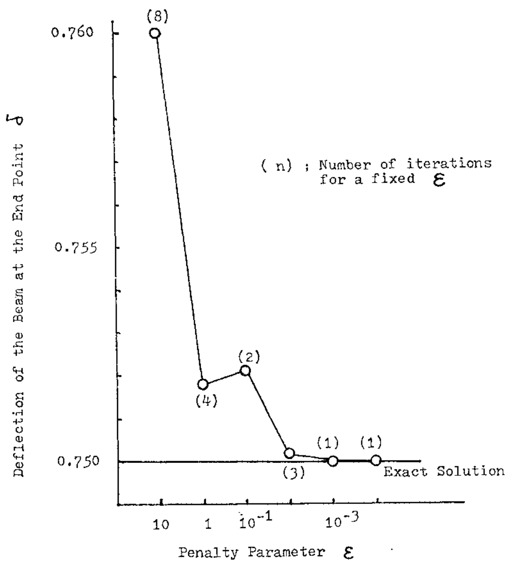


Fig. 1 Convergence of the Two-step Successive Iteration Method For Penalty Solver.

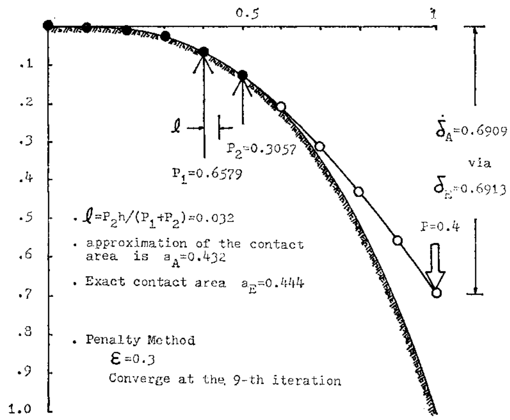


Fig. 2 Deformed Configuration of the Beam by Penalty Method for EXAMPLE 2.

$$w(x)=\begin{cases}-x^3 \text{ if } x\leq 0.5\\ x^3-3x^2+1.5x-0.25 \text{ if } x\geq 0.5\end{cases}$$

Convergence of the two-step successive iteration is shown in **Fig. 1** by the deflection at the end of cantilever. The penalty term $B_e(w_\epsilon)(v)$ is constructed by the four-point trapezoidal rule. While rather many iterations are necessary to converge for the larger penalty parameter ϵ, very few iterations are required for the smaller ϵ. Deformed configuration of the cantilever is given in **Fig. 2**. It is easily seen that the finite element approximation provides the exact solution is particular case.

(4) Contact Pressue p_ϵ

From the value of the penalty virtual work (13), an approximation of the contact pressure p_ϵ can be identified as

$$p_\epsilon(x)=-\epsilon^{-1}(w_\epsilon+s)(x)^- \quad\cdots\cdots(38)$$

if the analogous manner to the case of plane elasticity[12] is applied. However, this (38) may not be proper for the present beam contact problem, since concentrated reaction forces may exist at edges of contact surfaces. Such point reactions cannot possess the virtual work represented by the integral form. In order to include such point forces, the virtual work due to the contact pressure P should be given by the duality pairing $p(v)$ on $V'\times V$ for $p\in V'$ and $v\in V$. That is, $p(.)$ is a continuous linear functional on V. Thus, from the penalty virtual work (13), we identify

$$p_\epsilon(v)=-\epsilon^{-1}\int_0^L (w_\epsilon+s)(x)^- v(x)dx \quad\cdots\cdots(39)$$

instead of (38). By (32),

$$p_\epsilon(v)=\epsilon^{-1}B^e(w_\epsilon)(v)\,,$$

$$B^e(w)(v)=h\sum_{i=1}^{Ng}-g_i((w+s)(\xi_i))^- v(\xi_i)$$

Setting

$$p_\epsilon^{e,i}=-\epsilon^{-1}hg_i(w_\epsilon+s)(\xi_i)^- \quad\cdots\cdots(40)$$

yields

$$B^e(w_\epsilon)(v)=\sum_{i=1}^{Ng}p_\epsilon^{e,i}v(\xi_i)=\sum_{i=1}^{Ng}p_\epsilon^{e,i}(\delta(\xi_i)v)$$

where $\delta(.)$ is the Dirac delta function. Then

$$p_\epsilon(v)=\sum_{i=1}^{Ng}\sum_{e=1}^{E}p_\epsilon^{e,i}(\delta(\xi_i)v)=\sum_{I=1}^{N_I}p_\epsilon^{I}(\delta(x_I)v) \quad\cdots\cdots(41)$$

i.e.

$$p_\epsilon=\sum_{I=1}^{N_I}p_\epsilon^{I}\delta(x_I) \quad\cdots\cdots(42)$$

Thus p_ϵ^I, obtained by (40) and (41), is the value of the contact pressure p_ϵ (i.e., the equivalent nodal contact force) associated with the point $x=x_I$, where N_I is the number of all integration points in the model.

Table 2 Convergence of the Contact Forces as $\epsilon\to 0$.

x_I/ϵ	10	1	0.1	0.01	0.001
0.366 7	0.064 3	0.022 1	0.	0.	0.
0.4	0.082 9	0.067 7	0.	0.	0.
0.433 3	0.102 4	0.127 7	0.052 9	0.	0.
0.466 7	0.120 3	0.190 4	0.258 1	0.079 6	0.001 0
0.5	0.133 7	0.236 9	0.406 7	0.874 1	1.000 0
0.533 3	0.139 3	0.239 8	0.289 2	0.049 2	0.
0.566 7	0.131 3	0.135 2	0.	0.	0.
0.6	0.130 4	0.	0.	0.	0.
Σp^I	1.051 6	1.019 7	1.006 9	1.003 0	1.001 0
Number of Iterations	7	3	1	2	2

(*) Exact Solution is $P=1.$ at $x=0.5$.

We shall show p_ϵ^I for EXAMPLE 2 in **Table 2**. It can be realized from the table that p_ϵ at $x=0.5$ converges to p rapidly as $\epsilon\to 0$.

4. MIXED METHOD

We introduce a variational inequality of the mixed type, which is compatible to both penalty and relaxation methods, in this section.

Suppose that

$$w''=M/EI \quad\cdots\cdots(43)$$

in the inequality (1). If $M\in H_0^1(0, L)$, integration by parts in (1) yields

$$-\int_0^L M'(v-w)'dx\geq\int_0^L f(v-w)dx \quad\cdots\cdots(44)$$

A variational expression to (43) is

$$\int_0^L (M/EI-w'')(N-M)dx\geq 0$$

for every $N\in H_0^1(0, L)$. By integration by parts, we obtain

$$\int_0^L ((M/EI)(N-M)+w'(N-M)')dx\geq 0 \quad\cdots\cdots(45)$$

Adding (44) and (45) yields a mixed variational formulation to the primal variational inequality (1);

$$(w, M)\in K:\ \int_0^L ((M/EI)(N-M)+w'(N-M)' -M'(v-w)')dx\geq\int_0^L f(v-w)dx,$$

$$\text{for every } (v, N)\in K \quad\cdots\cdots(46)$$

where

$$K=\{(v, N)\in H_0^1(0, L)xH_0^1(0, 1):\ v+s\geq 0\} \quad\cdots\cdots(47)$$

The space $H_0^m(0, L)$ is the completion of $C_0^\infty(0, L)$ in the norm of $H^m(0, L)$, and $C_0^\infty(0, L)$ is the space of all infinitely differentiable functions with compact support in $(0, L)$[11].

Differences of (47) from (2) are that the deflection w is merely an element of $H^1(0, L)$ in (47) via $H^2(0, L)$ in (2), and that the moment M is assumed to be in H^1 $(0, L)$ while the moment EIw'' in (1) belongs to $L^2(0, L)$. The later fact implies that point moments cannot be applied on the beam within the framework of the mixed formulation (46). However, since (46) is defined on $H^1(0, L) \times H^1$ $(0, L)$, C^0-elements are applicable to discretize the variational inequality (46), while C^1-elements are required for (1).

Let the interval $(0, L)$ be decomposed into the set $\{I_e\} \underset{e=1}{E}$ of subintervals; $I_e = (x_e, x_{e+1})$ such that $\cup \bar{I}_e = [0, L]$. Let ξ be the local coordinate attached to a specific interval I_e such that $\xi = 0$ and $\xi = 1$ correspond to $x = x_e$ and $x = x_{e+1}$, respectively. Within an interval I_e, M and w are interpolated by

$$\left.\begin{aligned} &M(\xi) = \sum_{i=1}^{2} M^i \phi_i(\xi) \text{ and } w(\xi) = \sum_{i=1}^{2} w^i \phi_i(\xi) \\ &\phi_1(\xi) = 1 - \xi \text{ and } \phi_2(\xi) = \xi \end{aligned}\right\} \quad \cdots\cdots(48)$$

Substitution these into (46) yields

$$\sum_{i,j=1}^{2} (N^i - M^i)(k_{ij}^{e1} M^j + k_{ij}^{e2} w^j) - (v^i - w^i) k_{ij}^{e2} M \geq \sum_{i=1}^{2} (v^i - w^i) f_i \quad \cdots\cdots(49)$$

where

$$\left.\begin{aligned} &k_{ij}^{e1} = h \int_0^1 \phi_i \phi_j / EI d\xi, \quad k_{ij}^{e2} = \int_0^1 \phi_i' \phi_j' d\xi / h \\ &f_j^e = \int_0^1 f \phi_j d\xi \end{aligned}\right\} \quad \cdots\cdots(50)$$

Here h is the length of the element. Assembling of (49) all over the intervals $\{I_e\}$, we arrive at the global form of the discretization of (46):

$$(M, w) \in K_h: \quad \sum_{j,i=1}^{N} (\hat{N}^i - M^i)(K_{ij}^1 M^j + K_{ij}^2 w^j) - (v^i - w^i) K_{ij}^2 M^j \geq \sum_{i=1}^{N} (v^i - w^i) F_i,$$
$$\text{for every } (\hat{N}, v) \in K_h \quad \cdots\cdots(51)$$

where

$$K_h = \{(\hat{N}, v) \in R^N \times R^N: \hat{N}^1 = \hat{N}^N = v^1 = v^N = 0, \quad v^i + s^i \geq 0\} \quad \cdots\cdots(52)$$

Here $N = E + 1$ is the number of nodal points, $s^i = s(x_i)$, x_i is the coordinate of the i-th nodal point, and

$$K_{ij}^1 = \sum_{e=1}^{E} k_{ij}^{e1} \text{ and } K_{ij}^2 = \sum_{e=1}^{E} k_{ij}^{e2} \quad \cdots\cdots(53)$$

(1) Penalty Methods

Since the piecewise linear polynomial is used for the deflection, the contact constraint

$$(w + s)(x) \geq 0 \text{ in } (0, L)$$

is satisfied if it is satisfied at each nodal point of the finite element model. That is, provided with the distance s is given by the piecewise linear polynomial, K_h is a subset of the constrained set K, (47). Applying the penalty

$$-\epsilon^{-1} \sum_{i=1}^{N} (w^i + s^i)^- (v^i - w^i)$$

in the formulation (51), we have the penalized problem

$$\left.\begin{aligned} &\sum_{i=1}^{N} K_{ij}^1 M^j + K_{ij}^2 w^j = 0 \\ &-\sum_{i=1}^{N} K_{ij}^2 M^j - \epsilon^{-1}(w^i + s^i)^- = F_i \end{aligned}\right\} \quad \cdots\cdots(54)$$

Convergence of (54) to (51) as $\epsilon \to 0$ follows from the similar arguments in the previous section.

Nonlinear equations (54) is solved by the successive iteration similar to (36). In this case we may not need to use the two-step successive iteration (37), since the matrix K^2 in (54) is associated to the Laplace operator. Moreover, since the penalty term consists of only the deflection at nodal points. This leads much faster convergence of the successive interation is expected than the case (34) of the primal formulation.

As an example of the mixed method, we solve a similar problem to EXAMPLE 2 with the applied load $P = -0.4$ at the free end of the cantilever. The beam separates from the foundation at $x = 4/9$, and the deflection of the end point is $\delta = 0.69$ 13 in this case. When the beam is discretized by ten finite elements, the point at which the beam separate from the foundation, locates inside of an element. Thus, a super convergent result (i.e., the numerical solution which is exact at nodal points) might not be obtained,

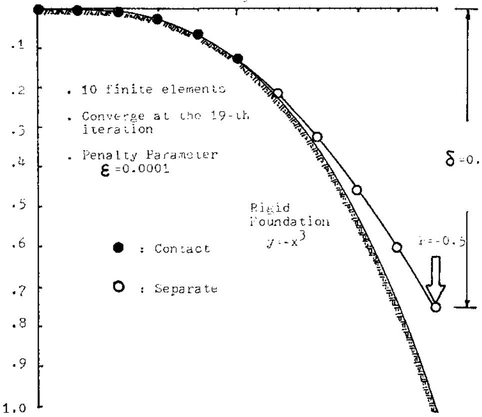


Fig. 3 Numerical Results of EXAMPLE 2 with $P = -0.4$ by the Mixed Penalty Method.

while it has been achieved in EXAMPLE 2. The numerical solution indicates that nodal points up to $x=0.5$ contact the foundation and that reacting pressure P_1 and P_2 are obtained at $x=0.4$ and 0.5 as shown in **Fig. 3**. Since the contact force exists only on the point $x=a$, P_1 and P_2 may imply an approximation of the separating point of the beam. Since moments should not be creased by the contact force about the point where the beam separates from the foundaation, we can obtain the approximation

$$a_A=x_1+P_2h/(P_1+P_2)=0.4+0.032=0.432$$

via the exact value $a_E=4/9$. The deflection of the end of cantilever is $\delta_A=0.6907$ via $\delta_E=0.6913$, i.e., only 0.09% error involves in the approximation.

(2) Relaxation Methods

While the penalty method has been applied to solve (51), the relaxation method is also applicable to solve the variational inequality (51). Indeed, since K^1 is invertible, we have

$$M^j=-\sum_{k,h=1}^{N}(K^1)^{-1}_{jk}K^2_{kh}w^h \quad \cdots\cdots(55)$$

from $(54)_1$. Substitution of (55) into (51) reduces the variational inequality

$$\sum_{i,j=1}^{N}(v^i-w^i)(K_{ij}w^j-F_i)\geq 0\,, \quad \forall v^i+s^i\geq 0 \quad \cdots\cdots(56)$$

where

$$K_{ij}=\sum_{k,h=1}^{N}K^2_{ih}(K^1)^{-1}_{hk}K^2_{kj} \quad \cdots\cdots(57)$$

It is noted that for any positive number ρ, (56) is equivalent to

$$\sum_{i,j=1}^{N}(v^i-w^i)(w^i-w^i+\rho(K_{ij}w^j-F_i))\geq 0$$

i.e.,

$$w^i=\mathrm{Max}\left(-s^i,\, w^i-\rho\left(\sum_{j=1}^{N}K_{ij}w^j-F_i\right)\right) \cdots(58)$$

The last step is clear from the consideration in R. For a given f, the problem

$$a\geq s:\ (a-f)(b-a)\geq 0\,,\quad b\geq s$$

has a unique solution $a=\mathrm{Max}(s,f)$. Thus, we need to solve the nonlinear equation (58) by the following projectional relaxation method:

(i) Pick $w_1{}^i\geq -s^i$, $1\leq i\leq N$

(ii) $$w^i_{l+1}=\mathrm{Max}\left\{-s^i,\,(1-\gamma)w_l{}^i+\gamma\left(-\sum_{j=1}^{i-1}K_{ij}w^j_{l+1}-\sum_{j=i+1}^{N}K_{ij}w_l{}^j+F_i\right)\Big/K_{(ii)}\right\} \cdots\cdots(59)$$

(iii) Repeat (ii) until a small tolerance

$$e_l=\sum_{i=1}^{N}|w^i_{l+1}-w_l{}^i|/\sum_{i=1}^{N}|w^i_{l+1}|$$

is obtained.

Here $\rho=\gamma/K_{(ii)}$ has been applied in (58). The iteration scheme (59) is called the projectional super over relaxation (S.O.R.) method and provides convergence for $0<\gamma<2$. Numerical experiments imply that the optimal factor γ exists in the interval (1.9, 2) for the mixed formulation (56).

5. RECIPROCAL METHOD

The methods discussed so far control the deflection of the beam in order to satisfy the unilateral contact condition. We here describe the method which control the contact pressure instead of the deflection of the beam. This method is certainly not the Lagrangian multiplier method which had been applied in the previous paper[1] and which could not provide the accurate contact pressure and fast convergence of Uzawa's algorithm to find a saddle point. The present method is based on the reciprocal formulation of the unilateral problem which is written by the contact pressure[1]. We first recall the unilateral contact condition to get the reciprocal formulation:

$$\left.\begin{array}{l} p\geq 0 \text{ if } w+s=0 \text{ (contact)} \\ p=0 \text{ if } w+s\geq 0 \text{ (separate)} \end{array}\right\} \quad \cdots\cdots(60)$$

and

$$w+s\geq 0 \text{ in } (0,L) \quad \cdots\cdots(61)$$

Here p is the contact pressure. It can be easily proved that the system of inequalities (60) and (61) is equivalent to

$$p\geq 0:\ (w+s)(q-p)\geq 0\,,\quad \forall q\geq 0 \quad \cdots\cdots(62)$$

If the contact pressure is known, the equilibrium equation of the beam under the applied force f is written by

$$(EIw'')''=f+p \text{ in } (0,L) \quad \cdots\cdots(63)$$

and the associated boundary conditions

$$w(0)=w(L)=0 \quad \cdots\cdots(64)$$

The boundary value problem (63) and (64) is uniquely solvable by the form

$$w=G(f)+G(p)\,. \quad \cdots\cdots(65)$$

Substituting (65) into (62), we have

$$p\geq 0:\ (G(p)+\hat{s})(q-p)\geq 0\,,\quad \forall q\geq 0 \cdots\cdots(66)$$

where $\hat{s}=s+G(f)$. We now extend the above pointwise arguments to the variational one.

Let q be an arbitrary continuous linear functional defined on a closed subspace

$$V=\{v\in H^2(0,L);\ v(0)=v(L)=0\} \quad \cdots\cdots(67)$$

of the Sobolev space $H^2(0,L)$. Let V' be the set of all continuous linear functionals on V, and be called the dual space of V. In general V' contains point and distributed moments and forces Let G be the inverse of linear operator A: $V\to V'$

defined by

$$A(w)(v)=\int_0^L EIw''v''dx \quad \cdots\cdots(68)$$

where $(.)(.)$ is the duality pairing on $V'\times V$.

The variational form to the pointwise relation (66) is then written by

$$p\in K:\ (q-p)(G(p)+\hat{s})\geq 0,\quad q\in K \quad \cdots\cdots(69)$$

where

$$K=\{q\in V':\ q\geq 0\} \quad \cdots\cdots(70)$$

The form (69) is called the *reciprocal variational inequality* to the primal one (1).

(1) Approximation of Green's Operator

If the stiffness EI of the beam is not constant, it is difficult to obtain the closed form of Green's operation to (63). Such difficulty is overcome by introducing an approximation of Green's operator, constructed by the inverse of stiffness matrix of the beam.

Suppose that the deflection of the beam is interpolated by the first order Hermite cubic polynomial in each finite element as in Section 3. Let $\underset{\sim}{G}$ is the inverse of the stiffness matrix $\underset{\sim}{K}$ under the simply supported condition $w(0)=w(L)=0$. Then the inverse relation (65) is written by

$$W=\underset{\sim}{G}F+\underset{\sim}{G}P \quad \cdots\cdots(71)$$

where generalized displacement $W^T=\{w^i,\theta^i\}$, generalized load $F^T=\{f_j, m_j\}$, and contact force $P^T=\{f_j{}^p, m_j{}^p\}$. The vectors F and P are equivalent nodal force and moment due to the applied force f and the unknown contact pressure p, respectively. The vector W is consists of nodal values of the deflection and its gradient. Combining (71) and (74) yields an approximation of Green's operator:

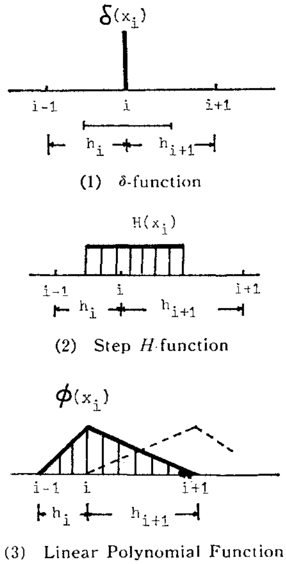
$$w(x)=G(f)(x)+G(p)(x) \quad \cdots\cdots(72)$$
$$=N(x)^T\underset{\sim}{G}F+N(x)^T\underset{\sim}{G}P \quad \cdots\cdots(73)$$

where $N(x)^T=\{N_i(x),\ M_i(x)\}_{i=1}^N$ is the vector of shape functions.

We next discretize the contact pressure p, which merely belongs to the dual space of V', (67), in the variational formulation. That is, p is a linear functional but not a function such as the deflection $w(x)$. This implies various manners of discretization of p. We shall show typical three kind of discretization of p, as shown in **Fig. 4**:

a) δ-function

For the one dimensional domain, the Dirac delta function $\delta(x_i)$ associated with the point x_i belongs to $H^{-1/2-\epsilon}(0,L)$, $\epsilon>0$, i.e., $\delta(x_i)\in V'$. Then the contact pressure p can be discretized by



(1) δ-function
(2) Step H function
(3) Linear Polynomial Function

Fig. 4 Basis Functions for the Contact Pressure.

$$p=\sum_{i=1}^N p^i\delta(x_i) \quad \cdots\cdots(74)$$

$$\delta(x_i)(v)=v(x_i) \quad \cdots\cdots(75)$$

The value p^i corresponds to the resultant of contact pressure on the interval $(x_i-h_i/2,\ x_i+h_{i+1}/2)$. Moreover, the vector P is formed by $f_j{}^p=p^j$ and $m_j{}^p=0$.

b) *H*-function

The second way of the discretization of p uses a kind of the Heaviside step function H. Let

$$p=\sum_{i=1}^N p^iH(x_i) \quad \cdots\cdots(76)$$

$$\left.\begin{aligned} &H(x_i)(v)=H_i{}^{-1}\int_{x_i-h_i/2}^{x_i+h_{i+1}/2} v(x)dx,\\ &H_i=(h_{i+1}+h_i)/2 \end{aligned}\right\} \quad \cdots\cdots(77)$$

The value p_i/H_i is the average of the contact pressure on the interval $(x_i-h_i/2,\ x_i+h_{i+1}/2)$. The equivalent nodal force and moment of p are

$$f_j{}^p=\sum_{i=1}^N p^iH(x_i)N_j \text{ and } m_j{}^p=\sum_{i=1}^N p^iH(x_i)M_j \quad \cdots\cdots(78)$$

c) Linear Polynomial

The third discretization of p is the one which is similar to the finite element interpolation. Here we consider the linear interpolation, but extension to more sophisticated interpolations seems to be straightforward. Let

$$p=\sum_{i=1}^N p^i\phi(x_i) \quad \cdots\cdots(79)$$

$$\phi(x_i)(v)=\int_{x_{i-1}}^{x_{i+1}} v(x)\phi_i(x)dx \Big/ \int_{x_{i-1}}^{x_{i+1}} \phi_i(x)dx \qquad \cdots\cdots(80)$$

where

$$\phi_i(x)=\begin{cases}(x-x_{i-1})/h_i & \text{if } x\in(x_{i-1},x_i)\\ (-x+x_{i+1})/h_{i+1} & \text{if } x\in(x_i,x_{i+1})\\ 0 & \text{if otherwise}\end{cases} \qquad \cdots\cdots(81)$$

The value of p^i is the resultant of the contact pressure on the interval (x_{i-1}, x_{i+1}) with respect to the weight function $\phi_i(x)$. The equivalent nodal force and moment are given by

$$f_j{}^p=\sum_{i=1}^N p^i\phi(x_i)N_j \text{ and } m_j{}^p=\sum_{i=1}^N p^i\phi(x_i)M_j \qquad \cdots\cdots(82)$$

We obtain the form of discretization of the reciprocal variational inequality (69). To do this, let

$$p=\sum_{i=1}^N p^iB(x_i) \qquad \cdots\cdots(83)$$

where $B(x_i)$ is one of $\delta(x_i)$, $H(x_i)$, and $\phi(x_i)$. From (73)

$$\left.\begin{array}{l}\hat{s}(x)=s(x)+N(x)^T\underset{\sim}{G}F\\ G(p)(x)=N(x)^T\underset{\sim}{G}\underset{\sim}{P}\end{array}\right\} \qquad \cdots\cdots(84)$$

Then we have

$$q(G(p))=\sum_{i,j=1}^N q^iC_{ij}p^j \text{ and } q(s)=\sum_{i=1}^N q^iS_i \qquad \cdots\cdots(85)$$

where

$$C_{ij}=\{B(x_i)N_k, B(x_i)M_k\}\cdot\begin{bmatrix}G_{2k-1,2h-1} & G_{2k-1,2h}\\ G_{2k,2h-1} & G_{2k,2h}\end{bmatrix}\begin{Bmatrix}B(x_j)N_h\\ B(x_j)M_h\end{Bmatrix} \qquad \cdots\cdots(86)$$

and

$$S_i=B(x_i)s+\{B(x_i)N_j, B(x_i)M_j\}\underset{\sim}{G}F \qquad \cdots\cdots(87)$$

Therefore, the discretization of the variational inequality (69) becomes

$$p^i\geq 0:\ \sum_{i,j=1}^N (q^i-p^i)(C_{ij}p^j+S_i)\geq 0,\quad \forall q^i\geq 0 \qquad \cdots\cdots(88)$$

For the case that $B(x_i)=\delta(x_i)$, we have

$$C_{ij}=G_{2i-1,2j-1} \qquad \cdots\cdots(89)$$

The system of inequalities (88) can be solved by the projectional S.O.R. method described in (59), since the matrix C is dominated by its diagonal and is positive definite. The same problem with EXAMPLE 1 is solved by the reciprocal method, and its numerical results are shown in **Fig. 5**. Total pressure of contact is $p_c=5.985$ via the exact one is 5.838. Moment by the contact pressure at the origin is 5.0638 while the exact one is 5.0. These numerical solutions indicate properness of the approximation of the reciprocal variational inequality.

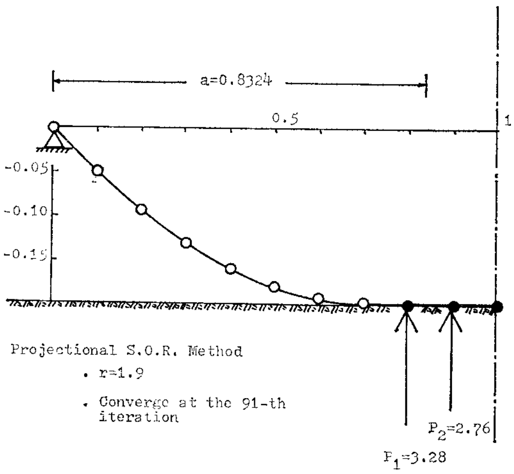


Fig. 5 Numerical Results of EXAMPLE 1 by Reciprocal Variational Inequality.

(2) Direct Discretization of Green's Operator

If the stiffness EI is constant over the beam, we can explicitly construct Green's operator for given boundary conditions. For example, if the beam is simply supported, we have

$$\left.\begin{array}{l}G(p)(x)=\int_0^L g(x;y)p(y)dy\\ g(x;y)=\begin{cases}(L-x)(2L-x)x(y-y^3)/6EIL & \text{if } x\leq y\\ (L-y)(2L-y)y(x-x^3)/6EIL & \text{if } x\geq y\end{cases}\end{array}\right\} \qquad \cdots\cdots(90)$$

If the contact pressure p is given by (84) with $B(x_i)=\delta(x_i)$, then

$$C_{ij}=g(x_i;x_j) \qquad \cdots\cdots(91)$$

Thus we can get the similar inequality (88) for the case that Green's operator can be obtained by the closed form.

6. CONCLUDING REMARKS

Unilateral problems of a beam have been formulated by the primal, mixed, and reciprocal variational inequalities, and have been solved numerically by the penalty and projectional relaxation methods after the finite element approximations. In this paper we have observed (1) the primal variational inequality of the Signorini problem can be solved by penalty methods (2) convergence of the penalized solutions to the one of the variational inequality is proved as the penalty parameter ϵ tends to zero, (3) the penalty term is discretized by the rule of numerical integration, (4) the nodalwise penalty (i.e., the two-point integration rule) may be enough for the constraint while the four-point integration rule of the elementwise penalty provides much more precise results, (5) the penalty parameter ϵ does not depend upon the mesh size h of the finite element model if the two-step successive iteration method is applied, (6) the mixed formulation of the Signorini problem leads the system of inequalities which can be solved by both penalty and projectional relaxation methods, (7) quality of numerical results of the mixed formulation is comparable to the one of the primal formulation with more sophisticated finite elements, (8) the contact pressure is controlled by the projectional relaxation method is the problem is formulated by the reciprocal variational inequality, (9) special discretization is necessary for the contact pressure which is a functional but is not a function, (10) expression of basis for discretization of functionals are given by the Dirac delta and Heaviside functions, (11) the reciprocal formulation is also obtained by the direct discretization of Green's function, and (12) quality of numerical solutions by reciprocal formulations is acceptable.

It is noted that the discussions in this paper can be extended to unilateral problems of linear and nonlinear plates[7] and shells without specific modifications. Furthermore, methodology of the penalty method, the mixed and reciprocal formulations with the projectional relaxation method are applicable to almost all unilateral problems in mechanics[3],[13].

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