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A TENSOR EXPANSION OF FINITE ROTATION AND MOMENT

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1. INTRODUCTION

In the linear theory of three-dimensional mechanics, the infinitesimal rotation may be treated in the linear vector space, similarly to the treatment of the moment, i.e. the rotation is usually resolved into components with respect to a set of three orthogonal axes. However, in the geometrically nonlinear problem, the three-dimensional rotation of finite magnitude can not be treated in the linear vector space, as is generally known.

This paper intends to analyze the three-dimensional finite rotation in the tensor field, and to define the general coordinate system of rotation. The expansion of this analysis is in complete correspondence to the tensor analysis of translation, which yields the curvilinear coordinate system. To indicate the analogy of this analysis to that of the curvilinear coordinate system, the Flügge's description about the tensor analysis of translation[2] will be summarized in Appendix I.

2. TENSORIZED EXPANSION OF ROTATION

In the following description, the theoretical expansion will be developed in the tensorized form, and the summation convention will be performed over the indices which mean the spatial components, unless the contrary is explicitly stated.

First, consider the usual vector notation of the moment and infinitesimal rotation. As well-known, it is only the moment and infinitesimal rotation which can be offered to this system of notation, since the finite rotation is not in the linear vector space. Denoting the spatial components of moment by $\{M_{(i)}\}=\{M_{(1)}, M_{(2)}, M_{(3)}\}$ with respect to a right-handed set of three orthogonal unit vectors $\{\boldsymbol{i}_{(i)}\}=\{\boldsymbol{i}_{(1)}, \boldsymbol{i}_{(2)}, \boldsymbol{i}_{(3)}\}$ under the rule of right-hand screw, the moment vector is usually written as

$$\boldsymbol{M}=M_{(i)}\boldsymbol{i}_{(i)} \quad \cdots\cdots(1\cdot\mathrm{a})$$

and, similarly, the vector of infinitesimal rotation is written by

$$\delta\boldsymbol{\theta}=\delta\theta_{(i)}\boldsymbol{i}_{(i)} \quad \cdots\cdots(1\cdot\mathrm{b})$$

where $\{\delta\theta_{(i)}\}$ are the components of rotation around the unit vectors $\{\boldsymbol{i}_{(i)}\}$ under the same rule. Here, the notation $\delta(\ )$ indicates an infinitesimal quantity or differential.

The above expressions for the moment and infinitesimal rotation mean the moment of magnitude $|\boldsymbol{M}|=(M_{(i)}M_{(i)})^{1/2}$ and the rotation of magnitude $|\delta\boldsymbol{\theta}|=(\delta\theta_{(i)}\delta\theta_{(i)})^{1/2}$ around the axes of direction cosigns

$$\left.\begin{array}{l}\cos\alpha_{(i)}=M_{(i)}/|\boldsymbol{M}|\\ \cos\beta_{(i)}=\delta\theta_{(i)}/|\delta\boldsymbol{\theta}|\end{array}\right\} \quad \cdots\cdots(2\cdot\mathrm{a,\ b})$$

respectively, and satisfy the following four equations;

$$\left.\begin{array}{l}\alpha(\boldsymbol{x}+\boldsymbol{y})=\alpha\boldsymbol{x}+\alpha\boldsymbol{y}\\ (\alpha+\beta)\boldsymbol{x}=\alpha\boldsymbol{x}+\beta\boldsymbol{x}\\ \alpha(\beta\boldsymbol{x})=(\alpha\beta)\boldsymbol{x}\\ 1\boldsymbol{x}=\boldsymbol{x}\end{array}\right\} \quad \cdots\cdots(3\cdot\mathrm{a}\sim\mathrm{d})$$

in which the Gothic letters $\boldsymbol{x}$ and $\boldsymbol{y}$ indicate the vectors defined by Eqs. (1·a, b), and the Greek letters α and β, scalars. The set of Eqs. (3·a～d) is the necessary condition to be in the linear vector space.[1] The proof of the above statement and equations for the moment vectors can be derived from both the rule of parallelogram of forces and the theorem of Varignon,[3] and that for the infinitesimal rotation vectors, from the geometrical expansions, respectively.

The dot products and cross products between the unit vectors $\{\boldsymbol{i}_{(i)}\}$ are, respectively, defined by

$$\left.\begin{array}{l}\boldsymbol{i}_{(i)}\cdot\boldsymbol{i}_{(j)}=\delta_{ij}\\ \boldsymbol{i}_{(i)}\times\boldsymbol{i}_{(j)}=e_{ijk}\boldsymbol{i}_{(k)}\end{array}\right\} \quad \cdots\cdots(4\cdot\mathrm{a,\ b})$$

where the symbol δ_{ij} means the Kronecker delta, and e_{ijk}, the permutation symbol. The unit vectors $\{\boldsymbol{i}_{(i)}\}$ may be regarded as a set of the base vectors of the linear vector space. Then, once the dot and cross products between $\{\boldsymbol{i}_{(i)}\}$ are defined, those products of any two vectors are specified under the rule of linear combination.

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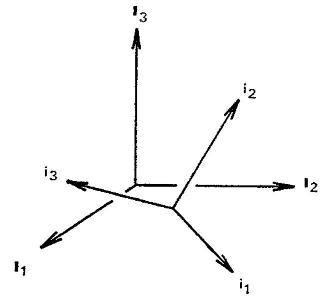

Fig. 1 Space-fixed $\{\boldsymbol{I}_{[I]}\}$ and rotatable $\{\boldsymbol{i}_{(i)}\}$.

It is evident that the dot products $\boldsymbol{M}\cdot\boldsymbol{M}$ and $\delta\boldsymbol{\theta}\cdot\delta\boldsymbol{\theta}$ indicate the squares of magnitude of $\boldsymbol{M}$ and $\delta\boldsymbol{\theta}$, respectively, and that the product $\boldsymbol{M}\cdot\delta\boldsymbol{\theta}$ means the infinitesimal work of $\boldsymbol{M}$ through $\delta\boldsymbol{\theta}$.

With the above preliminary, the tensor expansion of the three-dimensional finite rotation may be developed as follows: Consider the two sets of three orthogonal unit vectors of a right-hand system, one of which is fixed in the space and denoted by $\{\boldsymbol{I}_{[I]}\}=\{\boldsymbol{I}_{[1]}, \boldsymbol{I}_{[2]}, \boldsymbol{I}_{[3]}\}$, and the other is finitely rotatable from the space-fixed set and denoted $\{\boldsymbol{i}_{(i)}\}=\{\boldsymbol{i}_{(1)}, \boldsymbol{i}_{(2)}, \boldsymbol{i}_{(3)}\}$. The relativity of the rotatable $\{\boldsymbol{i}_{(i)}\}$ to the space-fixed $\{\boldsymbol{I}_{[I]}\}$ is completely represented by the orthonormal matrix $[T^{[J]}_{(i)}]$ which acts on the transformation from $\{\boldsymbol{I}_{[I]}\}$ to $\{\boldsymbol{i}_{(i)}\}$ as

$$\begin{Bmatrix} \boldsymbol{i}_{(1)} \\ \boldsymbol{i}_{(2)} \\ \boldsymbol{i}_{(3)} \end{Bmatrix} = \begin{bmatrix} T^{[1]}_{(1)}, & T^{[2]}_{(1)}, & T^{[3]}_{(1)} \\ T^{[1]}_{(2)}, & T^{[2]}_{(2)}, & T^{[3]}_{(2)} \\ T^{[1]}_{(3)}, & T^{[2]}_{(3)}, & T^{[3]}_{(3)} \end{bmatrix} \begin{Bmatrix} \boldsymbol{I}_{[1]} \\ \boldsymbol{I}_{[2]} \\ \boldsymbol{I}_{[3]} \end{Bmatrix} \cdots\cdots\cdots(5)$$

In the nine elements of $[T^{[J]}_{(i)}]$, there are the six normality and orthogonality conditions given by

$$T^{[K]}_{(i)}T^{[K]}_{(j)}=\delta_{ij}, \quad \text{or} \quad T^{[I]}_{(k)}T^{[J]}_{(k)}=\delta^{IJ}\cdots(6\cdot\text{a, b})$$

Then, the number of degrees of freedom of the rotation may be concluded to be three.

Suppose a general system which can specify the three-dimensional finite rotation by using three independent parameters $\{e^\alpha\}=\{e^1, e^2, e^3\}$, such as the Eulerian angle system. If the system can give any orthonormal matrix by varying the values of the parameters, then let the independent parameters $\{e^\alpha\}$ be called the general coordinates of rotation. Throughout such a procedure, the elements of $[T^{[J]}_{(i)}]$ may be regarded as functions of $\{e^\alpha\}$, holding the constraining conditions (6·a, b).

For independent infinitesimal variation of the coordinates $\delta\{e^\alpha\}$ from a finitely rotated state $\{e^\alpha\}$, the associated change of $\{\boldsymbol{i}_{(i)}\}$ against the space-fixed $\{\boldsymbol{I}_{[I]}\}$ may be derived by differentiating Eq. (5) as follows:

$$\delta\boldsymbol{i}_{(i)}=\delta T^{[J]}_{(i)}\boldsymbol{I}_{[J]}=T^{[J]}_{(i),\alpha}\delta e^\alpha\boldsymbol{I}_{[J]}\cdots\cdots\cdots\cdots\cdots\cdots(7)$$

where the notation $(\ \ )_{,\alpha}$ indicates the differentiation with respect to e^α. On the other hand, supposing the same infinitesimal rotation, which are resolved into components around the axes of the rotated unit vectors $\{\boldsymbol{i}_{(i)}\}$ in the same sense to Eq. (1·b), denoted by $\{\delta\theta_{(i)}\}$, the same changes of $\{\boldsymbol{i}_{(i)}\}$ may be also written as

$$\delta\boldsymbol{i}_{(i)}=e_{ijk}\delta\theta_{(k)}\boldsymbol{i}_{(j)}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(8)$$

After transforming $\{\boldsymbol{I}_{[I]}\}$ into $\{\boldsymbol{i}_{(i)}\}$ by the relations $\boldsymbol{I}_{[I]}=T^{[I]}_{(j)}\boldsymbol{i}_{(j)}$, equating the right-hand sides of Eqs. (7) and (8), the linear relations between $\{\delta\theta_{(i)}\}$ and $\delta\{e^\alpha\}$ are obtained as

$$e_{ijk}\delta\theta_{(k)}=T^{[K]}_{(i),\alpha}T^{[K]}_{(j)}\delta e^\alpha\cdots\cdots\cdots\cdots\cdots\cdots(9)$$

The solution of Eqs. (9) for $\{\delta\theta_{(i)}\}$ is written by

$$\left.\begin{aligned} &\delta\theta_{(i)}=\beta^{(i)}_\alpha\delta e^\alpha \\ &\beta^{(i)}_\alpha=(1/2)e_{ijk}T^{[K]}_{(j),\alpha}T^{[K]}_{(k)} \end{aligned}\right\}\cdots\cdots\cdots(10\cdot\text{a, b})$$

Here, expanding the concept of the usual vector expression of infinitesimal rotation, Eq. (1·b), the covariant base vectors $\{\boldsymbol{g}_\alpha\}=\{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3\}$ of the general coordinate system are now defined by the expression

$$\delta\boldsymbol{\theta}=\delta e^\alpha\boldsymbol{g}_\alpha\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(11)$$

By equating the right-hand sides of Eqs. (1·b) and (11) through the relations (10·a), the covariant base vectors may be explicitly represented in terms of $\{\boldsymbol{i}_{(i)}\}$, as follows:

$$\boldsymbol{g}_\alpha=\beta^{(i)}_\alpha\boldsymbol{i}_{(i)}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(12)$$

The physical meaning of the expression (11) may be intuitively understood as the inbinitesimal rotation resulting from the superposition of the rotations of magnitudes $\delta e^\alpha|\boldsymbol{g}_\alpha|$ around the respective unit vectors $\boldsymbol{i}_\alpha=\boldsymbol{g}_\alpha/|\boldsymbol{g}_\alpha|$ under the rule of right-hand screw, where not summed, $\alpha=1, 2, 3$. Here, the independence of the parameters $\{e^\alpha\}$ is equal to the conditions that any of the base vectors is not zero vector, and that they are not on a same plane element in the region of $\{e^\alpha\}$; that is,

$$\boldsymbol{g}_1\cdot(\boldsymbol{g}_2\times\boldsymbol{g}_3)\neq 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(13)$$

The contravariant base vectors $\{\boldsymbol{g}^\alpha\}$ are usually defined by the relations

$$\boldsymbol{g}^\alpha\cdot\boldsymbol{g}_\beta=\delta^\alpha_\beta\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(14)$$

Representing $\{\boldsymbol{g}^\alpha\}$ in terms of the unit vectors $\{\boldsymbol{i}_{(i)}\}$ as

$$\boldsymbol{g}^\alpha=\beta^\alpha_{(i)}\boldsymbol{i}_{(i)}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(15)$$

the transformation coefficients $\beta^\alpha_{(i)}$ can be determined by the substitution of Eqs. (12) and (15) into Eqs. (14) to yield the relations:

$$\beta^{\alpha}_{(i)}\beta^{(i)}_{\beta}=\delta^{\alpha}_{\beta} \qquad \cdots\cdots(16)$$

That is, the coefficient matrix $[\beta^{\alpha}_{(i)}]$ is obtained as the inverse of $[\beta^{(i)}_{\alpha}]$.

The other quantities related to the general coordinate system of rotation, such as the metric tensor and the permutation tensor, etc., and more advanced operations, for instance the covariant differentiation, are defined in the same way to the usual three-dimensional tensor analysis, such as in the curvilinear coordinate system, see Appendix I, or Ref. 2).

3. APPLICATION TO EULER ANGLES

The Eulerian angle system is one of the most usual specifications of the finite rotation. Let the angle parameters be denoted by $\{\varphi^{\alpha}\}=\{\varphi^1, \varphi^2, \varphi^3\}$: initially setting the rotatable $\{\boldsymbol{i}_{(i)}\}$ onto the space-fixed $\{\boldsymbol{I}_{[I]}\}$, φ^1 is the first angle of rotation around $\boldsymbol{i}_{(1)}$ under the rule of right-hand screw, and φ^2 and φ^3 are the second and the third around $\boldsymbol{i}_{(2)}$ and $\boldsymbol{i}_{(3)}$, respectively, as shown in **Fig. 2**. The transformation matrix from $\{\boldsymbol{I}_{[I]}\}$ to $\{\boldsymbol{i}_{(i)}\}$ may be written as

$$[T^{[J]}_{(i)}]=\begin{bmatrix} \cos\varphi^2\cos\varphi^3, & \sin\varphi^1\sin\varphi^2\cos\varphi^3+\cos\varphi^1\sin\varphi^3, & -\cos\varphi^1\sin\varphi^2\cos\varphi^3+\sin\varphi^1\sin\varphi^3 \\ -\cos\varphi^2\sin\varphi^3, & -\sin\varphi^1\sin\varphi^2\sin\varphi^3+\cos\varphi^1\cos\varphi^3, & \cos\varphi^1\sin\varphi^2\sin\varphi^3+\sin\varphi^1\cos\varphi^3 \\ \sin\varphi^2, & -\sin\varphi^1\cos\varphi^2, & \cos\varphi^1\cos\varphi^2 \end{bmatrix} \qquad \cdots\cdots(17)$$

When denoting the disturbance of $\{\boldsymbol{i}_{(i)}\}$ due to the infinitesimal increments of $\{\varphi^{\alpha}\}$ by $\delta\{\boldsymbol{i}_{(i)}\}=[\delta\Phi^{(j)}_{(i)}]\{\boldsymbol{i}_{(j)}\}$, the elements of $[\delta\Phi^{(j)}_{(i)}]$ may be obtained from the relations $[\delta\Phi^{(j)}_{(i)}]=[\delta T^{[K]}_{(i)}][T^{[K]}_{(j)}]^T$ as follows:

$$[\delta\Phi^{(j)}_{(i)}]=\begin{bmatrix} 0, & \sin\varphi^2\delta\varphi^1+\delta\varphi^3, & \cos\varphi^2\sin\varphi^3\delta\varphi^1-\cos\varphi^3\delta\varphi^2 \\ -\sin\varphi^2\delta\varphi^1-\delta\varphi^3, & 0, & \cos\varphi^2\cos\varphi^3\delta\varphi^1+\sin\varphi^3\delta\varphi^2 \\ -\cos\varphi^2\sin\varphi^3\delta\varphi'+\cos\varphi^3\delta\varphi^2, & -\cos\varphi^2\cos\varphi^3\delta\varphi^1-\sin\varphi^3\delta\varphi^2, & 0 \end{bmatrix} \qquad \cdots\cdots(18\cdot a)$$

The same matrix $[\delta\Phi^{(j)}_{(i)}]$ for the infinitesimal angles of rotation around the current $\{\boldsymbol{i}_{(i)}\}$, $\{\delta\theta_{(i)}\}=\{\delta\theta_{(1)}, \delta\theta_{(2)}, \delta\theta_{(3)}\}$, may be written as

$$[\delta\Phi^{(j)}_{(i)}]=\begin{bmatrix} 0, & \delta\theta_{(3)}, & -\delta\theta_{(2)} \\ -\delta\theta_{(3)}, & 0, & \delta\theta_{(1)} \\ \delta\theta_{(2)}, & -\delta\theta_{(1)}, & 0 \end{bmatrix} \qquad \cdots\cdots(18\cdot b)$$

Equating the elements of Eqs. (18·a) and (18·b), the coefficient matrix which relates the two sets of angle disturbances as $\{\delta\theta_{(i)}\}=[\beta^{(i)}_{\alpha}]^T\delta\{\varphi^{\alpha}\}$ is given by

$$[\beta^{(i)}_{\alpha}]=\begin{bmatrix} \cos\varphi^2\cos\varphi^3, & -\cos\varphi^2\sin\varphi^3, & \sin\varphi^2 \\ \sin\varphi^3, & \cos\varphi^3, & 0 \\ 0, & 0, & 1 \end{bmatrix} \qquad \cdots\cdots(19)$$

This matrix gives also the relation between the covariant base vectors of the Eulerian angle system $\{\boldsymbol{g}_{\alpha}\}$ and the rotatable unit vectors $\{\boldsymbol{i}_{(i)}\}$ by $\{\boldsymbol{g}_{\alpha}\}=[\beta^{(i)}_{\alpha}]\{\boldsymbol{i}_{(i)}\}$. Here, this relation between $\delta\{\varphi^{\alpha}\}$ and $\{\delta\theta_{(i)}\}$ was already indicated, for instance, in the description about the Eulerian angle system by Washizu[4], but with no insight into the extensibility in the tensor filed. The matrix of coefficients relating the contravariant base vectors to the rotatable unit vectors as $\{\boldsymbol{g}^{\alpha}\}=[\beta^{\alpha}_{(i)}]\{\boldsymbol{i}_{(i)}\}$ is obtained by inverting $[\beta^{(i)}_{\alpha}]$ as

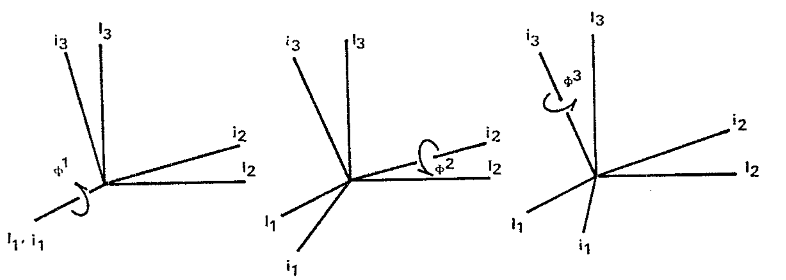

Fig. 2 Eulerian angle system.

$$[\beta^{\alpha}_{(i)}]=\begin{bmatrix} \dfrac{\cos\varphi^3}{\cos\varphi^2}, & \sin\varphi^3, & -\tan\varphi^2\cos\varphi^3 \\ -\dfrac{\sin\varphi^3}{\cos\varphi^2}, & \cos\varphi^3, & \tan\varphi^2\sin\varphi^3 \\ 0, & 0, & 1 \end{bmatrix} \quad \cdots\cdots(20)$$

Next, the covariant components of metric tensor $\{g_{\alpha\beta}\}$ are given from the relation $[g_{\alpha\beta}]=[\beta^{(i)}_{\alpha}][\beta^{(i)}_{\beta}]^T$ as

$$[g_{\alpha\beta}]=\begin{bmatrix} 1, & 0, & \sin\varphi^2 \\ & 1, & 0 \\ \text{Sym.} & & 1 \end{bmatrix} \quad \cdots\cdots(21)$$

Similarly, the contravariant components are given by

$$[g^{\alpha\beta}]=\begin{bmatrix} \dfrac{1}{(\cos\varphi^2)^2}, & 0, & -\dfrac{\sin\varphi^2}{(\cos\varphi^2)^2} \\ & 1, & 0 \\ \text{Sym.} & & \dfrac{1}{(\cos\varphi^2)^2} \end{bmatrix} \quad \cdots\cdots(22)$$

4. DISCUSSION

When the physical phenomenon, *rotation*, is offered to the tensor analysis, at the beginning of expansion, the phenomenon must be examined on its extensibility in the tensor field by the conditions that it is measurable and that its differentials satisfy Eqs. (3·a∼d). The latter is the necessary condition of being in the linear vector space. Under these conditions, the tensor field of the rotation can be supposed, and then the general coordinate system can be introduced. In the tensor field with a general coordinate system, there exist the base vectors, the activity of which is to connect the phenomenon and the coordinates in the differential form. The derivation of those vectors is usually attained by analyzing the infinitesimal increment of the phenomenon due to those of the coordinates. When, up to this stage, the expansion is developed, the phenomenon can be said to be extended in the tensor field.

While the concrete expansion for the phenomenon, *rotation*, is given in this paper, the above process is true of any phenomenon offered to the tensor analysis; e.g. the same expansion for the phenomenon, *translation*, yields the curvilinear coordinate system. The calculus, operations and theorems, etc. are common to any tensor field with three dimensions, where, of course, the meaning of the base vectors is peculiar to each phenomenon.

5. CONCLUSION

In the general coordinate system of rotation, as defined before, where the finite rotation is specified by the three independent parameters $\{e^{\alpha}\}$, it is spontaneous that any infinitesimal rotation $\delta\boldsymbol{\theta}$ added to a finitely rotated state $\{e^{\alpha}\}$ is represented in terms of the differentials of the general coordinates $\delta\{e^{\alpha}\}$ as given by Eq. (11), and, on the other hand, the moment $\boldsymbol{M}$ acting at the rotated state may be resolved into components with respect to any of the four reference frames $\{\boldsymbol{I}_{[I]}\}$, $\{\boldsymbol{i}_{(i)}\}$, $\{\boldsymbol{g}_{\alpha}\}$ or $\{\boldsymbol{g}^{\alpha}\}$.

As stated before, the dot product of $\boldsymbol{M}$ and $\delta\boldsymbol{\theta}$ gives the infinitesimal work of $\boldsymbol{M}$ through $\delta\boldsymbol{\theta}$, and, when choosing the covariant components from the four component expressions, from the relations $\boldsymbol{g}_{\alpha}\cdot\boldsymbol{g}^{\beta}=\delta^{\beta}_{\alpha}$, the work $\delta\psi_R$ may be written as

$$\delta\psi_R=\boldsymbol{M}\cdot\delta\boldsymbol{\theta}=M_{\alpha}\delta e^{\alpha} \quad \cdots\cdots(23)$$

Furthermore, suppose the case where the acting moment is related to the rotation, and where the components $\{M_{\alpha}\}$ are functions of $\{e^{\alpha}\}$. If the functions have the symmetric properties $M_{\alpha,\beta}=M_{\beta,\alpha}$ in the region of $\{e^{\alpha}\}$, thus Eq. (23) is in the exact differential form, and is integrable regardless of the path of integration. In this case, the moment having its potential $\psi_R(e^{\alpha})$ may be called conservative.

In the sense of the above statements, the covariant components $\{M_{\alpha}\}$ may be regarded as the only associated components of moment to the general coordinates $\{e^{\alpha}\}$. From Eqs. (5), (6·a, b), (12), (15), and (16), the covariant components are in the relations of transformation to the other three components $\{M_{[I]}\}$, $\{M_{(i)}\}$ and $\{M^{\alpha}\}$, as follows:

$$M_{\alpha}=M^{\beta}g_{\alpha\beta}=M_{(i)}\beta^{(i)}_{\alpha}=M_{[I]}T^{[I]}_{(j)}\beta^{(j)}_{\alpha} \quad \cdots\cdots(24)$$

APPENDIX I. CURVILINEAR COORDINATES OF TRANSLATION

To aid the tensor analysis of the translation to be developed, let a cartesian coordinate system be supposed in the space, the coordinates of which are denoted by $\{X_{[I]}\}=\{X_{[1]}, X_{[2]}, X_{[3]}\}$, and whose unit vectors, by $\{\boldsymbol{I}_{[I]}\}=\{\boldsymbol{I}_{[1]}, \boldsymbol{I}_{[2]}, \boldsymbol{I}_{[3]}\}$. In this system, the point position, which is specified by connecting the distances $X_{[I]}$ with the directions of $\boldsymbol{I}_{[I]}$, I=1, 2, 3, from the origin, is denoted by

$$\boldsymbol{r}=X_{[I]}\boldsymbol{I}_{[I]} \quad \cdots\cdots(\text{A. }1)$$

It is needless to say that this position vector is in the three-dimensional linear vector space.

The dot products and cross products between the unit vectors $\{\boldsymbol{I}_{[I]}\}$ are defined as follows:

$$\left.\begin{array}{l}\boldsymbol{I}_{[I]}\cdot\boldsymbol{I}_{[J]}=\delta_{IJ}\\ \boldsymbol{I}_{[I]}\times\boldsymbol{I}_{[J]}=e_{IJK}\boldsymbol{I}_{[K]}\end{array}\right\}\quad\cdots\cdots\cdots\cdots\text{(A. 2·a, b)}$$

where the symbol δ_{IJ} means the Kronecker delta, and e_{IJK}, the permutation symbol. Once those products of the unit vectors are defined, those of any two position vectors are specified by the rule of linear combination of vectors.

In general, if, existing a system to determine the spatial position of a point by means of three independent parameters $\{x^\alpha\}=\{x^1, x^2, x^3\}$, the spatial position is in the one-to-one correspondence to the three parameters, the system with $\{x^\alpha\}$ is usually called a curvilinear coordinate system of translation. The functional relationships between the curvilinear coordinates $\{x^\alpha\}$ and the cartesian coordinates $\{X_{[I]}\}$ may be supposed, and let the derivatives of the functions be denoted by

$$\beta_\alpha^{[I]}=\frac{\partial X_{[I]}}{\partial x^\alpha},\quad \beta_{[I]}^\alpha=\frac{\partial x^\alpha}{\partial X_{[I]}}\quad\cdots\cdots\text{(A. 3·a, b)}$$

Here, the preceding one-to-one correspondence between the spatial position and $\{x^\alpha\}$ is equal to the condition that the sign of the determinant of the derivative matrix $[\beta_\alpha^{[I]}]$, or $[\beta_{[I]}^\alpha]$, does not change in the region of $\{x^\alpha\}$.

For any infinitesimal increments of $\{x^\alpha\}$, the point position is correspondingly disturbed, and the disturbance may be obtained from the differentials of the above functions as $\delta\boldsymbol{r}=\boldsymbol{r}_{,\alpha}\delta x^\alpha$, where the notation $(\ )_{,\alpha}$ indicates the differentiation with respect to x^α. In the tensor analysis, the derivatives $\boldsymbol{r}_{,\alpha}$ are usually defined as the covariant base vectors, denoted by

$$\boldsymbol{g}_\alpha=\boldsymbol{r}_{,\alpha}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\text{(A. 4)}$$

Then, the disturbance of the position is written as

$$\delta\boldsymbol{r}=\delta x^\alpha\boldsymbol{g}_\alpha\quad(=\delta X_{[I]}\boldsymbol{I}_{[I]})\quad\cdots\cdots\cdots\cdots\text{(A. 5)}$$

By the substitution of Eq. (A. 3·a) into Eq. (A. 5), the relation $\delta x^\alpha\boldsymbol{g}_\alpha=\beta_\alpha^{[I]}\delta x^\alpha\boldsymbol{I}_{[I]}$ is obtained, and then the covariant base vectors may be written as

$$\boldsymbol{g}_\alpha=\beta_\alpha^{[I]}\boldsymbol{I}_{[I]}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\text{(A. 6)}$$

The contravariant base vectors $\{\boldsymbol{g}^\alpha\}$ are defined by the relations

$$\boldsymbol{g}^\alpha\cdot\boldsymbol{g}_\beta=\delta_\beta^\alpha\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\text{(A. 7)}$$

From the relations $\beta_{[I]}^\alpha\beta_\beta^{[I]}=\delta_\beta^\alpha$ and Eqs. (A. 6), $\{\boldsymbol{g}^\alpha\}$ may be explicitly given by

$$\boldsymbol{g}^\alpha=\beta_{[I]}^\alpha\boldsymbol{I}_{[I]}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\text{(A. 8)}$$

The covariant, contravariant and mixed components of metric tensor of the curvilinear coordinate system are, respectively, defined by

$$g_{\alpha\beta}=\boldsymbol{g}_\alpha\cdot\boldsymbol{g}_\beta,\quad g^{\alpha\beta}=\boldsymbol{g}^\alpha\cdot\boldsymbol{g}^\beta,\quad g_\beta^\alpha=\boldsymbol{g}^\alpha\cdot\boldsymbol{g}_\beta$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\text{(A. 9·a, b, c)}$$

The substitution of Eqs. (A. 6) and (A. 8) into the above definitions yields the explicit expressions for the metric tensor components, as follows:

$$g_{\alpha\beta}=\beta_\alpha^{[K]}\beta_\beta^{[K]},\quad g^{\alpha\beta}=\beta_{[K]}^\alpha\beta_{[K]}^\beta,\quad g_\beta^\alpha=\delta_\beta^\alpha$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\text{(A. 10·a, b, c)}$$

More detailed descriptions about the tensor analysis of translation, more advanced subjects, such as the covariant differentiation, and applications to the continuum mechanics are stated in *Tensor Analysis and Continuum Mechanics* by Flügge.[2] In this book, the applications of the tensor calculus to the solid mechanics are performed only to the extent of the small displacement theory, but it is of course that the results of the tensor analysis can be employed in the analysis of finite displacement problems, if desired.

NOTATION

The following symbols are used in this paper:

- $\delta\boldsymbol{\theta}$ = infinitesimal rotation;
- $\boldsymbol{M}$ = moment;
- $\boldsymbol{r}$ = position vector;
- δ_{ij} = Kronecker delta;
- e_{ijk} = permutation symbol;
- ψ_R = potential of moment;
- $\{\boldsymbol{I}_{[I]}\}$ = space-fixed set of three orthogonal unit vectors;
- $\{\boldsymbol{i}_{(i)}\}$ = rotatable set of three orthogonal unit vectors;
- $\{e^\alpha\}$ = general coordinate system of rotation;
- $\{\varphi^\alpha\}$ = Eulerian angle system;
- $\{x^\alpha\}$ = curvilinear coordinate system of translation;
- $\{X_{[I]}\}$ = cartesian coordinate system;
- $[T_{(i)}^{[J]}]$ = orthonormal transformation matrix;
- $[\delta\Phi_{(i)}^{(j)}]$ = matrix denoting disturbance of $\{\boldsymbol{i}_{(i)}\}$;
- $\{\beta_\alpha^{(i)}\}$, $\{\beta_{(i)}^\alpha\}$, $\{\beta_\alpha^{[I]}$, $\{\beta_{[I]}^\alpha\}$ = transformation coefficients;
- $\{M_{[I]}\}$ = moment components with respect to $\{\boldsymbol{I}_{[I]}\}$;
- $\{M_{(i)}\}$ = moment components with respect to $\{\boldsymbol{i}_{(i)}\}$;
- $\{M_\alpha\}$ = covariant components of moment;
- $\{M^\alpha\}$ = contravariant components of moment;
- $\{\boldsymbol{g}_\alpha\}$ = covariant base vectors;
- $\{\boldsymbol{g}^\alpha\}$ = contravariant base vectors; and
- $\{\delta\theta_{(i)}\}$ = components of infinitesimal rotation around $\{\boldsymbol{i}_{(i)}\}$.

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