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CATASTROPHE AND IMPERFECTION SENSITIVITY OF TWO-DEGREE-OF-FREEDOM SYSTEMS

By Yoshiji NIWA, Eiichi WATANABE** and Noboru NAKAGAWA****

SYNOPSIS

This paper is concerned with the static instability of two-degree-of-freedom system representing some of the typical civil engineering structures, in terms of catastrophe theory of René Thom and special attention is focused on the evaluation of the imperfection sensitivity.

Principal catastrophes that may be encountered in civil engineering structures may be thought to be Fold, Cusp, Dual Cusp, Hyperbolic Umbilic, or Elliptic Umbilic Catastrophes. Among them, Fold, Cusp, Dual Cusp constitute the basic elementary catastrophes; while Hyperbolic or Elliptic Umbilic is generated as the simultaneous occurrence of Fold and Dual Cusp Catastrophes, or simultaneous occurrence of Fold and Cusp Catastrophes.

The catastrophic characteristics can be determined by the properties of bifurcation set, i.e., the mapping to the control space at the singular points, which in civil engineering field, represents the imperfection sensitivity.

This paper briefly states the catastrophic properties and performs the identification of catastrophe of two-degree-of-freedom systems, and finally describes the interesting sensitivity interactions among two independent imperfections.

1. STATIC INSTABILITY AND CATASTROPHE

(1) General Remarks

The study on the static instability was initiated by Euler. Great development in this field was accomplished by the Branching Theory of Poincaré. The nonlinear branching theory was initiated by Koiter and further development has been established by researchers including Budiansky and Hutchinson. Thompson applied the nonlinear branching theory to discrete models and came up with some of quite interesting results.

On the other hand, in completely different field of science, a treatise titled as 'Stabilite structurelle et morphogenese (Structural stability and morphogenesis)' by R. Thom appeared[1]. In this treatise general mathematics of morphology are described with a philosophical tone and some specific applications are given to embryology and linguistics.

The first paper that incorporated the elastic stability with Thom's catastrophe theory was written by Thompson and Hunt[2]. This discussion has been followed by several papers[3)~6)].

A brief discussion and an interpretation of stability and catastrophe will be given in the following few sections, and the meaning of the initial imperfections is described.

(2) Classification of Elastic Stability

The potential energy of a structure can generally be given as a function of generalized coordinates, Q_i, and the control parameters, Λ^j:

$$V=V(Q_i, \Lambda^j) \quad (i=1, \cdots, n; j=0, 1, \cdots, l) \quad \cdots\cdots (1)$$

where, n and $l+1$ refer respectively the degree-of-freedom and the number of controlling parameters. For simplicity, let Λ^0 indicate the loading parameter, and $\Lambda^j (j=1, \ldots, l)$ indicate the initial imperfections of the structure.

Consider a small perturbation of δV of potential V at an arbitrary point c: $(Q_i, \Lambda^j)=(Q_{ic}, \Lambda_c^j)$ resulting from a small perturbation δQ_i of general coordinates Q_i, then the following approximate equation can be obtained using dummy indices:

$$\begin{aligned}\delta V &= V(Q_i+\delta Q_i, \Lambda^j)-V(Q_i, \Lambda^j)\\ &= V_{ic}\delta Q_i+\frac{1}{2!}V_{ijc}\delta Q_i\delta Q_j\\ &\quad +\frac{1}{3!}V_{ijkc}\delta Q_i\delta Q_j\delta Q_k+\delta(0^4) \quad \cdots\cdots (2)\end{aligned}$$

where

* Dr. Eng., Professor, Kyoto Univ.
** Ph. D., Associate Professor, Kyoto Univ.
*** Engineer, Aoki Kensetsu Construction Co.

$V_{ic}\equiv\partial V/\partial Q_i|_{\delta=0}$; $V_{ijc}\equiv\partial^2 V/\partial Q_i\partial Q_j|_{\delta=0}$;
$V_{ijkc}\equiv\partial^3 V/\partial Q_i\partial Q_j\partial Q_k|_{\delta=0}$

The first term of the right hand side vanishes because of equilibrium of the system, $V_i=0$, and the system remains stable when the second term of the right hand side is positive definite, and unstable when it is negative definite. The system is called critical when $det[V_{ij}]=0$.

Assume next that no imperfections exist, then $V=V(Q_i, \Lambda^0)$. Let Q_i be transformed into generalized coordinates, v_j, so that the second derivatives of V with respect to these generalized coordinates form diagonal matrix, and let D refer to the potential thus newly defined, then

$$D(v_i, \Lambda^0)\equiv V[Q_j(v_i), \Lambda^0] \quad \cdots\cdots(3\cdot a)$$

$$Q_j=\Phi_{ji}v_i: \text{ Affine transform} \quad \cdots\cdots(3\cdot b)$$

where Φ_{ji} is chosen so that

$$D_{ij}\equiv\frac{\partial^2 D}{\partial v_i\partial v_j}=C_i\delta_{ij} \quad \cdots\cdots(3\cdot c)$$

(i: not summed) (δ_{ij}: Kronecker's delta)

The stability then is determined in the following form[7]:

stable	when	$C_i>0, \forall i$
unstable	when	$C_i<0, \exists i$
critical	when	$C_i\geqq 0, \forall i$ and $C_j=0, \exists j$

Subspace coordinates v_i in general imply the buckling mode, and now assume that m coincident bucklings occur at the critical load, Λ_c^0. Moreover let Greek subscript α designates non-critical modes or passive modes; while let Roman subscript i designate the buckling, or active modes then, the following relationships hold:

$D_i=0$ for coincident buckling modes, (active modes) ......(4·a)

$D_\alpha=0$ for non-critical modes, (passive modes) ......(4·b)

$D_{iic}=0$ for m-fold coincident buckling modes, (i: not summed) ......(4·c)

$D_{\alpha\alpha c}>0$ for non-critical modes. (α: not summed) ......(4·d)

Since the equilibrium equation $D_\alpha=0$ is non-singular, the m-fold non-critical modes, v_β, can be expressed in terms of the critical modes, v_i, thus,

$$v_\beta=v_\beta(v_i, \Lambda^0) \quad \cdots\cdots(5)$$

Upon substitution of this expression into the expression for the potential, D, a new potential, A, can be defined as[7]:

$$A(v_i, \Lambda^0)\equiv D[v_i, v_\beta(v_i, \Lambda^0), \Lambda^0] \quad \cdots\cdots(6)$$

Moreover, it will be quite natural to assume that potential A can be expressed in terms of linear function of Λ^0. Then, the basic relationships of the derivatives of the potentials, A and D, can be obtained and they are listed in Appendix A:

From Appendix A, it will be easily seen that

At the coincident buckling load:

$$\left.\begin{aligned}
&A_i=A_i^0=A_i^{00}=A_i^{000}=A^{00}=A^{000}=0 \quad ('^0'\equiv\partial/\partial\Lambda^0)\\
&A^0=D^0\\
&A_{ij}=D_{ij}=0 \quad (\text{for } i\neq i)\\
&A_{ij}^0=D_{ij}^0=0 \quad (\text{for } i\neq j)\\
&A_{ii}=D_{ii}=0\\
&A_{ii}^0=D_{ii}^0\\
&A_{iii}=D_{iii}\\
&A_{iiii}=D_{iiii}-3\sum_{\alpha\neq i}\frac{(D_{\alpha ii})^2}{D_{\alpha\alpha}}
\end{aligned}\right\} \quad \cdots\cdots(7\cdot a)$$

At the limit point:

$$A_i^0\neq 0, \quad A_{iii}\neq 0 \quad (\text{definition}) \cdots\cdots(7\cdot b)$$

where subscripts of A, such as used for A_{ij}, refer to the partial differentiation of A with respect to generalized coordinates, v_i and v_j, just like in the case of V_{ij} and D_{ij}; while the super script $_0$ on A refers to the differentiation with respect to Λ^0.

(3) Catastrophe

Here a brief summary of the elementary catastrophe is provided[1),8)].

Let a potential, V be defined as a map from $(k+n)^{th}$ Euclidean space, R^{k+n}, which consists of the k^{th} Euclidean control space, R^k, and n^{th} Euclidean behaviour space, R^n to a linear Euclidean space, R:

$$\left.\begin{aligned}&V:\ R^k\times R^n \rightarrow R\\ &V=V(P_i, Z_i)\end{aligned}\right\} \quad \cdots\cdots(8)$$

P_i: control parameters,
Z_i: generalized coordinates

Let equilibrium space, M_V be defined as a subspace of R^{k+n}:

$$M_V=\{(P_i, Z_i)|V_k\equiv\partial V/\partial Z_k=0,\ k=1, \cdots, n\} \quad \cdots\cdots(9)$$

Furthermore, let us consider a map, χ from R^{k+n} to R^k:

$$\chi:\ R^{k+n} \rightarrow R^k$$

The catastrophe map, χ_V, then can be defined as:

$$\chi_V:\ M_V \rightarrow R^k \quad \cdots\cdots(10)$$

where subscript on both χ and M refer to the potential concerned.

In general, the catastrophe map, χ_V has various complex forms. However, according to Thom, under some appropriate transformation of coordinates, it can be represented by far more simple and finite numbered map, χ_F, corresponding to potential F of Table B.1 of Appendix B.

That is, M_V can be transformed to M_F under some transformation of coordinates, χ_F being *homeomorphic** to χ_V:

$$\chi_F:\ M_F \to R^k \qquad (11)$$

It has been proved that catastrophe can be classified into only seven elementary ones provided that the number of control parameters is less than equal to 4[1]. Those catastrophes are referred to as Fold, Cusp, Swallow Tail, Butterfly, Hyperbolic Umbilic, Elliptic Umbilic, and Parabolic Umbilic catastrophes. Their relations to the elastic instability will be described in Appendix B.

(4) Elastic Instability

In the previous section a brief explanation and an interpretation on the catastrophe were given. It will be seen from a simple comparison of Equations (1) and (8) that the phenomenon of elastic instability is a catastrophe. Here, the loading parameter Λ^0 and the imperfection parameters Λ^j constitute the control parameters, and the displacements Q_i correspond to the behaviour parameters.

The effect of the initial imperfections, $\Lambda^j (j=1, \cdots, l)$ may be in general replaced by that of the 'equivalent loadings' where the imperfections are interpreted as the elastic displacements due to these loadings. This implies that the imperfections may be treated as independent loading parameters to Λ^0.

In actual structures, the critical point does not necessarily result in the collapse, and thus in such a case elastic-plastic analysis may be necessary.

If the imperfections are also taken into account, Eq. (6) can be rewritten in more general form:

$$A(v_i, \Lambda^j) \equiv D[v_i,\ v_\alpha(v_i, \Lambda^j),\ \Lambda^j]\ \ (j=0, 1, \cdots, l) \qquad (12)$$

The equations of equilibrium thus can be obtained by:

$$A_i(v_j, \Lambda^m) \equiv \partial A(v_j, \Lambda^m)/\partial v_i = 0 \qquad (13)$$

This relation holds identically in the equilibrium space M_A and represent the relationships between v_j and Λ^m.

Now, assume that the behaviour space and the control space are functions of certain parameter s, then[7]

$$A_i = A_i[v_i(s), \Lambda^j(s)] = 0$$

Since this holds identically for the parameter, s, the n-th derivative with respect to s becomes

$$A_i^{(n)}(s) = d^n A_i(s)/ds^n = 0 \qquad (14)$$

The relationships corresponding to $n=1, 2, 3$ are given in the following manner:

$$\left.\begin{aligned}
A_i'=0:&\quad A_{ij}v_j' + A_i{}^j\Lambda^{j\prime} = 0\\
A_i''=0:&\quad A_{ijk}v_j'v_k' + A_{ij}v_j'' + 2A_{ij}^{\,k}v_j'\Lambda^{k\prime}\\
&\qquad + A_i{}^j\Lambda^{j\prime\prime} + A_i{}^{jk}\Lambda^{j\prime}\Lambda^{k\prime} = 0\\
A_i^{(3)}=0:&\quad A_{ijkl}v_j'v_k'v_l' + 3A_{ijk}v_j'v_k''\\
&\qquad + A_{ij}v_j^{(3)} + 3A_{ijk}^{\,l}v_j'v_k'\Lambda^{l\prime}\\
&\qquad + 3A_{ij}^{\,k}v_j''\Lambda^{k\prime} + 3A_{ij}^{\,k}v_j'\Lambda^{k\prime\prime}\\
&\qquad + A_i{}^j\Lambda^{j(3)} + 3A_{ij}^{\,kl}v_j'\Lambda^{k\prime}\Lambda^{l\prime}\\
&\qquad + 3A_i{}^{jk}\Lambda^{j\prime}\Lambda^{k\prime\prime}\\
&\qquad + A_i{}^{jkl}\Lambda^{j\prime}\Lambda^{k\prime}\Lambda^{l\prime} = 0
\end{aligned}\right\} \qquad (15)$$

where

$$(\,')\equiv d/ds; \quad A_i{}^j \equiv dA_i/d\Lambda^j$$

Now, let us consider a specific problem when imperfections do not exist, i.e., $\Lambda^j=0\ (j=1, \cdots, l)$. Further, assuming a single distinct root at the critical point: $(v_1, \Lambda^0)=(v_{1c}, \Lambda_c^0)$, and $A_{11c}=0$, the following relations can be derived by setting $s=v_1$:

$$\left.\begin{aligned}
&A_1{}^0\Lambda^{0\prime}|_c=0\\
&A_{111}+2A_{11}^0\Lambda^{0\prime}+A_1{}^0\Lambda^{0\prime\prime}|_c=0
\end{aligned}\right\} \qquad (16)$$

Thus, when $A_{1c}^0| \neq 0$,

$$\Lambda_c^{0\prime} = d\Lambda^0/dv_1|_c = 0\,, \qquad (17)$$

and moreover, when $A_{111c}\neq 0$, then

$$\Lambda_c^{0\prime\prime} = d^2\Lambda^0/dv_1{}^2|_c = -A_{111c}/A_{1c}^0 \neq 0 \qquad (18)$$

This implies that $\Lambda^0(v_1)$ has extremum value at the critical point, and in this case, the catastrophe is called Limit Point, and corresponds to Fold, according to the classification by Thom. The representation of the potential is given by the Taylor expansion about the critical point: $(v_1, \Lambda^0)=(v_{1c}, \Lambda_c^0)$, and thus

$$A(v_1, \Lambda^0) = \frac{1}{3!}A_{111c}(v_1-v_{1c})^3 + A_{1c}^0(\Lambda^0-\Lambda_c^0)(v_1-v_{1c}) \qquad (19)$$

The catastrophe map in this particular case yields nothing but a point $\Lambda^0=\Lambda_c^0$.

Next, assume that $A_1{}^0|_c=0$, at a critical point $A_{11}=0$, then, using the relations of Eqs. (7), the following equations can be derived:

$$\left.\begin{aligned}
&A_{111}+2A_{11}^0\Lambda^{0\prime}|_c=0\\
&A_{1111}+3A_{11}^0\Lambda^{0\prime\prime}|_c=0
\end{aligned}\right\} \qquad (20)$$

Thus, the slope of load-displacement curve can be given by:

$$\Lambda_c^{0\prime} = d\Lambda^0/dv_1|_c = -A_{111c}/(2A_{11c}^0) \qquad (21)$$

Thus, when $A_{111c}\neq 0$ as in the case of asymmetric buckling, the slope has non-zero value.

Now, consider the case when $l=1$ in Eq. (1)

* Two topological spaces X and Y have the same form if they are homeomorphic. The term homeomorphic implies that there is a map from X to Y which is bijective and bicontinuous.

and let $\Lambda^1=\epsilon$, where ϵ refers to a small imperfection parameter, corresponding to a distinct critical buckling mode v_1, then the potential $A(v_1, \Lambda^0, \epsilon)$ can be approximately written straightforward in terms of Taylor expansion about the critical point: $(v_1, \Lambda_c^0, \epsilon)=(0, \Lambda_c^0, 0)$, of the corresponding perfect system, and thus

$$A=\frac{1}{3!}A_{111c}v_1^3+\frac{1}{2!}A_{11c}^0(\Lambda^0-\Lambda_c^0)v_1^2+\dot{A}_{1c}\epsilon v_1 \qquad \cdots\cdots(22)$$

where $(\dot{\ })\equiv\partial/\partial\epsilon$.

Consider a mapping from (v_1, ϵ)-plane to (Λ^0, ϵ)-plane by taking into account the equilibrium $\partial A/\partial v_1=0$, and let Jacobian of the transformation be zero, that is,

$$J=\begin{vmatrix} \dfrac{\partial\Lambda^0}{\partial v_1} & \dfrac{\partial\Lambda^0}{\partial\epsilon} \\ \dfrac{\partial\epsilon}{\partial v_1} & \dfrac{\partial\epsilon}{\partial\epsilon} \end{vmatrix}=\frac{\partial\Lambda^0}{\partial v_1}=0 \qquad \cdots\cdots(23)$$

then the mapping becomes catastrophe map, to give the imperfection sensitivity of the form:

$$\Lambda_m^0=\Lambda_c^0\pm|A_{111c}|^{1/2}|2\dot{A}_{1c}\epsilon|^{1/2}/|A_{11c}^0| \qquad \cdots\cdots(24)$$

where the suffix m indicates the extremum value.

In the above equation, the sign $(+)$ corresponds to the unrealistic case.

On the other hand, consider an imperfection just as before, however in the case when $A_{111c}=0$, then, the potential can be written straightforward by means of Taylor expansion about the critical point in the following form:

$$A=\frac{1}{4!}A_{1111c}v_1^4+\frac{1}{2!}A_{11c}^0(\Lambda^0-\Lambda_c^0)v_1^2+\dot{A}_{1c}\epsilon v_1 \qquad \cdots\cdots(25)$$

The imperfection sensitivity can be obtained by the similar transformation and by letting the Jacobian be zero, that is, by Eq. (23)

$$\Lambda_m^0=\Lambda_c^0-\frac{1}{2}(A_{1111c})^{1/3}(3\dot{A}_{1c}\epsilon)^{2/3}/A_{11c}^0 \qquad \cdots\cdots(26)$$

Since A_{11c}^0 is negative in general, Eq. (26) represents realistic imperfection sensitivity, for unstable symmetric buckling characterized by $A_{1111c}<0$. In the case of stable symmetric buckling, $A_{1111c}>0$, this equation gives unrealistic sensitivity, and to obtain realistic imperfection sensitivity, some criterion on the plastic failure must be introduced.

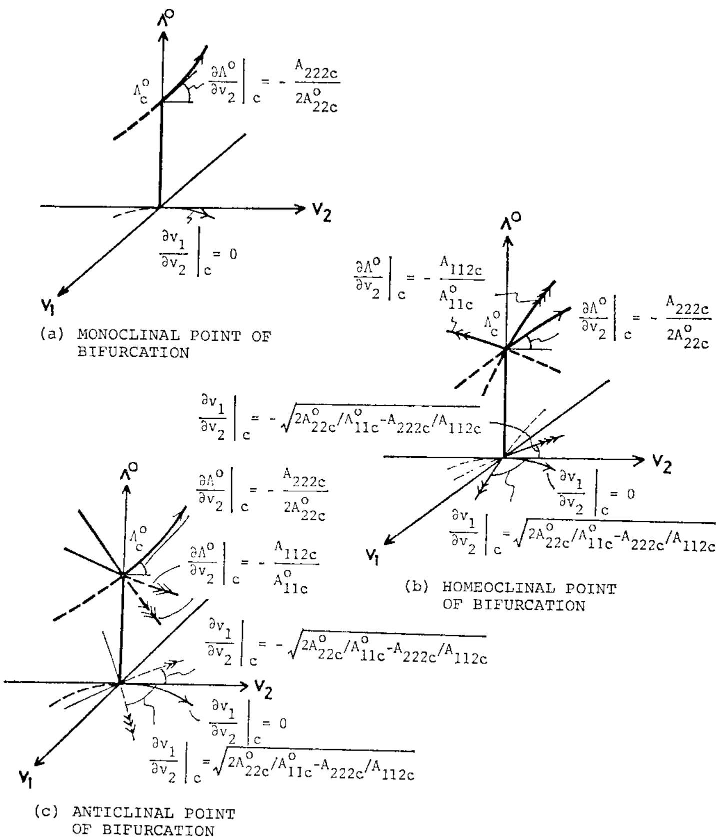

Fig. 1 Semi-Symmetric Point of Bifurcation.

Next, let us consider a case of coincident buckling of two-degree-of-freedom system. Assuming that the potential, A is a even function with respect to v_1. Such a case is called Semi-Symmetric Buckling[2]. Then, since $A_{11c}=A_{22c}=0$, the following relations will be obtained from Eqs. (15), provided that the initial imperfections are neglected:

$$\left.\begin{aligned} &2A_{112}v_1'v_2'+2A^0_{11}v_1'\Lambda^{0\prime}|_c=0\\ &A_{112}v_1'^2+A_{222}v_2'^2+2A^0_{22}v_2'\Lambda^{0\prime}|_c=0 \end{aligned}\right\}\cdots\cdots(27)$$

The non trivial solutions of Eq. (27) are given by the following: at critical point,

(i) $\Lambda^{0\prime}/v_2'|_c\equiv\partial\Lambda^0/\partial v_2|_c=-A_{222c}/(2A^0_{22c})$,
$v_1'/v_2'|_c\equiv\partial v_1/\partial v_2|_c=0$,

or (ii), (iii)

$$\Lambda^{0\prime}/v_2'|_c=-A_{112c}/A^0_{11c},$$
$$v_1'/v_2'|_c=\pm(2A^0_{22c}/A^0_{11c}-A_{222c}/A_{112c})^{1/2}\ \cdots(28)$$

The case when only (i) exists is called Monoclinal Point of bifurcation; while when (ii) and (iii) exist and $\partial\Lambda^0/\partial v_2$ have the same sign, then the phenomenon is called Homeoclinal Point of Bifurcation, and when $\partial\Lambda^0/\partial v_2$ have different sign, then it is called Anticlinal Point of Bifurcation[2]. The relationship between those semi-symmetric bucklings and René Thom's Umbilic Catastrophes are also given in Appendix B. The geometric meaning of the semi-symmetric bifurcation bucklings are illustrated in **Fig. 1**.

2. TWO-DEGREE-OF-FREEDOM SYSTEM

(1) General Remarks

As a numerical illustration, simple yet important two-degree-of-freedom systems comprising rigid links and springs are considered. Model 1 shows an asymmetric bifurcation model, Model 2 an unstable symmetric bifurcation model, and Model 3 an arch model. Those models may be interpreted as generalization of Thompson's single-degree-of-freedom systems, and the results of the analysis may be used as useful informations toward the analysis of multi-degree-of-freedom system. One of the most important relationships of these systems would be the sensitivity interactions among the initial imperfections of different modes upon the load-carrying capacity, known as the bifurcation set. Thus, numerical computations were performed to obtain the bifurcation sets, and in some cases where the critical point is not attained in the elastic range, a simple plastic criterion is used to obtain the imperfections as a realistic catastrophe criterion.

(2) Asymmetric Buckling Model

Fig. 2 shows the model. This model is thought to be representing trusses, rigid frames, shells, and stiffened plates.

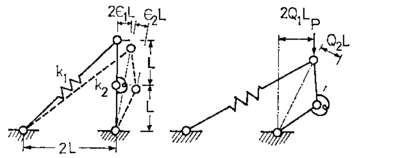

Fig. 2 Asymmetric Bifurcation Buckling Model.

First, the potential of the model will be obtained in a precise manner; then, this potential will be approximated by Taylor's expansions. Moreover, the discussions will be made on the perfect system to obtain the basic catastrophe characteristics; then, the imperfection study will be followed.

Let Q_1 and Q_2 denote the general coordinates representing the displacements, and let ϵ_1 and ϵ_2 denote the initial displacements corresponding to Q_1 and Q_2, respectively, then the potential energy, V can be written as:

$$\begin{aligned} V=&\frac{1}{2}k_1(2L)^2[\{(1+Q_1)^2+(1-Q_1^2-Q_2^2)\}^{1/2}\\ &-\alpha]^2+\frac{1}{2}k_2(2\sin^{-1}(Q_2-\epsilon_2))^2\\ &-P(2L)\{(1-\epsilon_1^2-\epsilon_2^2)^{1/2}-(1-Q_1^2-Q_2^2)^{1/2}\} \end{aligned}\quad\cdots\cdots(29)$$

where

$$\alpha\equiv\{(1+\epsilon_1)^2+(1-\epsilon_1^2-\epsilon_2^2)\}^{1/2}$$

This potential can be expanded into Taylor's series and rewritten as D if terms higher than 4th order are to be neglected and nondimensionalized through the division of V by k_1L^2: when $\epsilon_1=\epsilon_2=0$:

$$\begin{aligned} D=&\frac{1}{2}Q_1^2-\frac{1}{4}Q_1^3-\frac{1}{2}Q_1Q_2^2+\frac{5}{32}Q_1^4\\ &+\frac{3}{8}Q_1^2Q_2^2+\frac{1}{8}Q_2^4+\kappa\left(Q_2^2+\frac{1}{3}Q_2^4\right)\\ &-\Lambda^0\left\{\frac{1}{2}(Q_1^2+Q_2^2)+\frac{1}{8}(Q_1^2+Q_2^2)^2\right\} \end{aligned}$$

where

$$\left.\begin{aligned} &\kappa=k_2/(k_1L^2)\\ &\Lambda^0=P/(k_1L) \end{aligned}\right\}\ \cdots\cdots(30)$$

Since, the second derivatives of the potential, D_{ij}, has been already diagonalized, the critical loads of the system, that is, in the case of $\epsilon_1=\epsilon_2=0$, can be given by either of the equations:

$$\left.\begin{aligned} &\Lambda^0_{c1}=1 &&\text{from } D_{11c}=0\\ &\Lambda^0_{c2}=2\kappa &&\text{from } D_{22c}=0 \end{aligned}\right\}\ \cdots\cdots(31)$$

The catastrophe of the system can be classified into three according to the value of κ in the following manner:

a) When $0<\kappa<1/2$ $\quad \Lambda_c^0=\Lambda_{c2}^0=2\kappa$, and

$D_{2c}^0=0,\quad D_{11c}=1-2\kappa>0,\quad D_{22c}=0,\quad D_{22c}^0=-1,$

$D_{122c}=-1,\quad D_{222c}=0,\quad D_{2222c}=3+2\kappa$

Let Λ denote the potential newly defined by use of the equation, $D_1=0$, and upon elimination of Q_1, the following relationships will be obtained:

$\Lambda_{2c}^0=D_{2c}^0=0,\quad \Lambda_{22c}^0=D_{22c}^0=-1,$

$\Lambda_{222c}^0=D_{222c}^0=0,$

and

$$\Lambda_{2222c}=D_{2222c}-3(D_{122c})^2/D_{11c}=-\frac{4\kappa(1+\kappa)}{1-2\kappa}<0$$

Thus, the critical point is found to be *unstable* point of bifurcation, or *dual cusp* catastrophe.

b) When $\kappa=1/2$ $\quad \Lambda_c^0=\Lambda_{c1}^0=\Lambda_{c2}^0=1$, and

$\Lambda_{11c}^0=-1,\quad \Lambda_{22c}^0=-1,\quad \Lambda_{111c}=-3/2,$

$\Lambda_{122c}=-1,$

thus,

$$2\Lambda_{11c}^0/\Lambda_{22c}^0-\Lambda_{111c}/\Lambda_{122c}=1/2>0,\quad \Lambda_{111c}\cdot\Lambda_{122c}>0$$

Thus, the catastrophe is found to be *homeoclinal* point of bifurcation and *hyperbolic umbilic* catastrophe.

c) When $\kappa>1/2$ $\quad \Lambda_c^0=\Lambda_{c1}^0=1$, and

$D_{1c}^0=0,\quad D_{11c}^0=-1,\quad D_{111c}=-3/2.$

Thus, the catastrophe is found to be *asymmetric* point of bifurcation, i.e., *fold* catastrophe.

When the initial imperfections are taken into account, the following equations of equilibrium can be obtained using the potential given in Eq. (29)

$$D_1=2+2\sqrt{2}\,\alpha\left(-\frac{1}{2}+\frac{1}{4}Q_1-\frac{3}{16}Q_1^2-\frac{1}{8}Q_2^2+\frac{5}{32}Q_1^3+\frac{3}{16}Q_1Q_2^2\right)-\Lambda^0\left\{Q_1+\frac{1}{2}Q_1(Q_1^2+Q_2^2)\right\}=0$$

$$D_2=-2Q_2+2\sqrt{2}\,\alpha\left(\frac{1}{2}Q_2-\frac{1}{4}Q_1Q_2+\frac{1}{8}Q_2^3+\frac{3}{16}Q_1^2Q_2\right)+\kappa\left\{2(Q_2-\epsilon_2)+\frac{4}{3}(Q_2-\epsilon_2)^3\right\}-\Lambda^0\left\{Q_2+\frac{1}{2}Q_2(Q_1^2+Q_2^2)\right\}=0 \quad \cdots\cdots\cdots(32)$$

Eq. (32) can be solved using a perturbation method, and the load-displacement relationships in each of three cases: (i) $\kappa=0.25$, (ii) $\kappa=0.5$, and (iii) $\kappa=1.0$, are shown in **Figs. 3, 4,** and **5** where solid lines represent stable paths; while broken lines represent unstable paths. Furthermore, it is simply assumed that the system fails plastically when

$|Q_1|\geqq 0.5$ or $|Q_2|\geqq 0.5$

The maximum load is represented by Λ_m^0 which corresponds to the cases when either the local maximum load is attained, as can be seen from Eq. (23), or the plastic failure occurs. The values of Λ_m^0/Λ_c^0 are obtained in the aforementioned three different cases, respectively, and the final results of analysis are illustrated in **Figs. 6, 7,** and **8** by changing the coordinates of (ϵ_1, ϵ_2). These values of the imperfection sensitivity are noted to be unity at the origin of the coordinates, $(\epsilon_1, \epsilon_2)=(0, 0)$. These surfaces are also partly obtainable by Eqs. (24) and (26), for Case 3 and Case 1, respectively, and referred to as the bifurcation sets, or imperfection sensitivity. It is very important to know that sometimes it may be

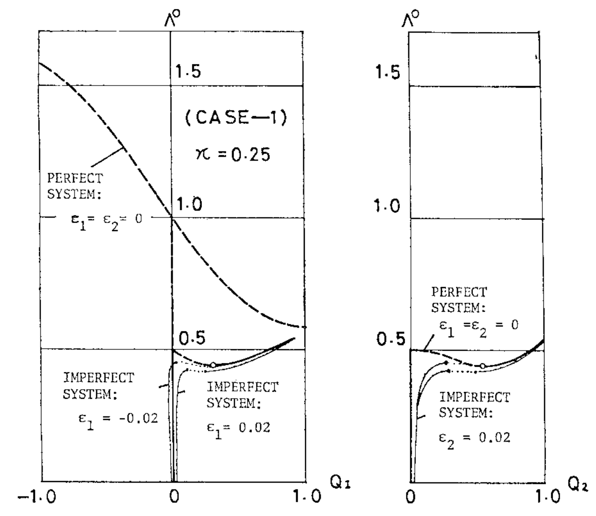

Fig. 3 Equilibrium Path of Unstable Symmetric Point of Bifurcation ($\kappa=0.25$).

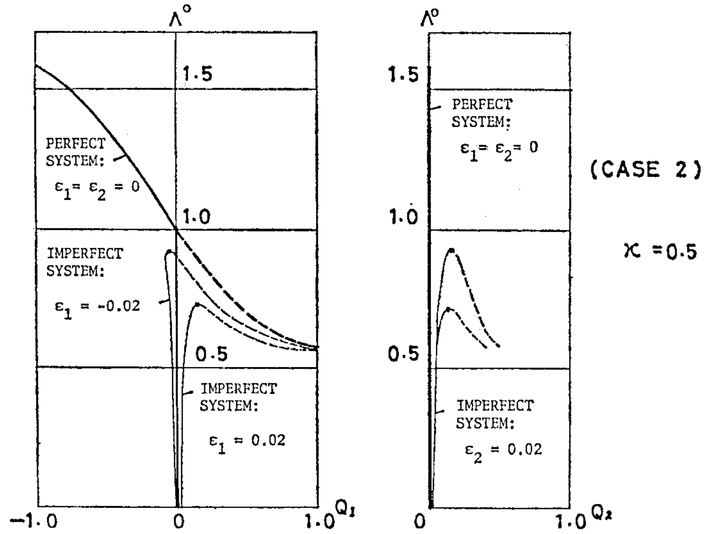

Fig. 4 Equilibrium Path of Homeoclinal Point of Bifurcation. (κ=0.5).

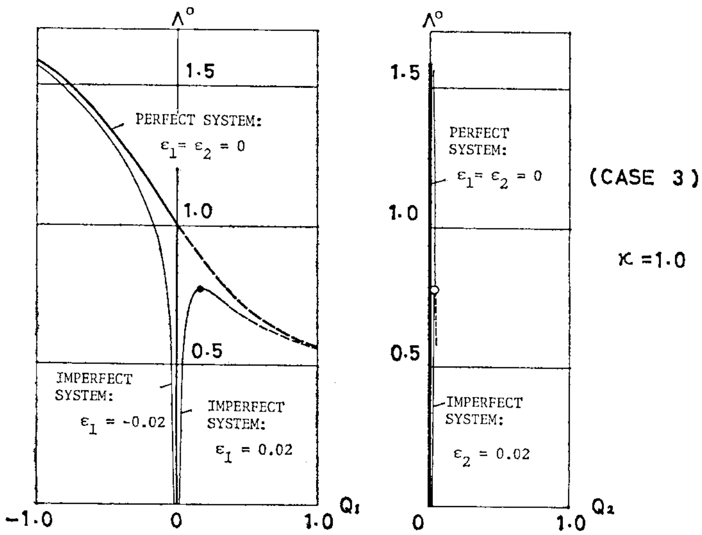

Fig. 5 Equilibrium Path of Asymmetric Point of Bifurcation. (κ=1.0).

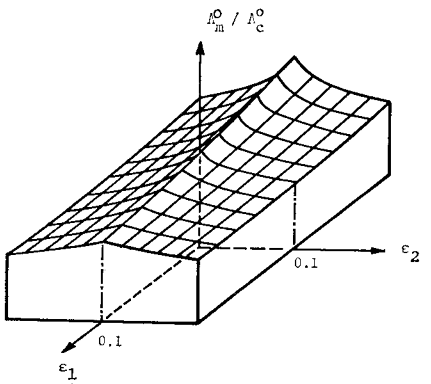

Fig. 6 Dual Cusp Bifurcation Set (κ=0.25). Partly Obtainable by Eq. (26).

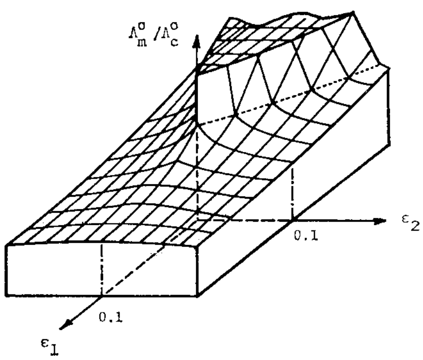

Fig. 7 Hyperbolic Umbilic Bifurcation Set (κ=0.5).

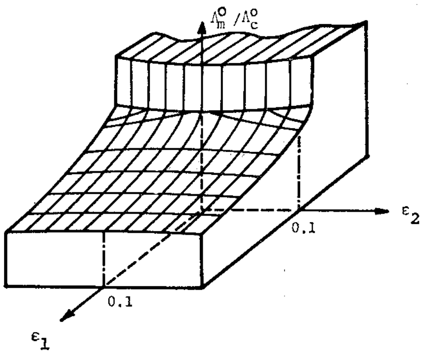

Fig. 8 Fold Bifurcation Set ($\kappa=1.0$). Partly Obtainable by Eq. (24).

quite dangerous to determine the load carrying capacity of the system by just knowing the lowest critical load and the corresponding initial imperfection only. Especially, the consideration of two imperfections at the coincident buckling, $\kappa=1/2$, will be seen mandatory for the ultimate load.

(3) Unstable Symmetric Buckling Model

Fig. 9 shows the model. This is thought to be representing rigid frames, lateral buckling of beams, struts on elastic foundation. Just like for the preceeding model, the discussions will be started on the ideally perfect system; afterwards, the discussions will be made on the imperfect system.

The potential of the system will be obtained precisely in the following equation:

$$V=\frac{1}{2}[2L(Q_1-\epsilon_1)]^2+\frac{1}{2}k_2[2\sin^{-1}(Q_2-\epsilon_2)]^2 -P(2L)\{(1-\epsilon_1^2-\epsilon_2^2)^{1/2}-(1-Q_1^2-Q_2^2)^{1/2}\} \quad \cdots\cdots(33)$$

where ϵ_1 and ϵ_2 refer to the initial displacements corresponding to displacements Q_1, and Q_2, respectively.

The Taylor's expansion of V yields a new potential function, D, expressed in terms of the 4th order polynomials of Q_1 and Q_2, and $\kappa=k_2/(k_1L^2)$, $\Lambda=P/(k_1L)$:

when $\epsilon_1=\epsilon_2=0$:

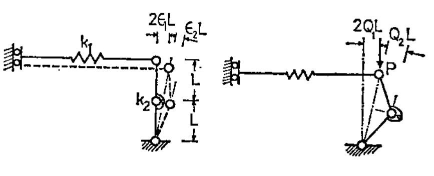

Fig. 9 Unstable Symmetric Bifurcation Buckling Model.

$$D=Q_1^2+\kappa\left(Q_2^2+\frac{1}{3}Q_2^4\right)-\Lambda^0\left\{\frac{1}{2}(Q_1^2+Q_2^2)+\frac{1}{8}(Q_1^2+Q_2^2)^2\right\} \quad \cdots\cdots(34)$$

The critical loads are given from $D_{11}=0$ and $D_{22}=0$:

$$\left.\begin{array}{ll}\Lambda_{c1}^0=2 & \text{from } D_{11c}=0\\ \Lambda_{c2}^0=2\kappa & \text{from } D_{22c}=0\end{array}\right\} \quad \cdots\cdots(35)$$

Therefore, the catastrophe of the system can be classified into the following according to the value of κ:

a) When $0<\kappa<1$ $\quad \Lambda_c^0=\Lambda_{c2}^0=2\kappa$, and

$D_{2c}^0=0$, $D_{11c}=2(1-\kappa)>0$, $D_{22c}=0$,

$D_{22c}^0=-1$, $D_{122c}=0$, $D_{2222c}=2\kappa$,

and

$A_{2c}^0=D_{2c}^0=0$, $A_{22c}^0=D_{22c}^0=-1$,

$A_{222c}=D_{222c}=0$,

$A_{2222c}=D_{2222c}-3(D_{122c})^2/D_{11c}=2\kappa>0$

Thus, the catastrophe is *stable symmetric* bifurcation and *cusp* catastrophe.

b) When $\kappa=1$ $\quad \Lambda_c^0=\Lambda_{c1}^0=\Lambda_{c2}^0=2$, and

$D_{1c}^0=0$, $D_{2c}^0=0$, $D_{11}^0=-1$, $D_{22}^0=-1$,

$D_{111c}=D_{112c}=D_{122c}=D_{222c}=0$, $D_{1111c}=-6$,

$D_{1122c}=-2$, $D_{2222c}=2$.

The catastrophe may be called *double symmetric* bifurcation buckling and may be called *coupled cusp* catastrophe.

c) When $\kappa>1$ $\quad \Lambda_c^0=\Lambda_{c1}^0=2$

$D_{1c}^0=0$, $D_{11c}=0$, $D_{22c}=2(\kappa-1)>0$,

$D_{111c}=D_{112c}=0$, $D_{1111c}=-6$,

thus

$A_{1111c}=-6$.

The catastrophe will be found to be *unstable symmetric* bifurcation buckling and *dual cusp* catastrophe.

When the initial imperfections are also considered, the following equations of equilibrium may be obtained from Eq. (33):

$$D_1=2(Q_1-\epsilon_1)-\Lambda^0\left\{Q_1+\frac{1}{2}Q_1(Q_1^2+Q_2^2)\right\}=0$$

$$D_2=2\kappa\left\{(Q_2-\epsilon_2)+\frac{2}{3}(Q_2-\epsilon_2)^3\right\}-\Lambda^0\left\{Q_2+\frac{1}{2}Q_2(Q_1^2+Q_2^2)\right\}=0 \quad \cdots\cdots(36)$$

Eq. (36) were solved by means of a perturbation method, and the results were obtained in each of the cases; while as in the previous case, the plastic conditions, $|Q_i|\leqq 0.5$ were used. **Figs. 10, 11,** and **12** show the load-displacement relationships in three different cases of (i) $\kappa=0.5$, (ii) $\kappa=1.0$, and (iii) $\kappa=1.5$, respectively.

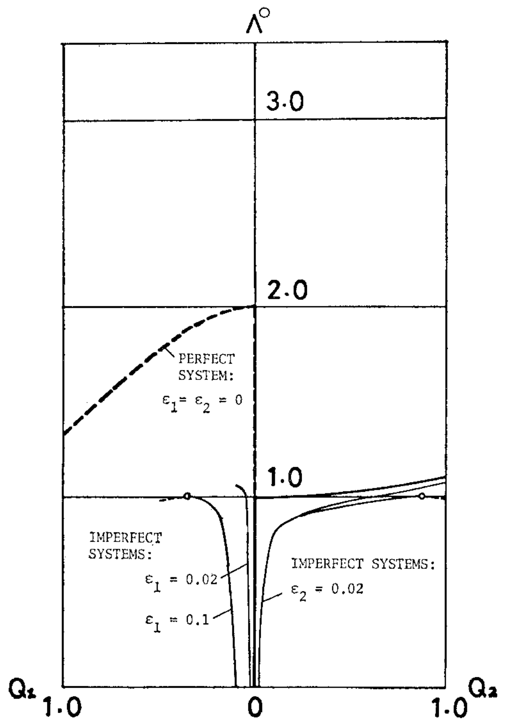

Fig. 10 Equilibrium Path of Stable Symmetric Point of Bifurcation ($\kappa=0.5$).

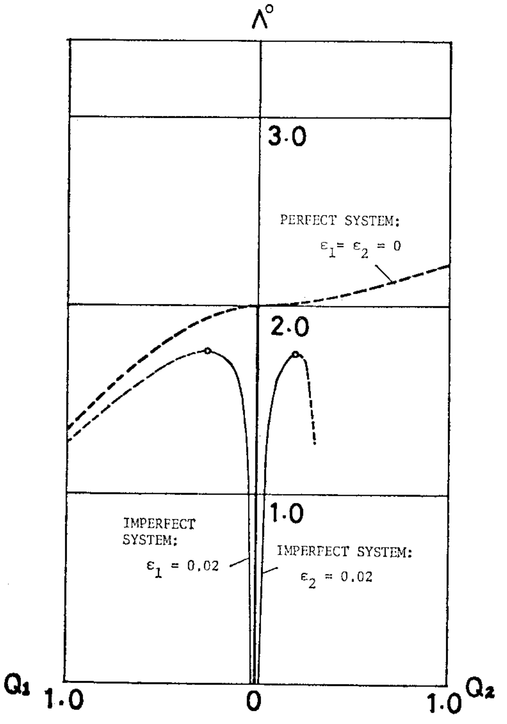

Fig. 11 Equilibrium Path of Double Symmetric Point of Bifurcation ($\kappa=1.0$).

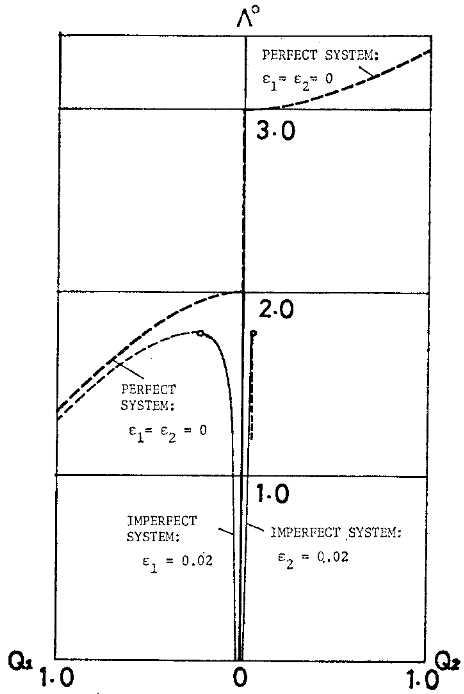

Fig. 12 Equilibrium Path of Unstable Symmetric Point of Bifurcation ($\kappa=1.5$).

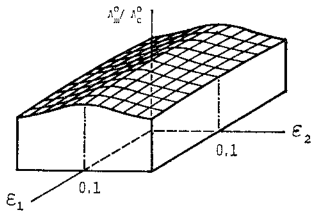

Fig. 13 Cusp Bifurcation Set ($\kappa=0.5$). Plastic Failure.

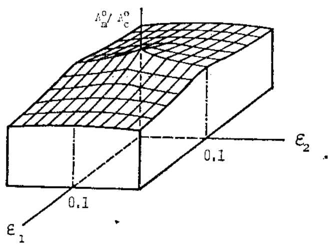

Fig. 14 Coupled Cusp Bifurcation Set ($\kappa=1.0$).

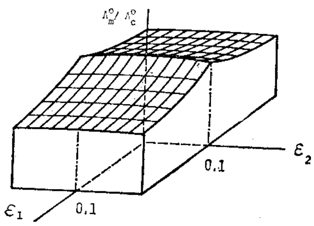

Fig. 15 Dual Cusp Bifurcation Set ($\kappa=1.5$). Partly Obtainable by Eq. (26).

The bifurcation set, or imperfection sensitivity of the system is given by **Figs. 13, 14,** and **15,** where the ordinates represent nondimensionalized load Λ_m^0/Λ_c^0 which have the value of unity at the origin: $(\epsilon_1, \epsilon_2)=(0, 0)$. Λ_m^0 indicates the local maximum load. Just like in the previous problem of asymmetric buckling, consideration of the lowest buckling mode only will be found to lead to insufficient results especially when the coincident buckling occurs.

(4) Arch Model

Fig. 16 shows an arch model consisting of an extentional spring and two flexural springs of constant k_1, and k_2, respectively. The arch is assumed to be subjected to two vertical concentrated loads, $2P$, and the deformation is indicated by the angle of rotation at each of the supports, ϕ_1 and ϕ_2, respectively.

The potential of the system, then can be given precisely by the following equation:

$$V=\frac{1}{2}k_1L^2\{b-\sqrt{(Q_1-Q_2)^2+(a-\sqrt{1-Q_1^2}-\sqrt{1-Q_2^2})^2}\}^2+\frac{1}{2}k_2\{(\phi_1-\tan^{-1}z-\phi_0)^2+(\phi_2+\tan^{-1}z-\phi_0)^2\}-PL(2Q_0-Q_1-Q_2) \quad \cdots\cdots(37)$$

where

$$z\equiv\frac{\sin\phi_1-\sin\phi_2}{a-\cos\phi_1-\cos\phi_2}\,; \qquad b=a-2\cos\phi_0$$

and

$$Q_1=\sin\phi_1, \quad Q_2=\sin\phi_2, \quad Q_0=\sin\phi_0$$

Let us consider transformation of coordinates:

$$v_1=\frac{1}{2}(\sin\phi_1+\sin\phi_2)\,; \quad v_2=\frac{1}{2}(\sin\phi_1-\sin\phi_2) \quad \cdots\cdots(38)$$

and upon Taylor's series expansion up to quartic terms, the following equations of equilibrium will be obtained:

$$D_1=c_1v_1+c_2v_1^3+c_3v_1v_2^2-\phi_0(2+v_1^2+v_2^2)+2\Lambda^0=0$$
$$D_2=v_2(c_4+c_5v_2^2+c_3v_1^2-2\phi_0v_1)=0$$

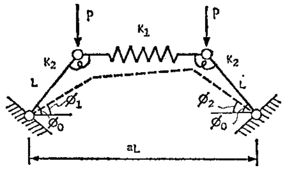

Fig. 16 Arch Model.

where

$$D(v_1, v_2, \Lambda^0)=V[\phi_1(v_1, v_2), \phi_2(v_1, v_2), P]/k_2,$$

and

$$\left.\begin{aligned}c_1&=2[1+\kappa(d-b)],\\ c_2&=4/3+\kappa[(1-b/d)a+2b/d],\\ c_3&=4(1+1/d-2/d^2-4/d^3)\\&\quad+\kappa[(1-d/b)(3a-4)+2ab/d^2],\\ c_4&=2[(1+2/d)^2+\kappa(1-d/b)a],\\ c_5&=4(1/3+2/3d-4/d^2-40/3d^3-32/3d^4)\\&\quad+a[(1-b/d)+2ab/d^3]\kappa\end{aligned}\right\} \quad \cdots\cdots(39)$$

and

$$\kappa\equiv k_1L^2/k_2\,; \qquad d\equiv a-2\,; \qquad \Lambda^0\equiv\frac{PL}{k_2}$$

Thus, derivatives will be found to be as follows:

$$\left.\begin{aligned}&D_{11}=c_1+3c_2v_1^2+c_3v_2^2-2\phi_0v_1,\\ &D_{12}=2v_2(c_3v_1-\phi_0),\\ &D_{22}=c_4+3c_5v_2^2+c_3v_1^2-2\phi_0v_1,\\ &D_1^0\equiv\partial^2D/\partial v_1\partial\Lambda^0=2\neq0,\\ &D_2^0=0, \quad D_{111}=6c_2v_1-2\phi_0,\\ &D_{222}=6c_5v_2, \quad D_{2222}=6c_5\\ &D_{112}=2c_3v_2, \quad D_{221}=(2c_3v_1-2\phi_0)\end{aligned}\right\} \quad \cdots\cdots(40)$$

Let v_{1ST} and Λ_{ST}^0 designate the snap-through displacement and load, respectively, then, they will be determined by the following equations:

$$\left.\begin{aligned}&D_{11}|_{v_2=0}=c_1+3c_2v_{1ST}^2-2\phi_0v_{1ST}=0\\ &D_1|_{v_2=0}=c_1v_{1ST}+c_2v_{1ST}^3-\phi_0(2+v_{1ST}^2)+2\Lambda_{ST}^0=0\end{aligned}\right\} \quad \cdots\cdots(41)$$

Thus,

$$v_{1ST}=\frac{1}{3c_2}(\phi_0+\sqrt{\phi_0^2-3c_1c_2}) \qquad \text{if} \quad \phi_0^2>3c_1c_2 \quad \cdots\cdots(42)$$

And surely at this snap-through point,

$$\left.\begin{aligned}&D_{111}=6c_2v_{1ST}-2\phi_0=2\sqrt{\phi_0^2-3c_1c_2}\neq0\\ &D_1^0=2\neq0\end{aligned}\right\} \cdots(43)$$

On the other hand, the bifurcation buckling displacement, v_{1B}, and load, Λ_B^0 are given by

$$\left.\begin{aligned}&D_{22}|_{v_2=0}=c_4+c_3v_{1B}^2-2\phi_0v_{1B}=0\\ &D_1|_{v_2=0}=c_1v_{1B}+c_2v_{1B}^3-\phi_0(2+v_{1B}^2)+2\Lambda_B^0=0\end{aligned}\right\} \quad \cdots\cdots(44)$$

where $D_2|_{v_2=0}=0$ is evidently satisfied.

The displacement corresponding to the bifurcation buckling, v_{1B} is obtained as

$$v_{1B}=\frac{1}{c_3}(\phi_0+\sqrt{\phi_0^2-c_3c_4}) \qquad \text{if} \quad \phi_0\geqq c_3c_4 \cdots(45)$$

At this point, the catastrophe can be identified by the sign of the value of A_{2222} of Eq. (7·a), i.e.

$$A_{2222}=D_{2222}-3\frac{D_{221}^2}{D_{11}} \quad \cdots\cdots(46)$$

It is noted that the constants c_1 and c_2 do not

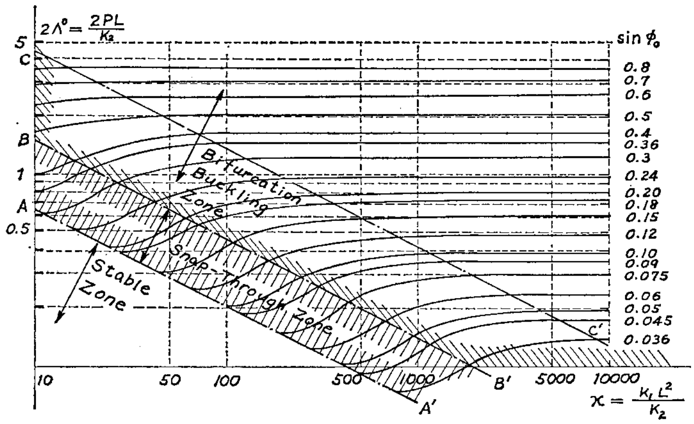

Fig. 17 Effects of $\sin\phi_0$ and Ratio of Spring Constants, κ, upon the Critical Load of Arch.

depend on the value of a, since

$$\left.\begin{array}{l} c_1 = 2[1-2\kappa(1-\cos\phi_0)] \\ c_2 = 4/3 + 2\kappa\cos\phi_0 \end{array}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots (47)$$

From Eq. (41), $\Lambda^0 = \Lambda^0_{ST}$ is seen independent of a; while, the other constants, c_3, c_4, and c_5, depend on the value of a. In other words, it will be understood that the snap-through is not influenced by the value of a; however, the bifurcation is influenced by the value of a.

Fig. 17 shows the effects of the rise ratio and the ratio of the spring constants, i.e., those of $\sin\phi_0$ and κ upon the critical load in case $a=4$. This example demonstrates that the domain is divided into three: (i) stable zone where instabilities do not occur at all, (ii) unstable zone where snap-through precedes to occur, and (iii) unstable zone where bifurcation buckling precedes to occur. These zones are given resepctively by

$$\left.\begin{array}{ll} \text{(i)} & \phi_0^2 < 3c_1c_2 \\ \text{(ii)} & \phi_0^2 \geqq 3c_1c_2 \text{ and either } \phi_0^2 < c_3c_4\,, \\ & \text{or } v_{1ST} > v_{1B} \\ \text{(iii)} & \phi_0^2 \geqq c_3c_4 \text{ and } v_{1B} \geqq v_{1ST} \end{array}\right\} \quad \cdots\cdots\cdots\cdots (48)$$

The results of numerical computation by Eq. (46) show that at the bifurcation buckling load

$$A_{2222} < 0$$

This implies that the catastrophe of the bifurcation buckling is *dual cusp*, or *unstable symmetric* buckling.

From the families of curves in **Fig. 17**, the following observations will be made: The snap-through load is significantly influenced by both $\sin\phi_0$ and κ; while, the bifurcation buckling zone may be divided into two sub-zones: In the sub-zone between straight lines BB' and CC', the buckling load may be equally influenced by both $\sin\phi_0$ and κ; while, in another sub-zone beyond the straight line CC', the buckling load does not depend on κ. In other words, in the last sub-zone, the buckling is seen to be *inextensional buckling*.

3. CONCLUSIONS

This paper is concerned with the catastrophe analysis of two-degree-of-freedom systems representing some of the civil engineering structures and at the same time, the bifurcation set, known as the imperfection sensitivity was obtained.

The first half of the paper is devoted for a brief summary of what the catastrophe is, and an interpretation of stability of structures in the light of catastrophe theory is given.

The second half of the paper is devoted for general discussion of two-degree-of-freedom structures in terms of the catastrophe theory. As a numerical illustration, the following models are considered: (i) asymmetric bifurcation buckling model, (ii) unstable symmetric bifurcation model, and (iii) an arch model.

It was found that the asymmetric bifurcation buckling model may be controlled by the *asymmetric buckling* as its name suggests, when the spring constant of the inclined spring is relatively small, *unstable buckling*, or *dual cusp* catastrophe when the constant is large enough, and *hyperbolic umbilic catastrophe*, or *homeoclinal buckling* when the constant takes a particular value.

On the other hand, the unstable buckling model was found to be controlled by *unstable buckling*, *coupled cusp*, and *stable buckling*, respectively as the ratio of the extensional spring to the flexural spring increases.

And thirdly, a detailed discussions were made with respect to the arch model, especially on the domains where whichever the *snap-through* or unstable-bifurcation buckling controls most significantly.

Actual structures are of course multi-degree-of-freedom systems and may not be nicely represented by two-degree-of-freedom systems. Nevertheless, as far as some buckling problems are concerned, these actual structures may be well represented in authors' opinion in small degree of freedom by use of the generalized coordinate systems, or model transforms. This is primarily because that the buckling is controlled mainly by the lowest possible load, and thus the structure may be influenced only by two modes at most, even in the worst case of coincident buckling.

Further study will be highly recommended to take into account the plastic deformations precisely and to find the correlation with analysis of multi-degree-of-freedom systems using discretization and modal transformation method.

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Last of all, the authors wish to express appreciation to Dr. T. Taniguchi of Okayama University for his valuable discussions and criticisms.

APPENDIX A. DERIVATIVES OF POTENTIAL

This appendix is intended to derive the basic relationships in Eq. (7). The substitution of Eq. (5) into Eq. (4·b) yields the identity

$$D_\alpha[v_i, v_\beta(v_i, \Lambda^0), \Lambda^0]=0 \quad \cdots\cdots(\mathrm{A}\cdot 1)$$

where Roman and Greek subscripts on v refer to the active and passive mode, respectively. The left-hand side now represents a function of totally $(m+1)$ independent active variables, v_i, and the loading parameter, Λ^0; thus, this left hand term can be differentiated as many times as pleased. Thus,

$$\left.\begin{aligned}&\frac{\partial D_\alpha}{\partial v_i}=D_{\alpha i}+D_{\alpha\beta}v_{\beta,i}=0\,,\\&\frac{\partial D_\alpha}{\partial \Lambda^0}=D_{\alpha\beta}v_\beta{}^0+D_\alpha{}^0=0\\&\text{where}\\&v_{\beta,i}\equiv\frac{\partial v_\beta}{\partial v_i}\,,\qquad v_\beta{}^0\equiv\frac{\partial v_\beta}{\partial \Lambda^0}\end{aligned}\right\}\quad\cdots\cdots(\mathrm{A}\cdot 2)$$

These equations are combined with Eq. (3·c) to lead to

$$v_{\alpha,i}\equiv\frac{\partial v_\alpha}{\partial v_i}=-\frac{D_{\alpha i}}{D_{\alpha\alpha}}=0 \quad (\alpha\text{: not summed}) \quad\cdots\cdots(\mathrm{A}\cdot 3)$$

$$v_\alpha{}^0\equiv\frac{\partial v_\alpha}{\partial \Lambda^0}=-\frac{D_\alpha{}^0}{D_{\alpha\alpha}} \quad (\alpha\text{: not summed}) \quad\cdots\cdots(\mathrm{A}\cdot 4)$$

Further differentiation of Eq. (A·2) will lead to

$$\frac{\partial^2 D_\alpha}{\partial v_i\partial v_j}=D_{\alpha ij}+D_{\alpha\beta j}v_{\beta,i}+D_{\alpha\beta}v_{\beta,ij}=0 \quad\cdots\cdots(\mathrm{A}\cdot 5)$$

In view of Eq. (A·3), this equation can be rewritten as

$$v_{\alpha,ij}\equiv\frac{\partial^2 v_\alpha}{\partial v_i\partial v_j}=-\frac{D_{\alpha ij}}{D_{\alpha\alpha}} \quad (\alpha\text{: not summed}) \quad\cdots\cdots(\mathrm{A}\cdot 6)$$

Partial differentiations of Eq. (6) with respect to active modes, v_i, will yield the following relations taking into consideration Eqs. (A·3) and (4·b):

$$A_i\equiv\frac{\partial A}{\partial v_i}=D_i+D_\alpha v_{\alpha,i}=0 \quad\cdots\cdots(\mathrm{A}\cdot 7)$$

$$\begin{aligned}A_{ij}\equiv\frac{\partial^2 A}{\partial v_i\partial v_j}&=D_{ij}+D_{i\alpha}v_{\alpha,j}+D_{\alpha j}v_{\alpha,i}\\&+D_{\alpha\beta}v_{\alpha,i}v_{\beta,j}+D_\alpha v_{\alpha,ij}=D_{ij} \quad\cdots\cdots(\mathrm{A}\cdot 8)\end{aligned}$$

$$\begin{aligned}A_{ijk}\equiv\frac{\partial^3 A}{\partial v_i\partial v_j\partial v_k}&=D_{ijk}+D_{ij\alpha}v_{\alpha,k}+D_{i\alpha k}v_{\alpha,j}\\&+D_{i\alpha\beta}v_{\alpha,j}v_{\beta,k}+D_{i\alpha}v_{\alpha,jk}+D_{\alpha jk}v_{\alpha,i}\\&+D_{\alpha j\beta}v_{\alpha,i}v_{\beta,k}+D_{\alpha j}v_{\alpha,ik}+D_{\alpha\beta k}v_{\alpha,i}v_{\beta,j}\\&+D_{\alpha\beta\gamma}v_{\alpha,i}v_{\beta,j}v_{\gamma,k}+D_{\alpha\beta}v_{\alpha,ik}v_{\beta,j}\\&+D_{\alpha\beta}v_{\alpha,i}v_{\beta,jk}+D_\alpha v_{\alpha,ijk}+D_{\alpha\beta}v_{\alpha,ij}v_{\beta,k}\\&=D_{ijk} \quad\cdots\cdots(\mathrm{A}\cdot 9)\end{aligned}$$

$$\begin{aligned}A_{ijkl}\equiv\frac{\partial^4 A}{\partial v_i\partial v_j\partial v_k\partial v_l}&=D_{ijkl}+D_{ij\alpha}v_{\alpha,kl}\\&+D_{ik\alpha}v_{\alpha,jl}+D_{il\alpha}v_{\alpha,jk}+D_{jk\alpha}v_{,il}\\&+D_{jl\alpha}v_{\alpha,ik}+D_{kl\alpha}v_{\alpha,ij}\\&+\sum_\alpha D_{\alpha\alpha}(v_{\alpha,ik}v_{\alpha,jl}+v_{\alpha,il}v_{\alpha,jk}+v_{\alpha,ij}v_{\alpha,kl})\\&=D_{ijkl}-\sum_\alpha\frac{1}{D_{\alpha\alpha}}(D_{ij\alpha}D_{kl\alpha}+D_{ik\alpha}D_{jl\alpha}\\&+D_{il\alpha}D_{jk\alpha}) \quad\cdots\cdots(\mathrm{A}\cdot 10)\end{aligned}$$

Furthermore, differentiations of A with respect to Λ^0 and v_i will yield the following relations taking into account Eqs. (3·c), (A·2), (A·3), (A·7), and (A·8):

$$A^0\equiv\frac{\partial A}{\partial \Lambda^0}=D_\alpha v_\alpha{}^0+D^0=D^0 \quad\cdots\cdots(\mathrm{A}\cdot 11)$$

$$\begin{aligned}A^{00}\equiv\frac{\partial^2 A}{\partial \Lambda^{02}}&=D_\alpha{}^0v_\alpha{}^0+D_{\alpha\beta}v_\alpha{}^0v_\beta{}^0+D_\alpha v_\alpha{}^{00}+D^{00}\\&=\sum_\gamma(D_\gamma{}^0+D_{\gamma\gamma}v_\gamma{}^0)v_\gamma{}^0=0 \quad\cdots\cdots(\mathrm{A}\cdot 12)\end{aligned}$$

$$\begin{aligned}A_i{}^0\equiv\frac{\partial^2 A}{\partial v_i\partial \Lambda^0}&=D_i{}^0+D_\alpha^0 v_{\alpha,i}+D_\alpha v_{\alpha,i}^0\\&+D_{\alpha\beta}v_{\alpha,i}v_{\beta,j}^0=D_i{}^0 \quad\cdots\cdots(\mathrm{A}\cdot 13)\end{aligned}$$

B. 1. Classification of Catastrophe.

By Thom		Buckling Analysis by Thompson		
Name	Potential, D	Name	Characteristics	
FOLD	$\frac{x^3}{3}+ux$	Limit Point	$D_{x}'\neq 0$,***	$D_{xxx}\neq 0$
		Asymmetric Buckling	$D_x'=0$,	$D_{xxx}\neq 0$
CUSP (DUAL CUSP)	$\pm\frac{x^4}{4}+\frac{u}{2}x^2+vx$	Stable Sym. Buckling	$D_x'=0$, $A_{xxxx}>0$	$D_{xxx}=0$
		Unstable Sym. Buckling	$D_x'=0$, $A_{xxxx}<0$	$D_{xxx}=0$
SWALLOW TAIL	$\frac{x^5}{5}+\frac{u}{3}x^3+\frac{v}{2}x^2+wx$	(rare)		
BUTTERFLY	$\pm\frac{x^6}{6}+\frac{u}{4}x^4+\frac{v}{3}x^3+\frac{w}{2}x^2+tx$	(rare)		
HYPERBOLIC UMBILIC	$x^3+y^3+wxy-ux-vy$	Monoclinal Homeoclinal	$D_{xxy}D_{yyy}>0$	$D_x'=0$ $D_y'=0$
ELLIPTIC UMBILIC	$\frac{x^3}{3}-xy^2+w(x^2+y^2)-ux-vy$	Anticlinal	$D_{xxy}D_{yyy}<0$	$D_{xxy}\neq 0$ $D_{yyy}\neq 0$
PARABOLIC UMBILIC	$x^2y\pm y^4+ux^2+vy^2+wx+ty$	(rare)		

* x, y: Behaviour Space ** u, v, w, t: Control Space
*** '′' implies differentiation with respect to control paremeters

$$A_{ij}^0 \equiv \frac{\partial^3 A}{\partial v_i \partial v_j \partial \Lambda^0} = D_{ij}^0 + D_\alpha{}^0 v_{\alpha,ij} + D_{\alpha j}^0 v_{\alpha,i} + D_{\alpha\beta}^0 v_{\alpha,i} v_{\beta,j} + D_\alpha{}^0 v_{\alpha,ij} + D_{\alpha\beta j} v_{\alpha,i} v_\beta{}^0 + D_{\alpha\beta\gamma} v_{\alpha,i} v_\beta{}^0 v_{\gamma,j} + D_{\alpha\beta} v_{\alpha,ij} v_\beta{}^0 + D_{\alpha\beta} v_{\alpha,i} v_{\beta,j}^0 + D_{\alpha j} v_{\alpha,i}^0 + D_{\alpha\beta} v_{\alpha,i}^0 v_{\beta,j} + D_\alpha v_{\alpha,ij}^0 = D_{ij}^0 + \sum_\gamma (D_\gamma{}^0 + D_{\gamma\gamma} v_\gamma{}^0) v_{\gamma,ij} = D_{ij}^0 \quad \text{(A·14)}$$

APPENDIX B. INTERACTION BETWEEN RENÉ THOM'S AND THOMPSON'S WORK

René Thom wrote a paper on morphogenesis in topological terms, which is now so-called Catastrophe Theory.

Let Q_i, Λ^j indicate the behavior space, and the control space, respectively, then he showed for the first time that the morphogenesis of Q_i can be classified into only seven catastrophes when the dimension of the control space, or in other words codimension, is less than or equal to four.

Table B. 1 shows the classifications of catastrophe by Thom, and Thmpson[1)~5)].

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