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ON DERIVATION OF TIMOSHENKO BEAM STIFFNESS EQUATION

By Tetsuo IWAKUMA*, Masahiro AI** and Fumio NISHINO***

1. INTRODUCTION

Since shear deformation is generally small in comparison with bending deformation, it is usually neglected in static analysis of beams. The effect of shear deformation increases with increase of the ratio of cross-sectional dimensions to beam length. Shear deformation plays an important role in such dynamic problems as beam vibration of higher modes and response of a beam to impact forces. Because of this, the influence of shear deformation has been extensively studied in relation to dynamic analysis of beams. Beams showing shear deformation are usually called Timoshenko beams. Timoshenko beam theory is based on the assumption that the plane normal to the beam axis before deformation is not normal to the axis after deformation but that it remains a plane. With a further increase in the ratio of the cross-sectional dimensions to length, the assumption that the plane section remains a plane no longer holds and, as a result, the so-called shear lag phenomenon starts to play a significant role. Timoshenko beam theory is applicable only for beams in which shear lag is insignificant. This implies that Timoshenko beam theory considers shear deformation, but that it should be small in quantity.

A number of finite element analyses have been reported for vibration of Timoshenko beams[1~6] based on energy principles. Although several kinds of polynomials and the homogeneous solution for static problems are used as interpolation functions resulting in different stiffness and mass matrices, there has been no discussion of the accuracy of the stiffness matrices themselves. Errors, however, have been discussed for natural frequencies resulting from approximations in both matrices. While energy principles are used, only strain energy and energy due to inertia force are evaluated and used to derive the stiffness and mass matrices. Since the work done by boundary forces is not included in energy expressions in these references, energy principles do not reveal components of boundary forces corresponding to the assumed interpolation functions, nor boundary conditions. Instead, they are derived from physical considerations, which have led to errors in boundary conditions and in the assembly of a global matrix from the local matrices[4].

It has been reported[7], for problems governed by ordinary differential equations, that the finite element method results in the exact stiffness matrix by employing the homogeneous solution as an interpolation function. In spite of the fact that the homogeneous solution of a Timoshenko beam equation is a polynomial, the finite element method using polynomials with a sufficient number of degrees of freedom as interpolation functions does not necessarily result in an exact stiffness equation, but results in a variety of stiffness matrices[1~6]. This is due to the energy expression for the Timoshenko beam, for which the selection of the polynomial identical to the homogeneous solution of the governing equation is not obvious. The exact stiffness equation for a prismatic Timoshenko beam can be easily obtained by integrating the governing differential equations[8,9], and thus it is not important to derive the stiffness matrix by the finite element technique. However, the energy expression for the Timoshenko beam is a good example in selection of an interpolation function for the finite element method, since the resulting stiffness matrix can be compared with the exact stiffness matrix. This paper discusses this selection of interpolation functions. The energy expression for the Timoshenko beam derived from basic assumptions on kinematic field and energy principles for a three-dimensional elastic continuum is also presented, and is used to clarify the confusion present in the literature in boundary conditions for the finite element formulation of the Timoshenko beam.

* Member of JSCE, M. Eng., Research Assistant, Dept. of Civil Eng., Northwestern Univ.
** Member of JSCE, Dr. Eng., Lecturer, Dept. of Civil Eng., Hosei Univ.
*** Member of JSCE, Ph. D., Professor, Dept. of Civil Eng., Univ. of Tokyo

2. GOVERNING EQUATIONS

The in-plane behavior of straight prismatic beams of which the cross-sectional deformation is small and can be neglected is considered in this paper. A rectangular Cartesian coordinate system (x_1, x_2, x_3) with a reference frame of unit vectors ($\boldsymbol{i}_1$, $\boldsymbol{i}_2$, $\boldsymbol{i}_3$) along the coordinate axes is defined with the x_3-axis parallel to the beam axis. From this selection of the coordinate and the problem statement, loading and displacement vectors having components only on the x_2-x_3 plane are considered. The assumptions for the kinematics of the Timoshenko beam can be expressed in strain tensor terms as[10]

$$\left.\begin{array}{l} e_{23}(x_1,x_2,x_3)=e_{32}(x_1,x_2,x_3)=\frac{1}{2}\gamma(z) \\ e_{11}(x_1,x_2,x_3)=e_{22}(x_1,x_2,x_3)=e_{12}(x_1,x_2,x_3)=0 \end{array}\right\} \quad \cdots\cdots(1\cdot\mathrm{a}\sim\mathrm{e})$$

where $\gamma(z)$ denotes a function of z, and represents a shear strain constant over a cross section at z. The notation (x, y, z) is used in lieu of x_i, together with (x_1, x_2, x_3) whenever convenient. The strain-displacement relations are

$$e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}) \quad \cdots\cdots(2)$$

where, $u_i=\boldsymbol{i}_i$ component of displacement vector $\boldsymbol{u}$, and $(\ )_{,i}$=derivative with respect to x_i. The displacement components satisfying Eq. (2) and the strain conditions of Eq. (1) can be written as

$$\left.\begin{array}{l} u_1(x_1,x_2,x_3)=0\ , \quad u_2(x_1,x_2,x_3)=v(z) \\ u_3(x_1,x_2,x_3)=w(z)+y\lambda(z)\ , \quad \lambda(z)=\gamma(z)-v'(z) \end{array}\right\} \quad \cdots\cdots(3\cdot\mathrm{a}\sim\mathrm{d})$$

where $(\ )'=d(\ )/dz$, and v and $w=u_2$ and u_3, respectively, at $x_2=0$ and $x_3=z$. $\lambda(z)$, defined in Eq. (3·d), denotes the change of angle of the beam axis due to bending. Substitution of Eq. (3) into Eq. (2) yields

$$e_{33}(x_1,x_2,x_3)=e_{zz}(y,z)=w'+y\lambda' \quad \cdots\cdots(4)$$

If the $\boldsymbol{i}_i$ component of body force vector is denoted by $\bar{p}_i$, and the surface force vector working on both ends by $\bar{t}_i$, the virtual work equation can be expressed as

$$\int_V \sigma_{ij}\delta e_{ij}dV-\int_V \bar{p}_i\delta u_i dV-\int_S \bar{t}_i\delta u_i dS=0 \quad \cdots\cdots(5)$$

where V is the body and S its surface of continuum, σ_{ij} are the components of the stress tensor, and δ denotes virtual kinematic quantities. Substituting Eqs. (1) to (4) into Eq. (5) gives the virtual work equation for a beam lying on $a\leqq z\leqq b$ as

$$\int_a^b \delta\boldsymbol{\varepsilon}^T\boldsymbol{\sigma}dz-\int_a^b \delta\boldsymbol{u}^T\bar{\boldsymbol{x}}dz-[\delta\boldsymbol{u}^T\bar{\boldsymbol{T}}]_{z=a}-[\delta\boldsymbol{u}^T\bar{\boldsymbol{T}}]_{z=b}=0 \quad \cdots\cdots(6)$$

where

$$\left.\begin{array}{ll} \boldsymbol{\sigma}^T=\lfloor N\ \ V\ \ M\rfloor\ , & \boldsymbol{\varepsilon}^T=\lfloor \epsilon\ \ \gamma\ \ \kappa\rfloor \\ \boldsymbol{u}^T=\lfloor w\ \ v\ \ \lambda\rfloor\ , & \bar{\boldsymbol{x}}^T=\lfloor \bar{p}_z\ \ \bar{p}_y\ \ \bar{m}\rfloor \\ \bar{\boldsymbol{T}}^T=\lfloor \bar{N}\ \ \bar{V}\ \ \bar{M}\rfloor & \end{array}\right\} \quad \cdots\cdots(7\cdot\mathrm{a}\sim\mathrm{e})$$

and the components of these vectors are defined by

$$\epsilon=w'\ ,\quad \gamma=\lambda+v'\ ,\quad \kappa=\lambda' \quad \cdots\cdots(8\cdot\mathrm{a}\sim\mathrm{c})$$

$$N=\int_A \sigma_{zz}dA\ ,\quad V=\int_A \sigma_{zy}dA\ ,\quad M=\int_A \sigma_{zz}y dA \quad \cdots\cdots(9\cdot\mathrm{a}\sim\mathrm{c})$$

$$\bar{P}_z=\int_A \bar{p}_z dA\ ,\quad \bar{P}_y=\int_A \bar{p}_y dA\ ,\quad \bar{m}=\int_A \bar{p}_z y dA \quad \cdots\cdots(10\cdot\mathrm{a}\sim\mathrm{c})$$

$$\bar{N}=\int_A \bar{t}_z dA\ ,\quad \bar{V}=\int_A \bar{t}_y dA\ ,\quad \bar{M}=\int_A \bar{t}_z y dA \quad \cdots\cdots(11\cdot\mathrm{a}\sim\mathrm{c})$$

A denotes cross-sectional area.

Integrating Eq. (6) by parts yields equilibrium equations and boundary conditions. The resulting equilibrium equations in (a, b) can be expressed as

$$N'+\bar{P}_z=0\ ,\quad V'+\bar{P}_y=0\ ,\quad M'-V+\bar{m}=0 \quad \cdots\cdots(12\cdot\mathrm{a}\sim\mathrm{c})$$

and the boundary conditions at $z=a$ and b can be expressed as

$$\left.\begin{array}{lll} w=\bar{w} & \text{or} & n_z N=\bar{N} \\ v=\bar{v} & \text{or} & n_z V=\bar{V} \\ \lambda=\bar{\lambda} & \text{or} & n_z M=\bar{M} \end{array}\right\} \quad \cdots\cdots(13\cdot\mathrm{a}\sim\mathrm{c})$$

where $\bar{w}$, $\bar{v}$, and $\bar{\lambda}$ are the prescribed displacement components at the boundaries, and n_z is the direction cosine between $\boldsymbol{i}_z$ and the outward normal vector of the cross sections of the ends.

$$n_z=\left\{\begin{array}{ll} -1\ , & z=a \\ 1\ , & z=b \end{array}\right\} \quad \cdots\cdots(14)$$

When a concentrated force $\bar{\boldsymbol{T}}_c^T=\lfloor N_c\ \ V_c\ \ M_c\rfloor$ acts at $z=c$ between a and b, addition to Eq. (6) of the virtual work done by this force results in the same equilibrium equation as Eq. (12) for the domains, a to c, and c to b, and the identical boundary conditions at $z=a$ and b as Eq. (13). In addition, the virtual work equation gives the internal boundary conditions at $z=c$ as

$$\left.\begin{array}{lll} w_c^-=w_c^+ & \text{and} & N_c^--N_c^+=\bar{N}_c \\ v_c^-=v_c^+ & \text{and} & V_c^--V_c^+=\bar{V}_c \\ \lambda_c^-=\lambda_c^+ & \text{and} & M_c^--M_c^+=\bar{M}_c \end{array}\right\} \quad \cdots\cdots(15\cdot\mathrm{a}\sim\mathrm{c})$$

where superscripts + and − denote the plus and minus sides of the point.

Considering only an elastic material, the following constitutive equations can be assumed with Young's modulus E, and shear modulus G

$$\sigma_{zz}=E\epsilon_{zz}\ ,\quad \sigma_{yz}=2G\epsilon_{yz} \qquad (16\cdot\text{a, b})$$

Integrating Eq. (9) with Eq. (16) and making use of Eqs. (1) and (4) gives the relation between generalized stress and strain as

$$\boldsymbol{\sigma}=\boldsymbol{C}\boldsymbol{\varepsilon} \qquad (17)$$

By selecting the z-axis on the centroidal axis, the matrix $\boldsymbol{C}$ becomes a diagonal matrix as

$$\boldsymbol{C}=\begin{bmatrix} EA & 0 & 0 \\ 0 & GkA & 0 \\ 0 & 0 & EI \end{bmatrix} \qquad (18)$$

where I=moment of inertia of a cross section. k is the so-called shear coefficient, and is a function of cross-sectional shape. The derivation of Eq. (18) results in k being equal to 1, regardless of cross sectional shape. Other considerations give a value of k other than 1[11].

The governing equations for a beam with shear deformation are given by Eqs. (8), (12), (13) and (17). Since the governing equations for axial deformation are independent of those for bending and shear, and they are not affected by the shear deformation, these equations are omitted in what follows.

Substituting Eq. (17) into Eqs. (12·b) and (12·c) gives two simultaneous ordinary differential equations in terms of displacement components. Elimination of λ from the equation results in

$$-EI(v^{iv}+\bar{P}_y''/GkA)+\bar{P}_y+\bar{m}'=0 \qquad (19)$$

Similarly, the boundary conditions are expressed in terms of displacement components by

$$v=\bar{v} \quad \text{or} \quad n_z\{-EI(v'''+\bar{P}_y/GkA)+\bar{m}\}=\bar{V}$$

$$-v'+\frac{1}{GkA}\{-EI(v'''+\bar{P}_y'/GkA)-m\}=\bar{\lambda}$$

$$\text{or} \quad n_z\{-EI(v''+\bar{P}_y/GkA)\}=\bar{M} \qquad (20\cdot\text{a, b})$$

The general solution of Eq. (19) is expressed as

$$v=c_0+c_1z+c_2z^2+c_3z_3+v_p \qquad (21)$$

where v_p is the particular solution, and $c_0\sim c_3$ are integration constants. Determining the integration constants of this general solution by use of the boundary conditions of Eq. (20), the stiffness equation of the Timoshenko beam can be easily obtained, and is written as

$$\boldsymbol{f}=\boldsymbol{k}\boldsymbol{q}-\boldsymbol{f}^0 \qquad (22)$$

where

$$\boldsymbol{f}^T=\lfloor \boldsymbol{T}^T(a) \quad \boldsymbol{T}^T(b)\rfloor\ ,\quad \boldsymbol{q}^T=\lfloor \boldsymbol{u}^T(a) \quad \boldsymbol{u}^T(b)\rfloor \qquad (23\cdot\text{a, b})$$

$\boldsymbol{f}^0$ is the equivalent nodal force vector, and $\boldsymbol{k}$ the stiffness matrix. The elements of these are given in **APPENDIX I**. These matrices coincide with the well-known results for the Timoshenko beam.[8] The stiffness matrix has a form modified to that for elementary beam theory by the coefficient ϕ defined by

$$\phi=EI/GkAl^2 \qquad (24)$$

where, l is the length of a beam element and hence

$$l=b-a \qquad (25)$$

With increasing GA, the stiffness matrix converges to that of elementary beam theory.

3. ENERGY DERIVATION OF STIFFNESS EQUATION

Substituting Eqs. (8) and (17) into (6), then assuming proper interpolation functions for displacement components, the stiffness equation can be derived by the finite element technique. Selection of interpolation functions and the resulting stiffness equations are examined and discussed.

There are two independent generalized displacement components, v and λ, as in Eq. (7·c). Thus, it is natural to assume two interpolation functions for the components. Functions are chosen in general to satisfy geometrical boundary conditions at the ends of an element. It is common practice to use polynomials as the interpolation functions, the orders of which are determined by the orders of the derivatives appearing in the virtual work equation, such that they satisfy the necessary condition for convergence that internal virtual work should not identically vanish. Since the highest order derivatives of the displacements appearing in the definition of generalized strains, γ and κ, are first order as can be seen in Eqs. (8·b) and (8·c) and since there are two degrees of freedom for the generalized displacement, v and λ, one at each end, as shown in Eq. (15), the first order polynomial is the lowest order polynomial interpolation function that satisfies the necessary condition for convergence. Thus expressing the displacement function $\boldsymbol{u}$ by

$$\boldsymbol{u}=\boldsymbol{N}\boldsymbol{q} \qquad (26)$$

this lowest order interpolation function can be expressed as

$$\boldsymbol{N}=\begin{bmatrix} 1-\xi & 0 & \xi & 0 \\ 0 & 1-\xi & 0 & \xi \end{bmatrix} \qquad (27)$$

where

$$\xi=(z-a)/l \quad \cdots\cdots(28)$$

Using the interpolation function of Eq. (27), the corresponding stiffness equation is

$$\boldsymbol{f}=\boldsymbol{k}_1\boldsymbol{q}-\boldsymbol{f}_1{}^0 \quad \cdots\cdots(29)$$

The elements of $\boldsymbol{k}_1$ and $\boldsymbol{f}$ are given in **APPENDIX II**.

With the interpolation function of Eq. (27), the generalized strains of Eq. (8) are expressed in terms of end displacements as

$$\left.\begin{aligned}\kappa&=\frac{\lambda(b)-\lambda(a)}{l}\\ \gamma&=\frac{v(b)-v(a)}{l}+\lambda(a)(1-\xi)+\lambda(b)\xi\end{aligned}\right\} \quad \cdots\cdots(30\cdot\text{a, b})$$

For arbitrary given constants κ and γ, there are no end displacements, $\lambda(a)$, λ(b), v(a) and v(b), to satisfy Eq. (30). This implies that κ and γ can not be independent. One example of this is that, when κ is not equal to zero, γ is also not equal to zero, and thus even for a constant moment, the shear strain can not vanish. In order to improve on this contradiction, higher order polynomials have to be considered. By selecting interpolation functions as expressed by

$$\boldsymbol{N}=\begin{bmatrix}(1-\xi) & -\frac{l}{2}(1-\xi) & \xi & \frac{l}{2}\xi(1-\xi)\\ 0 & (1-\xi) & 0 & \xi\end{bmatrix} \quad \cdots\cdots(31)$$

it is shown that both generalized strain can take simultaneously arbitrary constant values within an element. The stiffness matrix $\boldsymbol{k}_2$ derived from Eq. (31) is given in **APPENDIX III**. Since both Eqs. (27) and (31) are different from the homogeneous solution of the governing differential equations, the resulting stiffness matrices $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ do not agree with the exact stiffness matrix $\boldsymbol{k}$. While the stiffness matrix $\boldsymbol{k}$ reduces to that of elementary beam theory at the limit of increasing GkA, both $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ do not.

The components of generalized displacement for this Timoshenko beam, v and λ are related by γ, as can be seen in Eq. (8·b). For the special case where γ is equal to zero, λ is expressed by v', i.e., λ and v can not be independent. For Timoshenko beam theory based on assumption Eq. (1·a), i.e., the assumption that a plane section remains a plane, γ can be regarded as small as has been pointed earlier. With this consideration, the selection of interpolation functions for v and λ that are independent even when γ is equal to zero may be regarded as inadequate. Rather, it is better to select functions that become dependent when γ is equal to zero. To accomplish this, an interpolation function is assinged for γ, though γ is not a generalized displacement. In addition further higher order polynomials are selected for v and λ. Since no continuity condition exists for γ at the boundary and the order of the derivative of γ appearing in the expression of internal virtual work is zero, the lowest order polynomial satisfying the requirements as an interpolation function for γ becomes a constant, hence

$$\gamma=\Delta=\text{const.} \quad \cdots\cdots(32)$$

By selecting interpolation functions for γ and v, the interpolation function for λ is automatically determined by Eq. (8·b). Since two kinematic continuity conditions are present at each end of an element as can be understood from Eq. (7·c), a total of four degrees of freedom are necessary for displacement functions. Selecting a third order polynomial for v for this reason, and considering Eqs. (32) and (8·b) gives

$$\boldsymbol{u}=\boldsymbol{N}_\Delta\begin{Bmatrix}\boldsymbol{q}\\ \Delta\end{Bmatrix} \quad \cdots\cdots(33)$$

where

$$\boldsymbol{N}_\Delta=\begin{bmatrix}(1-3\xi^2+2\xi^3) & -l(\xi-2\xi^2+\xi^3) & (3\xi^2-2\xi^3) & -l(\xi^3-\xi^2) & \xi l(1-\xi)(1-2\xi)\\ 6(\xi-\xi^2)/l & (1-4\xi+3\xi^2) & 6(\xi^2-\xi)/l & (3\xi^2-2\xi) & 6(\xi-\xi^2)\end{bmatrix} \quad \cdots\cdots(34)$$

Using Eq. (33), the stiffness equation is

$$\begin{Bmatrix}\boldsymbol{f}\\ f_\Delta\end{Bmatrix}=\begin{bmatrix}\boldsymbol{k}_0 & \boldsymbol{k}_1\\ \boldsymbol{k}_1{}^T & k_2\end{bmatrix}\begin{Bmatrix}\boldsymbol{q}\\ \Delta\end{Bmatrix}-\begin{Bmatrix}\bar{\boldsymbol{f}}^0\\ f_\Delta{}^0\end{Bmatrix} \quad \cdots\cdots(35)$$

The third and fourth terms of Eq. (6), show there exists no nodal force, f_Δ, corresponding to Δ, thus

$$f_\Delta\equiv 0 \quad \cdots\cdots(36)$$

With this, the last row of Eq. (35) can be regarded as a constraint condition between $\boldsymbol{q}$ and Δ. Using this constraint, Eq. (35) can be condensed to

$$\boldsymbol{f}=\boldsymbol{k}_\Delta\boldsymbol{q}-\boldsymbol{f}_\Delta{}^0 \quad \cdots\cdots(37)$$

where

$$\boldsymbol{k}_\Delta=\boldsymbol{k}_0-\frac{1}{k_2}\boldsymbol{k}_1\boldsymbol{k}_1{}^T\ , \quad \boldsymbol{f}_\Delta{}^0=\bar{\boldsymbol{f}}^0-\frac{1}{k_2}\boldsymbol{k}_1 f_\Delta{}^0 \quad \cdots\cdots(38\cdot\text{a, b})$$

Eq. (37) agrees with Eq. (22), that is

$$\boldsymbol{k}_\Delta=\boldsymbol{k}\ , \quad \boldsymbol{f}_\Delta{}^0=\boldsymbol{f}^0 \quad \cdots\cdots(40\cdot\text{a, b})$$

For dynamic problems, the inertia force can be treated as distributed force, thus the virtual work equation can be expressed by replacing the second term of Eq. (6) by

$$\text{inertia term}=\int_V [(-\rho\ddot{u}_2)\delta u_2 + (-\rho\ddot{u}_3)\delta u_3]dV \quad\cdots\cdots\cdots\cdots(41)$$

Substituting Eq. (3) and omitting the term relating to axial displacement, Eq. (41) becomes

$$\text{inertia term}=-m\int_a^b \{\ddot{v}(z,t)\delta v(z,t) + r_0^2\ddot{\lambda}(z,t)\delta\lambda(z,t)\}dz \cdots\cdots\cdots(42)$$

where ρ=density, $(\ddot{\ })=\partial(\ \ )/\partial t^2$, m=mass per unit length, r_0,=radius of gyration, and m and r_0^2 are defined by

$$m=\rho A \cdots\cdots\cdots(43)$$

$$r_0^2=I/A \cdots\cdots\cdots(44)$$

Consider a solution of the form

$$v(z,t):=v(z)\,e^{i\omega t}\ ,\quad \lambda(z,t):=\lambda(z)\,e^{i\omega t} \cdots\cdots(45)$$

The same notations v and λ, used for the functions of z only, are used for simplicity in Eq. (45), and below for functions of z and t. Substituting Eq. (45) into Eq. (42) gives virtual inertia work as

$$\text{inertia term}=m\omega^2\int_z (v\delta v + r_0^2\lambda\delta\lambda)dz \cdots\cdots(46)$$

Substituting assumed displacement functions in Eq. (46) and integrating it results in the so-called consistent mass matrix. Since the accuracy of eigen values obtained by a discretized system generally increases with increasing matrix size, a function which has a larger number of degrees of freedom than Eq. (33) is selected by replacing $\gamma=\Delta$ with a linear function with $\gamma(a)$ and $\gamma(b)$ at both ends. With this function of γ, the displacement function, in view of Eq. (8·b) becomes

$$\boldsymbol{u}=\boldsymbol{N}_D\begin{Bmatrix}\boldsymbol{q}\\ \gamma(a)\\ \gamma(b)\end{Bmatrix}=\boldsymbol{N}_D\boldsymbol{q}_D \cdots\cdots\cdots(47)$$

where

$$\boldsymbol{N}_D=\begin{bmatrix}(1-3\xi^2+2\xi^3) & l(-\xi+2\xi^2-\xi^3) & (3\xi^2-2\xi^3) & l(\xi^2-\xi^3) & l(\xi-2\xi^2+\xi^3) & l(\xi^3-\xi^2)\\ (6\xi-6\xi^2)/l & (1-4\xi+3\xi^2) & (6\xi^2-6\xi)/l & (3\xi^2-2\xi) & (3\xi-3\xi^2) & (3\xi-3\xi^2)\end{bmatrix} \cdots\cdots\cdots(48)$$

Considering Eq. (46), Eq. (6) after substitution of Eq. (47) and integration gives the equation of motion of a discretized system

$$\boldsymbol{k}\boldsymbol{q}_D-m\omega^2\boldsymbol{m}\boldsymbol{q}_D=\begin{Bmatrix}\boldsymbol{f}\\ 0\\ 0\end{Bmatrix}=\boldsymbol{f}_D \quad\cdots\cdots\cdots(49)$$

The elements of $\boldsymbol{k}$ and $\boldsymbol{m}$ are given in **APPENDIX IV**.

The last two rows of Eq. (49) play the same roles as the last row of Eq. (35), however, with condensation of matrix sizes, the term ω_2 appears both outside the matrix as a coefficient and inside the matrices in the components. Since this operation makes numerical computation difficult, it is appropriate to superimpose Eq. (49) in its original form for the entire structure. It has to be noted that no continuity condition exists for $\gamma(a)$ and $\gamma(b)$, as can be seen in Eq. (15). Thus the terms corresponding to $\gamma(a)$ and $\gamma(b)$ are not to be superimposed.

4. DISCUSSION

In a beam problem governed by Eq. (12·c) under the loading condition $\bar{m}=0$, bending moment distribution is uniform when shear force is zero, and otherwise it is a function of z. With the employment of the displacement function of Eq. (27), however, non-zero shear strain is always present when bending strain exists, regardless of its distribution. Similarly, when $\lambda(a)$ is equal to $\lambda(b)$, shear strain is not necessarily zero in spite of the fact that bending strain distribution is zero inside the element. From these facts, it is obvious that deformation of an element can not be properly represented by Eq. (27). Similar remarks are applicable to the displacement functions of Eq. (31). It is known that displacement functions other than the homogeneous solution of the governing differential equation can not represent correct deformation. While the exact stiffness matrix is obtained by using, as interpolation function, the lowest order polynomials that satisfy the convergence criterion in the case of a beam element with no shear deformation, the use of the lowest order polynomial does not lead to the exact stiffness matrix for the Timoshenko beam element.

Eq. (27) has sufficient degrees of freedom to satisfy the geometrical boundary conditions and also satisfies necessary conditions of convergence determined from the order of the derivatives in the expression of internal virtual work. Eq. (7) indicates that γ and κ are the two components of generalized strains of the Timoshenko beam. The displacement function of Eq. (27) can not account for the situation in which both γ and κ take simultaneously arbitrary constant values, as has been discussed in Eq. (30). Given this fact, there remains uncertainty as to whether the solution of Eq. (29), which depends on Eq. (27), does converge to the exact solution of the problem. To check this, the discrete system of Eq. (29) is transformed into a consistent continuous system which can be compared with the governing differential equation of the Timoshenko beam.

Consider a nodal point O at $z=0$, where no load is applied, and two neighbouring elements A and B are of equal length l. By making use of Eq. (29) for elements A and B, the zero nodal force at O can be expressed in terms of $\boldsymbol{u}(-l)$, $\boldsymbol{u}(0)$ and $\boldsymbol{u}(l)$. Then, the displacement vectors at nodes $z=-l$, 0 and l, respectively, are as follows

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} -GkA/l & GkA/2 & 2GkA/l & 0 & -GkA/l & -GkA/2 \\ -GkA/2 & -EI/l+GkAl/6 & 0 & 2\{EI/l+GkAl/3 & GkA/2 & -EI/l+GkAl/6 \end{bmatrix} \cdot \begin{Bmatrix} \boldsymbol{u}(-l) \\ \boldsymbol{u}(0) \\ \boldsymbol{u}(l) \end{Bmatrix} - \begin{Bmatrix} \frac{1}{l}\int_0^l \bar{P}_{yA} z dz + \frac{1}{l}\int_0^l \bar{P}_{yB}(l-z)dz \\ \frac{1}{l}\int_0^l \bar{m}_A z dz + \frac{1}{l}\int_0^l \bar{m}_B(l-z)dz \end{Bmatrix} \quad \cdots(50)$$

where subscript A and B are quantities for elements A and B, respectively. The Taylor expansion of $\boldsymbol{u}$, $\bar{P}_y$ and $\bar{m}$ at O leads to

$$\left.\begin{aligned} &\boldsymbol{u}(\pm l)=\boldsymbol{u}(0)\pm\boldsymbol{u}'(0)l+\frac{1}{2}\boldsymbol{u}''(0)l^2\pm\frac{1}{6}\boldsymbol{u}'''(0)l^3+\cdots \\ &\bar{P}_{yA}=\bar{P}_y(0)+\bar{P}_y'(0)(z-l)+\frac{1}{2}\bar{P}_y''(0)(z-l)^2 \\ &\quad +\frac{1}{6}\bar{P}_y'''(0)(z-l)^3+\cdots \\ &\bar{P}_{yB}=\bar{P}_y(0)+\bar{P}_y'(0)z+\frac{1}{2}\bar{P}_y''(0)z^2 \\ &\quad +\frac{1}{6}\bar{P}_y'''(0)z^3+ \end{aligned}\right\} \quad \cdots(51\cdot a\sim c)$$

and so on. Substituting Eq. (51) into Eq. (50) and integrating terms with distributed forces results in

$$\left.\begin{aligned} &-[GkA(v''+\lambda')+\bar{P}_y]-\frac{l^2}{12} \\ &\quad\cdot[GkA(v^{iv}+2\lambda''')+\bar{P}_y'']+O(l^4)=0 \\ &-[EI\lambda''-GkA(v'+\lambda)+\bar{m}]-\frac{l^2}{12} \\ &\quad\cdot[EI\lambda^{iv}-2GkA(v'''+\lambda'')+\bar{m}'']+O(l^4)=0 \end{aligned}\right\} \quad \cdots(52\cdot a, b)$$

where the subscript 0 denoting values at O is omitted for simplification. $O(\ )$ denotes Landau's symbol, and $f=O\{g(l)\}$ implies

$$\lim_{l\to 0}\left|\frac{f}{g}\right|<\infty \quad \cdots(53)$$

For the limit as l approaches 0 in Eq. (52), the consistent differential equation to the discrete system expressed by Eq. (29) becomes

$$\left.\begin{aligned} &\{GkA(v'+\lambda)\}'+\bar{P}_y=0 \\ &\{EI\lambda'\}'-GkA(v'+\lambda)+\bar{m}=0 \end{aligned}\right\} \cdots(54\cdot a, b)$$

Since Eq. (54) agrees with the governing equation for the Timoshenko beam, as can be seen substituting Eqs. (17), (8·b) and (8·c) into Eqs. (12·b) and (12·c), the solution of the stiffness equation, Eq. (29), converges to the exact solution. From the nature of Eq. (27) which has a larger number of degrees of freedom for shear deformation than for bending deformation, the stiffness equation based on Eq. (27) may result in a more accurate solution for a beam with larger shear force than bending moment ratio for the same number of elements. In order to check this numerically, two cantilever beams of different lengths are considered. The properties of the beams are

$$\left.\begin{aligned} &E=2.1\times10^6\,\text{kg/cm}^2\ (210\ \text{GPa}) \\ &G=7.0\times10^5\,\text{kg/cm}^2\ (70\ \text{GPa}) \\ &A=30.0\,\text{cm}^2,\quad I=250.0\,\text{cm}^4,\quad k=0.8333 \end{aligned}\right\} \quad \cdots(55)$$

with $l=40$ cm and 100 cm. The shear coefficients as defined by Eq. (24) are 1.88×10^{-2} and 3.0×10^{-3}, respectively, for the beams. The equations for the beams are solved for boundary conditions of $v=\lambda=0$ at the fixed end, and $\bar{M}=0$ and $\bar{V}\neq0$ at other end. The displacement at the loaded end versus the number of elements relations for the two beams are shown in **Fig. 1**. Better accuracy is obtained with the same number of elements for the beam of shorter span, which agrees with the above expectation.

As shown by Eq. (21), the homogeneous solution for the Timoshenko beam is a polynomial, thus the exact stiffness equation, Eq. (22) can be easily derived for it. It is also natural to expect that the exact stiffness equation can

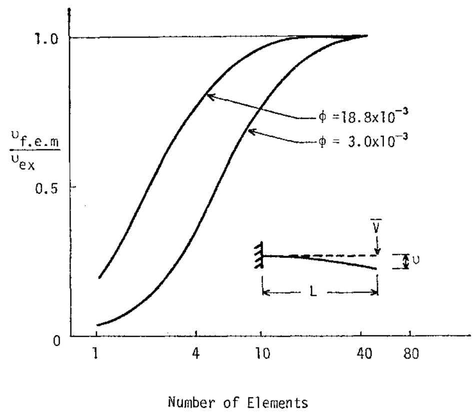

Fig. 1 Number of elements and end displacement.

also be easily obtained by finite element technique from energy principles. The use of the displacement function of Eq. (27), though it satisfies all the requirements for the finite element method, does not give Eq. (22), but gives an approximate equation, Eq. (29). One reason for this is that the generalized strain γ and κ can not simultaneously take arbitrary constant values. This has been improved in the displacement function of Eq. (31), however, the use of Eq. (31) also does not result in Eq. (22). In order to obtain Eq. (22), a still higher order polynomial has to be used for the displacement function. The reason for this may be the presence of the Eq. (8·b), constraint relation between v and λ. For the case in which γ is equal to zero, it may be inadequate to treat both v and λ as completely independent functions. In order to derive an accurate stiffness equation, it may be necessary to select displacement functions for which independence between v and λ is lost when γ equals zero. A displacement function, Eq. (34), was introduced adding the dummy displacement Δ, as a techniques to obtain this dependence.

In order to obtain Eq. (22) without introducing a dummy displacement function, it is necessary to employ third and second order polynomials for v and λ, respectively. This can be seen from Eq. (54). Separating v and λ, Eq. (54) can be transformed into two independent equations

$$\left.\begin{array}{l} EI\lambda'''+\bar{P}_y+\bar{m}'=0 \\ -EI(v^{iv}+\bar{P}_y''/GkA)+\bar{P}_y+\bar{m}'=0 \end{array}\right\} \quad \cdots\cdots(56\cdot\mathrm{a,\ b})$$

Clearly, the homogeneous solutions for v and λ are third and second order polynomials. Assuming forms of these functions that satisfy the four geometrical boundary conditions at both ends, a stiffness equation is obtained that is equivalent to Eq. (22) but in a different form. The four geometrical boundary conditions are not sufficient to determine the seven coefficients present in these polynomials, and hence the displacement functions include three dummy displacements resulting in a stiffness equation consisting of seven simultaneous equations. With static condensation of these three dummy displacements, the stiffness equation reduces to Eq. (22). In order to derive Eq. (22) directly, it is necessary to express all seven coefficients of the polynomials for the four existing geometric boundary conditions. The fact that the necessary and sufficient number of boundary conditions to solve the governing differential equation for Timoshenko beam are four, and hence the assignment of four geometrical boundary conditions, two at each end, leads to a unique solution, implies that all of the seven coefficients can be expressed uniquely by the four conditions. In order for this to be true, there must be three relations between v and λ. Since homogeneous solutions are of interest, if the distributed force terms are set equal to zero, there result

$$\lambda'+v''=0\ ,\quad EI\lambda''-GkA(\lambda+v')=0 \quad \cdots\cdots(57\cdot\mathrm{a,\ b})$$

which become the two constraint conditions. From the order of the polynomials of the homogeneous solutions, one more condition can be obtained by differentiating Eq. (57·a). Given these conditions, all seven coefficients can be determined for four geometrical boundary conditions. The resulting displacement function agrees with the homogeneous solutions of Eq. (54), and use of this function leads directly to Eq. (22)[12].

Timoshenko beam theory plays an important role in dynamic problems. In deriving stiffness equations for dynamic problems by finite element technique, attention has to be paid to the selection of displacement functions and kinematic conditions at nodes. In numerical considerations, simple examples are solved. In view of the fact that, in the analysis of a continuous system as a discrete system, better accuracy is generally obtained by using a stiffness equation with a larger number of degrees of freedom, numerical solutions are obtained using Eq. (49) which has the largest degree of freedom of the three stiffness equations discussed in this paper.

One report[4] deals with free vibration of the Timoshenko beam by the finite element method, in which the continuity condition is imposed on shear deformation at nodes. This treatment was not explained, however, it may have been used because it is continuous when an exact solution of a continuous system is obtained for a prismatic Timoshenko beam. There are only four kinds of boundary conditions at a node of a Timoshenko beam elment as expressed by Eq. (15), in which no continuity condition for γ is present. Imposing no continuity condition for γ at nodes, Eq. (49) is assembled for a whole beam. The solution of this system gives γ's with different values at both sides of a node even for a case of free vibration of a prismatic beam. This discrepancy, compared with the physically exact solution, is due to the fact that, contrary to the smooth distribution of inertia force in a continuous system, the distributed inertia force of a discrete system is treated as an equivalent nodal inertia force, as given by the second term on the left hand side of Eq. (49). By this reason, it is natural that γ is not continuous in a finite element solution based on Eq. (6). By contrast, imposing a continuity condition for γ gives a discrete

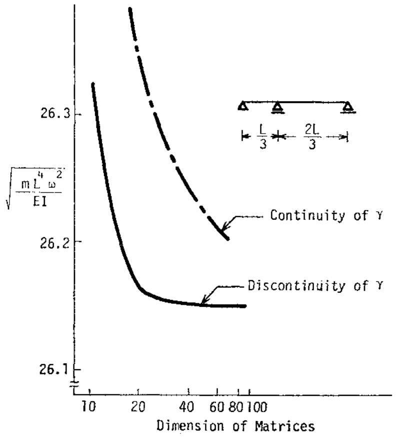

Fig. 2 Dimension of matrices and natural frequency.

system other than that based on Eq. (6). In order to see the numerical difference due to the difference in conditions of γ at nodes, the natural frequency of a two span continuous beam is evaluated. A beam of total length 90 cm is supported at a one third point as shown in **Fig. 2**. The same properties as given in Eq. (55) are used. Putting external loads equal to zero in Eq. (49), the equation for free vibration is,

$$\boldsymbol{k}\boldsymbol{q}_D = m\omega^2 \boldsymbol{m}\boldsymbol{q}_D \quad \cdots\cdots(58)$$

Numerical solutions of ω are obtained under two conditions, one imposing a continuity condition for γ, as reported in the literature[4], and the other without this condition. The accuracy of the natural frequency is plotted in **Fig. 2** for this to matrix size. In view of the facts that the finite element method leads to an upper bound solution for ω and that any kinematic constraint leads to a larger value of ω, it is concluded from the results shown in **Fig. 2** that the accuracy of ω is decreased by imposing continuity conditions for γ.

There exists some confusion in the literature on the treatment of boundary conditions. For Timoshenko beams, that is, for the continuum for which Eq. (1) is assumed as kinematic constraints, Eq. (13) is the boundary conditions compatible with the constraints. Hence, for example, the boundary conditions for the free end of a cantilever are $V=0$ and $M=0$, while there is a report[5] in which $\gamma=0$ is added as the third condition. In free vibration of a cantilever, it is true that γ is equal to zero at the free end since shear force at the free end is equal to zero. It does not hold true, however, in a discrete model, since the equivalent nodal force does not necessarily vanish at the free end. A zero nodal force, corresponding to γ in Eq. (47), shows that q is dependent on γ. Thus unnecessary prescription of γ has an unnecessary influence on $\boldsymbol{q}$.

5. CONCLUSION

Since the exact stiffness matrix for Timoshenko beam elements can be obtained easily by solving the governing differential equation, it is not significant to derive it by the finite element technique. In spite of the fact that the homogeneous solution of a prismatic Timoshenko beam problem is given by a polynomial, the finite element method using polynomials as displacement functions does not lead to an exact stiffness matrix unless special attention has been paid to the selection of the polynomial functions. Taking advantage of the known exact stiffness matrix, discussions on the selection of displacement functions have been presented that compare the resulting stiffness matrices with the exact matrix. The discussions of this paper may be of help in selecting displacement functions for problems for which no exact solutions are known, and may be applicable directly to plate and shell problems with shear deformation.

The discussions and conclusions of this paper are summarized as follows:

(1) In spite of the fact that the homogeneous solutions for generalized displacement, v and λ, are polynomials of third and second order, the finite element solutions using the lower order polynomial displacement functions that have sufficient degrees of freedom to satisfy geometrical boundary conditions and do not make strain energy identically equal to zero, converge to the exact solution of the continuous system. For this convergence, the displacement functions do not necessarily account for the fact that the generalized strains can take arbitrary constants simultaneously.

(2) As can be seen in the strain displacement relation of Eq. (8·b), the generalized displacements, v and λ, are dependent when γ is equal to zero. Selection of displacement functions which satisfy this dependence in addition to ordinary requirements in finite element technique leads to an exact stiffness matrix.

(3) While the lowest order polynomial displacement functions yield a 4×4 stiffness matrix, the function satisfying the dependence of **(2)** yields a 5×5 stiffness matrix. In view of the approximate nature of the former stiffness matrix, it may be of advantage in numerical computation to use the latter even though the number of degrees of freedom is larger than in the former.

(4) The most appropriate displacement functions for a Timoshenko beam element are a third order polynomial for v, and a second order polynomial for λ, whose total number of degrees of freedom is reduced from seven to four by considering the restriction of Eq. (57). This is possible only when the governing differential equation is simple and its homogeneous solution is a polynomial. In most cases for which the finite element method is the only feasible solution procedure because of the complexity of obtaining analytical solution, the consideration stated in **(4)** above is impractical but the selection stated in **(3)** may be suitable.

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6. APPENDICES

(1) Analytically Exact Matrix of Eq. (22) and Equivalent Nodal Forces

$$\boldsymbol{k}=\frac{EI}{(1+12\phi)l^3}\begin{bmatrix} 12 & -6l & -12 & -6l \\ & (4+12\phi)l^2 & 6l & (2-12\phi)l^2 \\ & & 12 & 6l \\ \text{Sym.} & & & (4+12\phi)l^2 \end{bmatrix} \quad \cdots\cdots(\text{A}\cdot 1)$$

$$\left.\begin{aligned}
\boldsymbol{f}^{0T}&=\lfloor f_1 \quad f_2 \quad f_3 \quad f_4 \rfloor \\
f_1&=\frac{1}{1+12\phi}\left\{\frac{1}{l^3}\int_0^l \bar{P}_y(l-z)^2(l+2z)dz +\frac{12\phi}{l}\int_0^l \bar{P}_y(l-z)dz+\frac{6}{l^3}\int_0^l \bar{m}z(l-z)dz\right\} \\
f_2&=\frac{1}{1+12\phi}\left\{-\frac{1}{l^2}\int_0^l \bar{P}_yz(l-z)^2dz +\frac{6\phi}{l}\int_0^l \bar{P}_yz(z-l)dz +\frac{1}{l^2}\int_0^l \bar{m}(l-z)(l-3z)dz +\frac{12\phi}{l}\int_0^l \bar{m}(l-z)dz\right\} \\
f_3&=\frac{1}{1+12\phi}\left\{\frac{1}{l^3}\int_0^l \bar{P}_yz^2(3l-2z)dz +\frac{12\phi}{l}\int_0^l \bar{P}_yzdz+\frac{6}{l^3}\int_0^l \bar{m}z(z-l)dz\right\} \\
f_4&=\frac{1}{1+12\phi}\left\{\frac{1}{l^2}\int_0^l \bar{P}_yz^2(l-z)dz +\frac{6\phi}{l}\int_0^l \bar{P}_yz(l-z)dz+\frac{1}{l^2}\int_0^l \bar{m}z(3z-2l)dz +\frac{12\phi}{l}\int_0^l \bar{m}zdz\right\}
\end{aligned}\right\} \quad \cdots\cdots(\text{A}\cdot 2)$$

(2) Stiffness Matrix and Equivalent Nodal Forces by Eq. (27)

$$\boldsymbol{k}_1=\begin{bmatrix} GkA/l & -GkA/2 & -GkA/l & -GkA/2 \\ & EI/l+GkAl/3 & GkA/2 & -EI/l+GkAl/6 \\ & & GkA/l & GkA/2 \\ \text{Sym.} & & & EI/l+GkAl/3 \end{bmatrix} \quad \cdots\cdots(\text{A}\cdot 3)$$

$$\boldsymbol{f}^{0T}=\lfloor f_1 \quad f_2 \quad f_3 \quad f_4 \rfloor$$

$$\left.\begin{aligned} f_1&=\frac{1}{l}\int_0^l \bar{P}_y(l-z)dz\ , & f_2&=\frac{1}{l}\int_0^l \bar{m}(l-z)dz \\ f_3&=\frac{1}{l}\int_0^l \bar{P}_yzdz\ , & f_4&=\frac{1}{l}\int_0^l \bar{m}zdz \end{aligned}\right\} \quad \cdots\cdots(\text{A}\cdot 4)$$

(3) Stiffness Matrix by Eq. (31)

$$\boldsymbol{k}_2=\begin{bmatrix} GkA/l & -GkA/2 & -GkA/l & -GkA/2 \\ & EI/l+GkAl/4 & GkA/2 & -EI/l+GkAl/4 \\ & & GkA/l & GkA/2 \\ \text{Sym.} & & & EI/l+GkAl/4 \end{bmatrix} \quad \cdots\cdots(\text{A}\cdot 5)$$

(4) Stiffness and Mass Matrices by Eq. (48)

$$\boldsymbol{k}=\frac{EI}{l^2}\begin{bmatrix} 12 & -6 & -12 & -6 & 6 & 6 \\ & 4 & 6 & 2 & -3 & -3 \\ & & 12 & 6 & -6 & -6 \\ & & & 4 & -3 & -3 \\ & & & & 3+1/3\phi & 3+1/6\phi \\ \text{Sym.} & & & & & 3+1/3\phi \end{bmatrix} \quad \cdots\cdots(\text{A}\cdot 6)$$

$$\boldsymbol{m}=l^2\begin{bmatrix} 13/35 & -11/210 & 9/70 & 13/420 & 11/210 & -13/420 \\ & 1/105 & -13/420 & -1/140 & -1/105 & 1/140 \\ & & 13/35 & 11/210 & 13/420 & -11/210 \\ & & & 1/105 & 1/140 & -1/105 \\ & & & & 1/105 & -1/140 \\ \text{Sym.} & & & & & 1/105 \end{bmatrix}$$

$$
+l^2\psi \begin{bmatrix} 6/5 & -1/10 & -6/5 & -1/10 & 3/5 & 3/5 \\ & 2/15 & 1/10 & -1/30 & -1/20 & -1/20 \\ & & 6/5 & 1/10 & -3/5 & -3/5 \\ & & & 2/15 & -1/20 & -1/20 \\ & & & & 3/10 & 3/10 \\ \text{Sym.} & & & & & 3/10 \end{bmatrix} \quad \cdots\cdots(A\cdot 7)
$$

$$
\psi = \left(\frac{r_0}{l}\right)^2 = \frac{Gk}{E}\phi \quad \cdots\cdots(A\cdot 8)
$$

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