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MODIFIED THEORY OF TORSION BENDING FOR A CIRCULARLY CURVED BOX GIRDER BRIDGE*

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1. INTRODUCTION

The use of the box girder bridge on curved alignment has become increasingly popular when constructing an urban express highway in the congested area in a large city. This has stimulated considerable interest for research in the field of the statical analysis of a circularly curved box girder bridge. As can be seen from an extensive list of the references given by P. F. McManus, G. A. Nasir, and C. G. Culver[1], a large amount of literature on this subject has been published all over the world in the last few decades. As far as the author knows, however, it was by such Japanese investigators as I. Konishi and S. Komatsu[2], S. Kuranishi[3], and Y. Fukasawa[4] that the conventional theories of torsion bending were formulated in a highly accurate manner in terms of the cylindrical coordinates.

As is well known, the conventional theories of torsion bending rest on the basic assumption that the warping displacements due to restrained torsion can be evaluated from the formula in Saint-Venant's theory of torsion corresponding to the local value of the rate of twist at the section considered. The assumption may be allowwable if the variation of twist along the girder axis is small or moderate, in other words, if the twist is almost uniform along the girder axis[5]. Conversely, a doubt is raised about the validity of the assumption in the case in which the cross sections are highly restrained against torsion.

Accordingly, it seems that the conventional theories deal with the secondary shear deformation due to restrained torsion as being a negligibly small quantity. In the case of a thin-walled girder with closed cross sections being undeformable in its own plane, however, Th. von Kármán and W. Z. Chien[5], and R. Heilig[6] pointed out that disregard of the secondary shear deformation would lead to an unsatisfactory result, especially, at the sections highly restrained against torsion. The reason for this is that in these sections the corresponding shear stresses appear, in general, to be about the same amount as, or, according to circumstances, to be considerably greater than the primary shear stresses evaluated from the formula similar to Saint-Venant's solution of torsion problem.

In the case of a straight girder, some attempts to improve the weakness of the conventional theories mentioned above were made by introducing a "warping shear correction factor" by such investigators as E. Reissner[7], S. U. Benscoter[8], R. Heilig[6], W. Graße[9], E. Schlechte[10], and K. Roick and G. Sedlacek[11]. E. Reissner's work, however, was performed about a cylindrical rod with one end built in. The theory presented by S. U. Benscoter was based on the assumed displacement field regarding the warping shear deformation in a direct manner. Whereas, R. Heilig developed his theory on the basis of the statically equivalent shearing stress field in equilibrium with the normal stresses due to restrained warping. The works performed by W. Graße, E. Schlechte, and K. Roick and G. Sedlacek can be placed in the category of the theory presented by R. Heilig, although these works were done from a different physical view point. D. Schade[12] generalized the basic constitutive equation derived by R. Heilig by introducing five more "shear correction factors". N.

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Saeki[13] proposed a modified theory of torsion bending in which account was taken of the circumferential displacement component of the secondary shear deformation. Recently, F. Nishino, A. Hasegawa, and E. Natori[14] and M. Hirashima and T. Usuki[15] presented a theory which made it possible to analyse the structural behaviour of a box girder interrelated with cross-sectional distorsion and shear deformation.

It should be noted that in the case of a straight girder bridge torsional moments are developed along the girder axis only when it carries the live loads with an eccentricity from the line of the shear center, and that its own weight does not contribute to produce torsional moment owing to its cross-sectional shape symmetrical about a vertical axis. In the case of a curved girder bridge, on the other hand, the transverse loadings due to its own weight and live loads may produce much higher torsional moments along the girder axis owing to the curvature effects of the axis. Accordingly, it can be considered that torsional moments play a still more important role in the analysis of stresses in a curved girder rather than in a straight girder. In fact, they will become increasingly important with higher curvature of the girder axis.

Moreover, it should be noted that the curved girder bridge under transverse loadings is, in general, in a condition of complicated response to flexure and twisting.

Therefore, it seems desirable and significant to extend the modified theories of torsion bending so as to cover the circularly curved box girder bridges. The extension of the modified theory proposed by S. U. Benscoter was made by R. Dabrowski[16]. However, none other than the author succeeded in extending the modified theory of R. Heilig's type.

So, the major objective of this paper is to present the developement of the theory of the latter type.

2. THEORETICAL DEVELOPMENT

(1) Assumptions

Throughout this paper, the following assumptions will be made with regard to the cross-sectional dimensions and deformation of a circularly curved box girder bridge.

(1) Cross-sectional dimensions and shape of the girder are constant throughout its length.

(2) Any characteristic dimension of the cross section (its width or height) is small compared with the developed span length.

(3) The contour of cross section is non-deformable in its own plane.

(4) The transverse distribution of the warping is the same as would occur in pure torsion and its intensity is expressed as a certain unknown function χ instead of the rate of twist ψ_x of the reference girder axis, unlike the conventional theories of torsion bending[2)~4)].

For the reason mentioned previously, the theory based on the assumption (4) can be considered improved in the way the intensity of the warping in a cross section is expressed.

Hereafter, all of the quantities marked with an asterisk indicate that they stand in some relation to the shear center axis. We will define the symbols for these quantities where they first appear in the following sections.

(2) Coordinate Systems

We define the geometry of the thin-walled girder by establishing four kinds of the coordinate systems as shown in **Fig. 1**.

(1) We take the system of cylindrical coordinates $(\bar{O}-\rho\theta\zeta)$, with its origin located at the center of curvature $\bar{O}$ of the generatrix of the girder, ρ and θ being in a plane of curvature and ζ extending perpendicularly downward. The angular coordinate θ is measured from the right end of the girder for an observer facing it from the radial inward side.

(2), (3) For any cross section of the girder defined by the angle θ, two right-handed systems of rectangular coordinates $(S-\tilde{x}\tilde{y}\tilde{z})$, $(O-xyz)$ with the origins located at the shear center S and the center of figure O, respectively, are established.

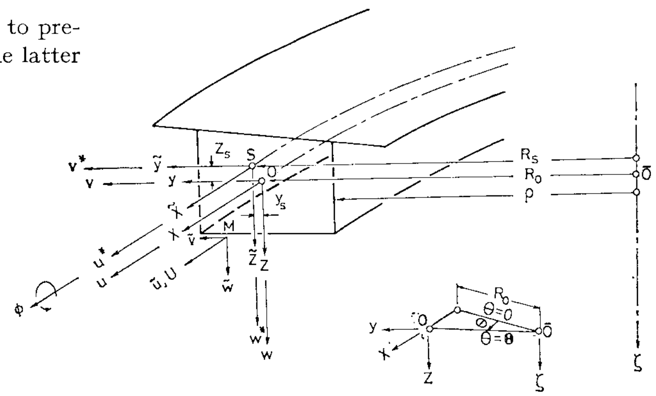

Fig. 1 Coordinate systems and displacement components.

The axes are so directed that $\bar{y}(y)$ and $\bar{z}(z)$ axes coincide with the radial outward and vertical downward directions, respectively, while $\bar{x}(x)$ axis coincides with the tangent drawn to the girder axis passing through the shear center S (center of figure O) and is taken positive in the direction corresponding to an increase of the angle θ.

(4) In the same cross section, the circumferential coordinate s is also taken along the midsurface contour of the cross section.

(3) Kinematics of Deformation

The translational displacement components of the shear center S, the center of figure O, and arbitrary point M on the midsurface contour of cross section are shown in **Fig. 1.** The angle of rotation of the cross section is denoted by the symbol ϕ and is taken positive when rotation is in the direction indicated in the figure. Let U be the warping displacement due to restrained torsion. Then, the assumption (4) determines U in the form:

$$U=-\omega^*(s)\chi^*(\theta) \qquad \cdots(1)$$

where ω^* is the warping function† with respect to the shear center. Now, guided by the well-known torsion-bending theory, the components of the displacement field are rewritten in the form:

$$\left.\begin{aligned}\bar{u}&=u^*-\bar{y}\phi_z^*+\bar{z}\phi_y^*-\omega^*\chi^*\\ \bar{v}&=v^*-\bar{z}\phi_x^*\\ \bar{w}&=w^*+\bar{y}\phi_x^*\end{aligned}\right\} \qquad \cdots(2)$$

where ϕ^*_x, ϕ_y^*, and ϕ_z^* are the rotational displacement components of the shear center axis in the directions of the x, y, and z axes, respectively. The components are related to the displacement components in the form:

$$\left.\begin{aligned}\phi_x^*&=\phi\\ \phi_y^*&=-\frac{1}{R_s}\frac{dw^*}{d\theta}\\ \phi_z^*&=\frac{1}{R_s}\left(\frac{dv^*}{d\theta}-u^*\right)\end{aligned}\right\} \qquad \cdots(3)$$

where R_s is the radius of curvature of the shear center axis.

Introducing the last two in Eqs. (2) into the appropriate strain-displacement relation in cylindrical coordinates, we get the expression for normal strain ϵ_θ of any longitudinal fiber of the girder. This can be written in terms of the kinematical quantities associated with the deformation of the shear center axis as follows:

$$\epsilon_\theta=\frac{R_s}{\rho}\left(\epsilon_x^*-\bar{y}\psi_z^*+\bar{z}\psi_y^*-\omega^*\frac{1}{R_s}\frac{d\chi^*}{d\theta}\right) \qquad \cdots(4)_1$$

where

ϵ_x^* = rate of stretch of the shear center axis,

ψ_y^* = change in curvature in the $\bar{x}\bar{z}$-plane,

ψ_z^* = change in curvature in the $\bar{x}\bar{y}$-plane.

These kinematical quantities are related to the translational displacement components of the shear center and the rotation of the cross section by the formulae:

$$\left.\begin{aligned}\epsilon_x^*&=\frac{1}{R_s}\left(\frac{du^*}{d\theta}+v^*\right)\\ \psi_y^*&=-\frac{1}{R_s}\left(\phi+\frac{1}{R_s}\frac{d^2w^*}{d\theta}\right)\\ \psi_z^*&=\frac{1}{R_s^2}\left(\frac{d^2v^*}{d\theta^2}-\frac{du^*}{d\theta}\right)\end{aligned}\right\} \qquad \cdots(5)$$

It will be convenient for the subsequent analysis to transform Eq. $(4)_1$ into an alternative form by the use of the relationship between the rate of stretch of the shear center axis, ϵ_x^*, and that of the centroidal axis, ϵ_x. The relationship is

$$\frac{R_o}{R_s}\epsilon_x=\epsilon_x^*+y_s\psi_z^*-z_s\psi_y^{*} \text{ [4)]} \qquad \cdots(4)_2$$

where R_o is the radius of curvature of the centroidal axis, and y_s and z_s are the coordinates of the shear center referenced from the center of figure. Substituting Eq. $(4)_2$ into Eq. $(4)_1$ yields the required expression for ϵ_θ as follows:

$$\epsilon_\theta=\frac{R_o}{\rho}\epsilon_x-\frac{R_s}{\rho}\left(y\psi_z^*-z\psi_y^*+\omega^*\frac{1}{R_s}\frac{d\chi^*}{d\theta}\right) \qquad \cdots(4)_3$$

The quantity ϵ_x in Eq. $(4)_3$ is related to the displacement components of the center of figure in the form:

$$\epsilon_x=\frac{1}{R_o}\left(\frac{du}{d\theta}+v\right) \qquad \cdots(6)$$

Next, the expression for shearing strain $\gamma_{s\theta}$ is obtained from the displacement field given by Eqs. (2), which, after some tedious and lengthy calculations, gives

$$\gamma_{s\theta}=\frac{R_s}{\rho}r_t^*\psi_x^*-\rho\frac{\partial}{\partial s}\left(\frac{\omega^*}{\rho}\right)\chi^* \qquad \cdots(7)$$

where r_t^* is the distance from the shear center to the tangent at any point on the midsurface contour; ψ_x^* is the rate of twist of the shear center axis and is expressed as

† Detailed derivation of the expression for ω^* in terms of curvilinear coordinates is given in Reference [2, 4].

$$\psi_x^*=\frac{1}{R_s}\left(\frac{d\phi}{d\theta}-\frac{1}{R_s}\frac{dw^*}{d\theta}\right) \quad \cdots\cdots(8)$$

(4) Normal Stress

According to Hooke's law, the axial normal stress, σ_θ, acting on a cross-sectional plane is given by the formula

$$\sigma_\theta=\frac{1}{n_e}E_s\epsilon_\theta \quad \cdots\cdots(9)$$

where n_e is the ratio of Young's modulus of elasticity of steel, E_s, to that of the material of a plate element forming the cross section, E. When this stress distribution is integrated over the entire cross-sectional area, F, it yields four independent stress resultant and stress couples: a normal force N_x, two bending moment M_y and M_z acting about the y and z axes, and a warping moment M_ω^* with respect to the shear center axis (**Fig. 2**).

For the subsequent analysis, we should note that the basic functions 1, $y(s)$, $z(s)$, and $\omega^*(s)$ are chosen so that they satisfy the following curvature and elasticity modulus weighted orthogonality relations:

$$\left.\begin{aligned}
&\int_F \frac{1}{n_e}\frac{1}{\rho}y\,dF=0\\
&\int_F \frac{1}{n_e}\frac{1}{\rho}z\,dF=0\\
&\int_F \frac{1}{n_e}\frac{1}{\rho}\omega^*\,dF=0\\
&\int_F \frac{1}{n_e}\frac{1}{\rho}\omega^*y\,dF=0\\
&\int_F \frac{1}{n_e}\frac{1}{\rho}\omega^*z\,dF=0
\end{aligned}\right\} \quad \cdots\cdots(10)$$

where $dF=tds$, and t is the thickness of a plate element. Then, performing the integrations of the defining equations for stress resultant and stress couples (**Fig. 2**)

$$\left.\begin{aligned}
&N_x=\int_F \sigma_\theta\,dF, \qquad M_y=\int_F \sigma_\theta z\,dF\\
&M_z=-\int_F \sigma_\theta y\,dF, \qquad M_\omega^*=\int_F \sigma_\theta \omega^*\,dF
\end{aligned}\right\} \quad \cdots\cdots(11)$$

by the use of Eqs. $(4)_3$ and (9), along with Eqs. (10), we obtain a set of equations after some algebraic manipulation:

$$\left.\begin{aligned}
&\epsilon_x=\frac{N_x}{E_sF_o}\\
&\psi_y^*=\frac{R_o}{R_s}\frac{J_zM_y+J_{yz}M_z}{E_s(J_yJ_z-J_{yz}^2)}\\
&\psi_z^*=\frac{R_o}{R_s}\frac{J_{yz}M_y+J_yM_z}{E_s(J_yJ_z-J_{yz}^2)}\\
&\frac{1}{R_s}\frac{d\chi^*}{d\theta}=-\frac{M_\omega^*}{E_sC_\omega^*}
\end{aligned}\right\} \quad \cdots\cdots(12)$$

where F_o is the cross-sectional area; J_y, J_z, and J_{yz} are, respectively, the moments and product of inertia with respect to a set of in-plane coordinate axes passing through the center of figure O; C_ω^* is the sectorial moment of inertia (warping constant) with respect to the shear center S. These quantities are curvature and elasticity modulus weighted cross-sectional properties[2] defined by

$$\left.\begin{aligned}
&F_o=R_o\int_F \frac{1}{n_e}\frac{1}{\rho}dF\\
&J_y=R_o\int_F \frac{1}{n_e}\frac{1}{\rho}z^2\,dF\\
&J_z=R_o\int_F \frac{1}{n_e}\frac{1}{\rho}y^2\,dF\\
&J_{yz}=R_o\int_F \frac{1}{n_e}\frac{1}{\rho}yz\,dF\\
&C_\omega^*=R_s\int_F \frac{1}{n_e}\frac{1}{\rho}\omega^{*2}\,dF
\end{aligned}\right\} \quad \cdots\cdots(13)$$

Combining Eqs. $(4)_3$, (9), and (12), we have the formula for σ_θ expressed in terms of the stress resultant N_x and stress couples, M_y, M_z, and M_ω^*:

$$\sigma_\theta=\frac{1}{n_e}\frac{R_o}{\rho}\left\{\frac{N_x}{F_o}-\frac{J_{yz}M_y+J_yM_z}{J_yJ_z-J_{yz}^2}y+\frac{J_zM_y+J_{yz}M_z}{J_yJ_z-J_{yz}^2}z+\frac{R_s}{R_o}\frac{M_\omega^*}{C_\omega^*}\omega^*\right\} \quad \cdots\cdots(14)$$

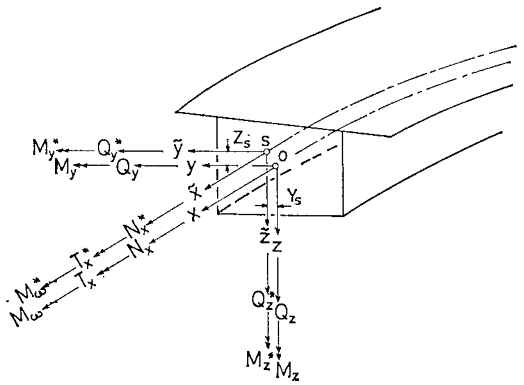

Fig. 2 Stress resultants and stress couples.

(5) Shear Flows[2),4)]

To obtain a possible shearing stress distribution on a cross-sectional plane, we shall not start from applying Hooke's law to the shearing strain given by Eq. (7), but employ the condition of longitudinal equilibrium for forces acting on a differential plate element. Let us now denote by q_σ the shear flow in equilibrium with the normal stress flux $\sigma_\theta t$. Then, the required equilibrium equation is given by

$$\frac{1}{\rho}\frac{\partial}{\partial\theta}(\sigma_\theta t)+\frac{1}{\rho^2}\frac{\partial}{\partial s}(\rho^2 q_\sigma)+\bar{p}_x=0 \quad \cdots\cdots\cdots(15)$$

where $\bar{p}_x$ is the x-component of the external force per unit area of the middle surface of a plate element. Solving Eq. (15) for q_σ, we find

$$q_\sigma=\frac{R_o^2}{\rho^2}\tilde{q}_\sigma-\frac{1}{\rho^2}\int_0^s\rho\frac{\partial}{\partial\theta}(\sigma_\theta t)ds -\frac{R_o^2}{\rho^2}\int_0^s\frac{\rho^2}{R_o^2}\bar{p}_x ds \quad \cdots\cdots\cdots(16)$$

where $\tilde{q}_\sigma$ is an integration constant which can be written in the form:

$$\tilde{q}_\sigma=\begin{cases}\tilde{q}^0_{\sigma,k}-\tilde{q}^0_{\sigma,k-1} & \text{on the wall of the plate element common to cells } k-1 \text{ and } k,\\ \tilde{q}^0_{\sigma,k} & \text{on the wall of the plate element belonging to } k\text{-th cell only,}\\ \tilde{q}^0_{\sigma,k}-\tilde{q}^0_{\sigma,k+1} & \text{on the wall of the plate element common to cells } k \text{ and } k+1.\end{cases} \quad \cdots\cdots\cdots(17)$$

The integration constant $\tilde{q}^0_{\sigma,k}$'s must be determined from the following conditions of compatibility

$$\oint_k\frac{1}{\rho}q_\sigma\frac{n_g}{t}ds=0 \quad (k=1,\cdots,n) \quad \cdots\cdots\cdots(18)$$

where n_g is the ratio of shearing modulus of elasticity of steel, G_s, to that of the material of a plate element, G, n is the number of cells making up the cross section, and the subscript k on the circuital integral indicates that the integration is to be carried around the k-th cell. We then substitute Eq. (16), along with Eq. (17), into Eqs. (18) to get a set of equations for determining the unknown quantities $\tilde{q}^0_{\sigma,k}$ $(k=1,\cdots,n)$:

$$-\tilde{q}^0_{\sigma,k-1}\int_{k-1,k}\frac{1}{\rho^3}\frac{n_g}{t}ds+\tilde{q}^0_{\sigma,k}\oint_k\frac{1}{\rho^3}\frac{n_g}{t}ds -\tilde{q}^0_{\sigma,k+1}\int_{k,k+1}\frac{1}{\rho^3}\frac{n_g}{t}ds =\frac{1}{R_o^2}\oint_k\frac{1}{\rho^3}\left\{\left[\int_0^s\rho\frac{\partial}{\partial s}(\sigma_\theta t)ds\right]\frac{n_g}{t}\right\}ds \quad (k=1,\cdots,n) \quad \cdots\cdots\cdots(19)$$

in which the contribution from the last term in Eq. (16) is omitted as is usually done in engineering beam theory. The subscript, say, $k-1, k$ on the integral in Eqs. (19) indicates that the integration is to be carried over the path along the wall of the plate element common to cells $k-1$ and k.

In view of Eq. (14), the shear flow q_σ determined in this way can be regarded as the sum of two parts—the shear flow q_b due to stretching and bending and the shear flow q_ω due to restrained torsion:

$$q_\sigma=q_b+q_\omega \quad \cdots\cdots\cdots(20)$$

After rearrangement of the resulting expression for q_σ, we get the formulae for q_b and q_ω written in the form:

$$\left.\begin{aligned} q_b&=K_n\frac{R_o^2}{\rho^2}(\tilde{q}_n-S_n)+K_y\frac{R_o^2}{\rho^2}(\tilde{q}_y-S_y)+K_z\frac{R_o^2}{\rho^2}(\tilde{q}_z-S_z)\\ q_\omega{}^*&=K_\omega{}^*\frac{R_s^2}{\rho^2}(\tilde{q}_\omega-S_\omega{}^*)\end{aligned}\right\} \quad \cdots\cdots\cdots(21)$$

where

$$\left.\begin{aligned} K_n&=\frac{1}{F_o}\frac{1}{R_o}\frac{dN_x}{d\theta}\\ K_y&=-\frac{J_{yz}}{J_yJ_z-J_{yz}^2}\frac{1}{R_o}\frac{dM_y}{d\theta}-\frac{J_y}{J_yJ_z-J_{yz}^2}\frac{1}{R_o}\frac{dM_z}{d\theta}\\ K_z&=\frac{J_z}{J_yJ_z-J_{yz}^2}\frac{1}{R_o}\frac{dM_y}{d\theta}+\frac{J_{yz}}{J_yJ_z-J_{yz}^2}\frac{1}{R_o}\frac{dM_z}{d\theta}\\ K_\omega{}^*&=\frac{1}{C_\omega{}^*}\frac{1}{R_s}\frac{dM_\omega{}^*}{d\theta}\end{aligned}\right\} \quad \cdots\cdots\cdots(22)$$

and S_n, S_y, S_z, and $S_\omega{}^*$ are the functions of the circumferential coordinate s defined by

$$\left.\begin{aligned} S_n(s)&=\int_0^s\frac{t}{n_e}ds\,, & S_y(s)&=\int_0^s y\frac{t}{n_e}ds\\ S_z(s)&=\int_0^s z\frac{t}{n_e}ds\,, & S_\omega(s)^*&=\int_0^s\omega^*\frac{t}{n_e}ds\end{aligned}\right\} \quad \cdots\cdots\cdots(23)$$

Moreover, $\tilde{q}_n$, $\tilde{q}_y$, $\tilde{q}_z$, and $\tilde{q}_\omega$ are the quantities which can be expressed in terms of $\tilde{q}^0_{n,k}$'s, $q^0_{y,k}$'s, $\tilde{q}^0_{z,k}$'s and $\tilde{q}^0_{\omega,k}$'s in exactly the same manner as $\tilde{q}_\sigma$ is related to $\tilde{q}_{\sigma,k}$'s by Eq. (17), respectively. It is immediately evident from the theory of linear algebraic equation that the quantities $\tilde{q}^0_{n,k}$'s, $\tilde{q}^0_{y,k}$'s, $\tilde{q}^0_{z,k}$'s, and $\tilde{q}^0_{\omega,k}$'s are the solutions to a set of equations obtained by replacing the integrals within braces in Eqs. (19), divided by R_0^2, by $S_n(s)$, $S_y(s)$, $S_z(s)$, and $S_\omega{}^*(s)$, respectively.

In the theory of torsion bending, the shear flow q_t due to torsion is assumed to be the sum of two parts—a primary shear flow q_s which is evaluated from the formula similar to Saint-Venant's solution in torsion problem, and a secondary shear flow q_ω which is in equilibrium with the normal stress flux caused by the restrained warping:

$$q_t = q_s + q_\omega \quad \cdots\cdots(24)$$

Then, the primary shear flow q_s is given by the formula†

$$q_s = \frac{T_s{}^*}{J_T{}^*}\frac{R_s{}^2}{\rho^2}\tilde{q}_s \quad \cdots\cdots(25)$$

where $\tilde{q}_s$, called Saint-Venant's torsion function, is written in the form:

$$\tilde{q}_s = \begin{cases} \tilde{q}^0_{s,k} - \tilde{q}^0_{s,k-1} & \text{on the wall of the plate element common to celles } k-1 \text{ and } k, \\ \tilde{q}^0_{s,k} & \text{on the wall of the plate element belonging to } k\text{-th cell only,} \\ \tilde{q}^0_{s,k} - \tilde{q}^0_{s,k+1} & \text{on the wall of the plate element common to cells } k \text{ and } k+1. \end{cases} \quad \cdots\cdots(26)$$

These quantities are determined by solving the following set of equations:

$$-\tilde{q}^0_{s,k-1}\int_{k-1,k}\frac{1}{\rho^3}\frac{n_g}{t}ds + \tilde{q}^0_{s,k}\oint_k \frac{1}{\rho^3}\frac{n_g}{t}ds - \tilde{q}^0_{s,k+1}\int_{k,k+1}\frac{1}{\rho^3}\frac{n_g}{t}ds = \frac{1}{R_s}\oint_k\frac{1}{\rho^2}\gamma_t{}^* ds \quad (k=1,\cdots,n) \quad \cdots\cdots(27)$$

The quantities $T_s{}^*$ and $J_T{}^*$ are called primary torsional moment and torsion constant, respectively. They are given by the formulae

$$T_s{}^* = \int_C q_s\gamma_t{}^* ds = G_s J_T{}^* \psi_x{}^* \quad \cdots\cdots(28)$$

and

$$J_T{}^* = R_s{}^2\int_C\frac{1}{\rho^2}\tilde{q}_s\gamma_t{}^* ds = R_s{}^3\int_C\frac{1}{\rho^3}\tilde{q}_s{}^2\frac{n_g}{t}ds \quad \cdots\cdots(29)$$

where the subscript C on the integral indicates that the integration is to be taken over the entire contour of the cross section.

Combining the second in Eqs. (21), the last in Eqs. (22), and Eq. (49), we obtain the expression for q_ω written in the form:

$$q_\omega = \frac{T_\omega{}^*}{C_\omega{}^*}\frac{R_s{}^2}{\rho^2}(\tilde{q}_\omega - S_\omega{}^*)\,\dagger \quad \cdots\cdots(30)$$

where $T_\omega{}^*$ is a stress couple called warping torque or secondary torsional moment and defined by the equation

$$T_\omega{}^* = \int_C q_\omega\gamma_t{}^* ds \quad \cdots\cdots(31)$$

Introducing Eqs. (25) and (30) into Eq. (24), we arrive at the formula for q_t:

$$q_t = \frac{T_s{}^*}{J_T{}^*}\frac{R_s{}^2}{\rho^2}\tilde{q}_s + \frac{T_\omega{}^*}{C_\omega{}^*}\frac{R_s{}^2}{\rho^2}(\tilde{q}_\omega - S_\omega{}^*) \quad \cdots\cdots(32)$$

(6) Orthogonality Relations

Let us denote the components of the shear flow q_b given by the first, the second, and the third terms on the right side of the first in Eqs. (21) by q_n, q_y, and q_z, respectively. It will then be shown that there exists an orthogonality relation between Saint-Venant's torsion function $\tilde{q}_s$ and any component of q_σ [12]. The relation can be written as

$$\left.\begin{aligned} &\int_C \tilde{q}_s q_n \frac{1}{\rho}\frac{n_g}{t}ds = 0, \quad \int_C \tilde{q}_s q_y \frac{1}{\rho}\frac{n_g}{t}ds = 0 \\ &\int_C \tilde{q}_s q_z \frac{1}{\rho}\frac{n_g}{t}ds = 0, \quad \int_C \tilde{q}_s q_\omega \frac{1}{\rho}\frac{n_g}{t}ds = 0 \end{aligned}\right\} \quad \cdots\cdots(33)$$

According to Eqs. (20) and (21), these relations are summarized as

$$\int_C \tilde{q}_s q_\sigma \frac{1}{\rho}\frac{n_g}{t}ds = 0 \quad \cdots\cdots(34)$$

To begin with, we shall take the last relation in Eqs. (33) as an example. As explained in preceding section, the condition of compatibility for the shear flow q_ω developed in the k-th cell of the cross section can be written in the form:

$$-\tilde{q}^0_{\omega,k-1}\int_{k-1,k}\frac{1}{\rho^3}\frac{n_g}{t}ds + \tilde{q}^0_{\omega,k}\oint_k\frac{1}{\rho^3}\frac{n_g}{t}ds - \tilde{q}^0_{\omega,k+1}\int_{k,k+1}\frac{1}{\rho^3}\frac{n_g}{t}ds = \oint_k\frac{1}{\rho^3}S_\omega{}^*\frac{n_g}{t}ds \quad \cdots\cdots(35)$$

Multiplying through both sides of Eq. (35) by $\tilde{q}^0_{s,k}$ and summing up the like expressions for all the cells obtained in this way, and then manipu-

† Detailed discussion on this subject is given in References [2, 4].

* After the analogy of a way of deriving the expression for q_σ, the influence of distributed external warping moment $m_\omega{}^*$ (defined later) on the value of q_ω is disregarded.

lating the left hand side of the resulting expression, we find

$$\int_C \tilde{q}_s \tilde{q}_\omega \frac{1}{\rho^3}\frac{n_g}{t}ds = \sum_{k=1}^{n} \tilde{q}^0_{s,k} \oint_k \frac{1}{\rho^3} S_\omega{}^* \frac{n_g}{t}ds \quad \cdots\cdots(36)$$

At the same time, if attention is paid to the positive directions of the flow type of quantities $S_\omega{}^*$ (which can be interpreted physically as the quantities related to the statically determinate parts of the shear flows q_ω), the right hand side of Eq. (36) can be rewritten in the form:

$$\sum_{k=1}^{n} \tilde{q}^0_{s,k} \oint_k \frac{1}{\rho^3} S_\omega{}^* \frac{n_g}{t}ds = \int_C \tilde{q}_s S_\omega{}^* \frac{1}{\rho^3}\frac{n_g}{t}ds \quad \cdots\cdots(37)$$

It then follows from Eqs. (36) and (37) that

$$\int_C \tilde{q}_s(\tilde{q}_\omega - S_\omega{}^*)\frac{1}{\rho^3}\frac{n_g}{t}ds = 0 \quad \cdots\cdots(38)$$

which, by virtue of the last of Eqs. (21), reduces to the last orthogonality relation in Eqs. (33).

Similarly, if the same general procedure as in the preceding case is followed the remaining orthogonality relations can be easily verified.

It should be noted that the orthogonality relations (33) hold for any shape of such hybrid cross sections as twin-celled or multicelled cross sections with overhangs. This is clear from the fact that Saint-Venant's torsion function $\tilde{q}_s$ vanishes on the open parts of the cross section.

(7) Governing Equation for Warping Moment and Torque

To derive the required governing equation, we shall employ the principle of virtual displacement[17]. To this effect, we multiply the internal equilibrium equation (15) by the virtual displacement, $\delta u = \omega^* \delta\chi^*$, and integrate over the entire middle surface S of a circularly curved girder, with $dS = \rho\, d\theta\, ds$ in mind. This gives

$$0 = \int_S \left\{ \frac{1}{\rho}\frac{\partial}{\partial\theta}(\sigma_\theta t) + \frac{1}{\rho^2}\frac{\partial}{\partial s}(\rho^2 q_\sigma) + \tilde{p}_x \right\} \delta u\, dS$$
$$= \int_0^\Theta \left\{ \int_C \frac{\partial}{\partial\theta}(\sigma_\theta t)\omega^* ds + \int_C \frac{1}{\rho}\frac{\partial}{\partial s}(\rho^2 q_\sigma)\omega^* ds + R_s \int_C \frac{\rho}{R_s}\tilde{p}_x \omega^* ds \right\} \delta\chi^* d\theta \quad \cdots\cdots(39)$$

where Θ is the opening angle of the curved girder.

Introducing Eq. (14) into the first integral within braces in Eq. (39) and integrating with the last three in Eqs. (10) and the last in Eqs. (13) in mind, we have

$$\int_C \frac{\partial}{\partial\theta}(\sigma_\theta t)\omega^* ds = \frac{dM_\omega{}^*}{d\theta} \quad \cdots\cdots(40)$$

The last integral within braces in Eq. (39) is defined as

$$\int_C \frac{\rho}{R_s}\tilde{p}_x \omega^* ds = m_\omega{}^* \quad \cdots\cdots(41)$$

where $m_\omega{}^*$ is the intensity of external warping moment with respect to the shear center axis. To evaluate the second integral, we perform an integration by parts on the integral to get

$$\int_C \frac{1}{\rho}\frac{\partial}{\partial s}(\rho^2 q_\sigma)\omega^* ds = (\rho^2 q_\sigma)\frac{\omega^*}{\rho}\Big|_C - \int_C \rho^2 q_\sigma \frac{\partial}{\partial s}\left(\frac{\omega^*}{\rho}\right)ds \quad \cdots\cdots(42)$$

Let us now introduce the summation signs $\sum_i$ and $\sum_j{}'$ which indicate that the summations are taken all over the walls i, j of the plate segments and the nodal points j of the cross section, respectively. Moreover, let Δ_j be an operator[8] which indicates that any "flow" type of quantities are summed algebraically at the nodal point j. Then, the fact that the first term on the right side of Eq. (42) vanishes can be proved in the following way: By noting that the shear flows q_σ are determined in such a way that at any nodal point the sum of the "inflows" balances with that of the "outflows", we can rearrange the term to yield

$$(\rho^2 q_\sigma)\frac{\omega^*}{\rho}\Big|_C = \sum_i \left[(\rho^2 q_\sigma)\frac{\omega^*}{\rho} \right]_i^j = \sum_j{}' \left\{ \frac{\omega^*}{\rho}\Delta_j(\rho^2 q_\sigma) \right\} = 0 \quad \cdots\cdots(43)$$

It is known that the differential equation for determining the warping function $\omega^*(s)$ is given by

$$\frac{R_s{}^2}{\rho^2}\tilde{q}_s = \left\{ \frac{R_s}{\rho} r_t{}^* - \rho\frac{\partial}{\partial s}\left(\frac{\omega^*}{\rho}\right) \right\} \frac{t}{n_g} \quad {}^{2),4)} \quad \cdots\cdots(44)$$

We then substitute the expression for $\partial/\partial s(\omega^*/\rho)$ obtained from Eq. (44), along with Eq. (20), into the remaining integral on the right side of Eq. (42). This gives

$$\int_C \rho^2 q_\sigma \frac{\partial}{\partial s}\left(\frac{\omega^*}{\rho}\right)ds = R_s \int_C q_\omega r_t{}^* ds + R_s \int_C q_b r_t{}^* ds - R_s \int_C \tilde{q}_s q_\sigma \frac{1}{\rho}\frac{n_g}{t}ds \quad \cdots\cdots(45)$$

According to Eq. (31), the first integral on the right side of Eq. (45) is nothing else but warping torque, and the last integral vanishes on account of Eq. (34). Substituting now the expressions for $r_t{}^*$ obtained from Eq. (44) and for q_b from the first in Eqs. (21) into the second integral on the right side of Eq. (45) and performing an integration by parts on the integral with the first three in Eqs. (23) in mind, we have

$$\int_C q_b r_t^* ds = \frac{1}{R_s}\sum_j{}'\left\{\frac{\omega^*}{\rho}\Delta_j(\rho^2 q_b)\right\} + R_s\int_C \tilde{q}_s q_b \frac{1}{\rho}\frac{n_g}{t}ds + \frac{R_o^2}{R_s}\left[K_n\int_C \frac{1}{\rho}\omega^*\frac{t}{n_e}ds + K_y\int_C \frac{1}{\rho}\omega^* y\frac{t}{n_e}ds + K_z\int_C \frac{1}{\rho}\omega^* z\frac{t}{n_e}ds\right] \quad \cdots\cdots(46)$$

It is evident from the afore-mentioned argument and the first three in Eqs. (33) and the last three in Eqs. (10) that the right side of Eq. (46) vanishes. In view of Eqs. (42), (43), (45), and (46) and from the argument mentioned above, we finally arrive at the result

$$\int_C \frac{1}{\rho}\frac{\partial}{\partial s}(\rho^2 q_o)\omega^* ds = -R_s T_\omega^* \quad \cdots\cdots(47)$$

Then, introducing Eqs. (40), (41), and (47) into Eq. (39), we have

$$\int_0^\Theta \left[\frac{dM_\omega^*}{d\theta} - R_s T_\omega^* + R_s m_\omega^*\right]\delta\chi^* d\theta = 0 \quad \cdots\cdots(48)$$

In order that Eq. (48) holds for arbitrary variation $\delta\chi^*$, the equation within the parentheses must vanish. Therefore,

$$\frac{dM_\omega^*}{d\theta} - R_s T_\omega^* + R_s m_\omega^* = 0 \quad \cdots\cdots(49)$$

(8) Constitutive Equation for Warping Torque

To derive the required constitutive equation, we shall employ the principle of the complementary virtual work. The structural analysis with the aid of this principle introduces, in general, the stress resultants and stress couples as the argument functions. Not all of them, however, are independent because they are related with one another through the conditions of equilibrium. As is well known, the Lagrangian multiplier method makes it possible to treat the statical quantities as the independent argument functions if the conditions of equilibrium are taken as the auxiliary conditions.

Then, the mathematical statement of the principle is written in the form:

$$0 = \delta J^* \equiv \delta U_i^* + \delta W_e^* + \delta\int_0^\Theta \sum_j \lambda_j \Gamma_j d\theta \quad \cdots\cdots(50)$$

[18]

where

δU_i^* = the total internal complementary virtual work,

δW_e^* = the total external complementary virtual work,

$\lambda_j's$ = Lagrange multipliers,

$\Gamma_j's$ = conditions of equilibrium.

a) Total Internal Complementary Virtual Work

Provided that the effect of the shear flow q_b on the result can be disregarded, the total internal complementary virtual work δU_i^* is given by the equation

$$\delta U_i^* = \int_0^\Theta \int_C \left[\frac{n_e}{E_s}t\sigma_\theta\delta\sigma_\theta + \frac{n_g}{G_s}\frac{1}{t}q_t\delta q_t\right]\rho\, ds\, d\theta \quad \cdots\cdots(51)$$

We then substitute the values of σ_θ from Eq. (14) and q_t from Eq. (32) and their variations into the expression within parentheses in Eq. (51) and integrate the result over the entire contour C of the cross section by noting Eqs. (10), (13), (29), and the last in Eqs. (33). This gives

$$\delta U_i^* = R_o\int_0^\Theta \frac{N_x}{E_s F_o}\delta N_x d\theta + R_o\int_0^\Theta \frac{J_z M_y + J_{yz} M_z}{E_s(J_y J_z - J_{yz}^2)}\delta M_y d\theta + R_o\int_0^\Theta \frac{J_{yz} M_y + J_y M_z}{E_s(J_y J_z - J_{yz}^2)}\delta M_z d\theta + R_s\int_0^\Theta \frac{M_\omega^*}{E_s C_\omega^*}\delta M_\omega d\theta + R_s\int_0^\Theta \frac{T_s^*}{G_s J_T^*}\delta T_s^* d\theta + R_s\int_0^\Theta \frac{1}{\nu^*}\frac{T_\omega^*}{G_s J_T^*}\delta T_\omega^* d\theta \quad \cdots\cdots(52)$$

in which the following notation is introduced:

$$\frac{1}{\nu^*} = R_s^3\frac{J_T^*}{C_\omega^{*2}}\int_C (\tilde{q}_\omega - S_\omega^*)^2\frac{1}{\rho^3}\frac{n_g}{t}ds \quad \cdots\cdots(53)$$

The quantity $1/\nu^*$ is a dimensionless cross-sectional property (hereafter referred as to the warping-shear correction factor).

b) Total External Complementary Virtual Work

Denoting by $\tau_{\theta\rho}$ and $\tau_{\theta\zeta}$ the shearing stress components acting on the cross-sectional plane and by $\bar{p}_y$ and $\bar{p}_z$, respectively, the projections of the external surface load on the y and z axes, we find the formula for δW_e^*, the total external complementary virtual work, to be written as

$$\delta W_e^* = -\int_S (\bar{u}\delta\bar{p}_x + \bar{v}\delta\bar{p}_y + \bar{w}\delta\bar{p}_z)dS - n_x\int_{F_e}(\bar{u}\delta\sigma_\theta + \bar{v}\delta\tau_{\theta\rho} + \bar{w}\delta\tau_{\theta\zeta})dF \quad \cdots\cdots(54)$$

where the subscript F_e on the integral indicates that the integration is to be carried out over the entire cross-sectional area at both the end cross sections and

$$n_x=\begin{cases} -1 & \text{on the end cross section located at } \theta=0\,, \\ 1 & \text{on the end cross section located at } \theta=\Theta\,. \end{cases} \qquad \cdots(55)$$

Here, we shall introduce the following symbols for the stress resultants and stress couple produced by $\tau_{\theta\rho}$ and $\tau_{\theta\zeta}$ and for the external resultant forces and couples produced by $\bar{p}_x$, $\bar{p}_y$, and $\bar{p}_z$:

$$\left.\begin{aligned} &Q_y{}^*=\int_F \tau_{\theta\rho}dF\,, \qquad Q_z{}^*=\int_F \tau_{\theta\zeta}dF\,, \\ &T_x{}^*=\int_F (\tau_{\theta\zeta}\bar{y}-\tau_{\theta\rho}\bar{z})\,dF \end{aligned}\right\} \qquad \cdots(56)$$

and

$$\left.\begin{aligned} &p_x{}^*=\int_C \frac{\rho}{R_s}\bar{p}_x ds\,, \qquad p_y{}^*=\int_C \frac{\rho}{R_s}\bar{p}_y ds\,, \\ &p_z{}^*=\int_C \frac{\rho}{R_s}\bar{p}_z ds \end{aligned}\right\} \qquad \cdots(57)$$

also,

$$\left.\begin{aligned} &m_x{}^*=\int_C \frac{\rho}{R_s}(\bar{p}_z\bar{y}-\bar{p}_y\bar{z})\,ds\,, \\ &m_y{}^*=\int_C \frac{\rho}{R_s}\bar{p}_x\bar{z}\,ds\,, \qquad m_z{}^*=-\int_C \frac{\rho}{R_s}\bar{p}_x\bar{y}\,ds \end{aligned}\right\} \qquad \cdots(58)$$

Then, substituting Eqs. (2) into Eq. (54) and collecting the resulting terms by the use of the symbols defined in Eqs. (11), (56) to (58), together with Eq. (41), yields

$$\begin{aligned} \delta W_e{}^* =& [u^*\delta N_x{}^*]_0^\Theta+[v^*\delta Q_y{}^*]_0^\Theta+[w^*\delta Q_z{}^*]_0^\Theta \\ &+[\phi_x{}^*\delta T_x{}^*]_0^\Theta+[\phi_y{}^*\delta M_y{}^*]_0^\Theta \\ &+[\phi_z{}^*\delta M_z{}^*]_0^\Theta-[\chi^*\delta M_\omega{}^*]_0^\Theta \\ &-\int_0^\Theta (u^*\delta p_x{}^*+v^*\delta p_y{}^*+w^*\delta p_z{}^*)R_s d\theta \\ &-\int_0^\Theta (\phi_x{}^*\delta m_x{}^*+\phi_y{}^*\delta m_y{}^*+\phi_z{}^*\delta m_z{}^* \\ &-\chi^*\delta m_\omega{}^*)R_s d\theta \end{aligned} \qquad \cdots(59)$$

c) Auxiliary Conditions

We find from simple statics that the equilibrium equations, taken as the auxiliary conditions, can be expressed in terms of the stress resultants and stress couples as follows:

i) From the requirement of equilibrium for forces in the $\bar{x}$, $\bar{y}$, and $\bar{z}$ directions, it follows that

$$\left.\begin{aligned} &\Gamma_1\equiv\frac{dN_x{}^*}{d\theta}+Q_y{}^*+R_s p_x{}^*=0 \\ &\Gamma_2\equiv\frac{dQ_y{}^*}{d\theta}-N_x{}^*+R_s p_y{}^*=0 \\ &\Gamma_3\equiv\frac{dQ_z{}^*}{d\theta}+R_s p_z{}^*=0 \end{aligned}\right\} \qquad \cdots(60)$$

ii) From the requirement of equilibrium for moments about the x, y, and z axes, it follows that

$$\left.\begin{aligned} &\Gamma_4\equiv\frac{dT_x{}^*}{d\theta}+M_y{}^*+R_s m_x{}^*=0 \\ &\Gamma_5\equiv\frac{dM_y{}^*}{d\theta}-T_x{}^*-R_sQ_z{}^*+R_s m_y{}^*=0 \\ &\Gamma_6\equiv\frac{dM_z{}^*}{d\theta}+R_sQ_y{}^*+R_s m_z{}^*=0 \end{aligned}\right\} \qquad \cdots(61)$$

iii) One more auxiliary condition is given by Eq. (49), that is,

$$\Gamma_7\equiv\frac{dM_\omega{}^*}{d\theta}-R_sT_\omega{}^*+R_s m_\omega{}^*=0 \qquad \cdots(62)$$

Then, using Eqs. (60) to (62) and making use of the integration by parts we find from the calculus of variation that the last variation in Eq. (50) can be written in the form:

$$\begin{aligned} \delta\int_0^\Theta \sum_{j=1}^{7}\lambda_j\Gamma_j d\theta =& [\lambda_1\delta N_x{}^*]_0^\Theta+\int_0^\Theta\Big[-\frac{d\lambda_1}{d\theta}\delta N_x{}^* \\ &+\lambda_1\delta Q_y{}^*+\lambda_1R_s\delta p_x{}^*\Big]d\theta+[\lambda_2\delta Q_y{}^*]_0^\Theta \\ &+\int_0^\Theta\Big[-\frac{d\lambda_2}{d\theta}\delta Q_y{}^*-\lambda_2\delta N_x{}^*+\lambda_2R_s\delta p_y{}^*\Big]d\theta \\ &+[\lambda_3\delta Q_z{}^*]_0^\Theta+\int_0^\Theta\Big[-\frac{d\lambda_3}{d\theta}\delta Q_z{}^*+\lambda_3R_s\delta p_z{}^*\Big]d\theta \\ &+[\lambda_4\delta T_x{}^*]_0^\Theta+\int_0^\Theta\Big[-\frac{d\lambda_4}{d\theta}\delta T_x{}^*+\lambda_4\delta M_y{}^* \\ &+\lambda_4R_s\delta m_x{}^*\Big]d\theta+[\lambda_5\delta M_y{}^*]_0^\Theta \\ &+\int_0^\Theta\Big[-\frac{d\lambda_5}{d\theta}\delta M_y{}^*-\lambda_5\delta T_x{}^*-\lambda_5R_sQ_z{}^* \\ &+\lambda_5R_s\delta m_y{}^*\Big]d\theta+[\lambda_6\delta M_z{}^*]_0^\Theta \\ &+\int_0^\Theta\Big[-\frac{d\lambda_6}{d\theta}\delta M_z{}^*+\lambda_6R_s\delta Q_y{}^* \\ &+\lambda_6R_s\delta m_z{}^*\Big]d\theta+[\lambda_7\delta M_\omega{}^*]_0^\Theta \\ &+\int_0^\Theta\Big[-\frac{d\lambda_7}{d\theta}\delta M_\omega{}^*-\lambda_7R_s\delta T_\omega{}^* \\ &+\lambda_7R_s\delta m_\omega{}^*\Big]d\theta \end{aligned} \qquad \cdots(63)$$

d) Final Result

Referring now to **Fig. 2**, we see that the following relationships hold:

$$\left.\begin{array}{l} N_x^*=N_x\,,\qquad Q_y^*=Q_y\,,\qquad Q_z^*=Q_z\,,\\ T_x^*=T_s^*+T_\omega^*,\qquad M_y^*=M_y-z_sN_x\,,\\ M_z^*=M_z+y_sN_x \end{array}\right\} \quad\cdots\cdots(64)$$

Finally, substituting Eqs. (52), (59), and (63) into Eq. (50) and collecting the resulting terms by the use of Eqs. (64), we arrive at the following equation:

$$\begin{aligned} 0=\delta J^*&=[(\lambda_1-z_s\lambda_5+y_s\lambda_6-u^*+z_s\phi_y^*\\ &-y_s\phi_z^*)\delta N_x]_0^\theta+[(\lambda_2-v^*)\delta Q_y^*]_0^\theta\\ &+[(\lambda_3-w^*)\delta Q_z^*]_0^\theta+[(\lambda_4-\phi_x^*)\delta T_s^*]_0^\theta\\ &+[(\lambda_4-\phi_x^*)\delta T_\omega^*]_0^\theta+[(\lambda_5-\phi_y^*)\delta M_y]_0^\theta\\ &+[(\lambda_6-\phi_z^*)\delta M_z]_0^\theta+[(\lambda_7+\chi^*)\delta M_\omega^*]_0^\theta\\ &+\int_0^\theta\left\{\left[\frac{N_x}{E_sF_o}-\frac{1}{R_s}\frac{d\lambda_1}{d\theta}+\frac{z_s}{R_s}\frac{d\lambda_5}{d\theta}-\frac{y_s}{R_s}\frac{d\lambda_6}{d\theta}\right.\right.\\ &\left.-\frac{1}{R_s}\lambda_2-\frac{z_s}{R_s}\lambda_4\right]\delta N_x+\left[-\frac{1}{R_s}\frac{d\lambda_2}{d\theta}\right.\\ &\left.+\frac{1}{R_s}\lambda_1+\lambda_6\right]\delta Q_y^*+\left[-\frac{1}{R_s}\frac{d\lambda_3}{d\theta}-\lambda_5\right]\delta Q_z^*\\ &+\left[\frac{T_s^*}{G_sJ_T^*}-\frac{1}{R_s}\frac{d\lambda_4}{d\theta}-\frac{1}{R_s}\lambda_5\right]\delta T_s^*\\ &+\left[\frac{1}{\nu^*}\frac{T_\omega^*}{G_sJ_T^*}-\frac{1}{R_s}\frac{d\lambda_4}{d\theta}-\frac{1}{R_s}\lambda_5-\lambda_7\right]\delta T_\omega^*\\ &+\left[\frac{R_o}{R_s}\frac{J_zM_y+J_{yz}M_z}{E_s(J_yJ_z-J_{yz}^2)}-\frac{1}{R_s}\frac{d\lambda_5}{d\theta}+\lambda_4\right]\delta M_y\\ &+\left[\frac{R_o}{R_s}\frac{J_{yz}M_y+J_yM_z}{E_s(J_yJ_z-J_{yz}^2)}-\frac{1}{R_s}\frac{d\lambda_6}{d\theta}\right]\delta M_z\\ &\left.+\left[\frac{M_\omega^*}{E_sC_\omega^*}-\frac{1}{R_s}\frac{d\lambda_7}{d\theta}\right]\delta M_\omega^*\right\}R_sd\theta\\ &+\int_0^\theta\left\{(\lambda_1-u^*)\delta p_x^*+(\lambda_2-v^*)\delta p_y^*\right.\\ &+(\lambda_3-w^*)\delta p_z^*+(\lambda_4-\phi_x^*)\delta m_x^*\\ &+(\lambda_5-\phi_y^*)\delta m_y^*+(\lambda_6-\phi_z^*)\delta m_z^*\\ &\left.+(\lambda_7+\chi^*)\delta m_\omega^*\right\}R_sd\theta \quad\cdots\cdots(65) \end{aligned}$$

Since δJ^* vanishes for arbitrary variations δN_x, δQ_y^*, …, δM_z, δM_ω^*, δp_x^*, δp_y^*, …, δm_z^*, δm_ω^*, each term inside the parentheses in Eq. (65) must vanish independently. Therefore, the following system of equations is obtained:

$$\left.\begin{array}{l} \dfrac{N_x}{E_sF_o}-\dfrac{1}{R_s}\dfrac{d\lambda_1}{d\theta}+\dfrac{z_s}{R_s}\dfrac{d\lambda_5}{d\theta}-\dfrac{y_s}{R_s}\dfrac{d\lambda_6}{d\theta}\\ \qquad-\dfrac{1}{R_s}\lambda_2-\dfrac{z_s}{R_s}\lambda_4=0\,,\\ -\dfrac{1}{R_s}\dfrac{d\lambda_2}{d\theta}+\dfrac{1}{R_s}\lambda_1+\lambda_6=0\,,\\ -\dfrac{1}{R_s}\dfrac{d\lambda_3}{d\theta}-\lambda_5=0\,,\\ \dfrac{T_s^*}{G_sJ_T^*}-\dfrac{1}{R_s}\dfrac{d\lambda_4}{d\theta}-\dfrac{1}{R_s}\lambda_5=0\,,\\ \dfrac{1}{\nu^*}\dfrac{T_\omega^*}{G_sJ_T^*}-\dfrac{1}{R_s}\dfrac{d\lambda_4}{d\theta}-\dfrac{1}{R_s}\lambda_5-\lambda_7=0\,,\\ \dfrac{R_o}{R_s}\dfrac{J_zM_y+J_{yz}M_z}{E_s(J_yJ_z-J_{yz}^2)}-\dfrac{1}{R_s}\dfrac{d\lambda_5}{d\theta}+\lambda_4=0\,,\\ \dfrac{R_o}{R_s}\dfrac{J_{yz}M_y+J_yM_z}{E_s(J_yJ_z-J_{yz}^2)}-\dfrac{1}{R_s}\dfrac{d\lambda_6}{d\theta}=0\,,\\ \dfrac{M_\omega^*}{E_sC_\omega^*}-\dfrac{1}{R_s}\dfrac{d\lambda_7}{d\theta}=0\,,\\ \lambda_1-u^*=0\,,\qquad \lambda_2-v^*=0\,,\\ \lambda_3-w^*=0\,,\qquad \lambda_4-\phi_x^*=0\,,\\ \lambda_5-\phi_y^*=0\,,\qquad \lambda_6-\phi_z^*=0\,,\\ \lambda_7+\chi^*=0 \end{array}\right\} \quad\cdots\cdots(66)$$

Although the bracketed terms in Eq. (65) yield the natural boundary conditions, they are not essential to the subject under consideration.

Finally, substituting into the fifth in Eqs. (66) the values of λ_4, λ_5, and λ_7, from Eqs. (66), along with the first two in Eqs. (3), we get the required constitutive equation for T_ω^* as follows:

$$\begin{aligned} T_\omega^*&=\nu^*G_sJ_T^*\left\{\frac{1}{R_s}\left(\frac{d\lambda_4}{d\theta}+\lambda_5\right)+\lambda_7\right\}\\ &=\nu^*G_sJ_T^*\left\{\frac{1}{R_s}\left(\frac{d\phi}{d\theta}-\frac{1}{R_s}\frac{dw^*}{d\theta}\right)-\chi^*\right\} \quad\cdots\cdots(67)_1 \end{aligned}$$

or, in view of Eq. (8),

$$T_\omega^*=\nu^*G_sJ_T^*(\psi_x^*-\chi^*) \quad\cdots\cdots(67)_2$$

Similarly, if the values of the Lagrange multipliers determined in Eqs. (66) are introduced into the remaining equations, we find, as a matter of course, that the second and third in Eqs. (66) are identically satisfied and the remaining ones are in agreement with the results obtained earlier.

(9) Method of Solution for Practical Case

We shall restrict our study to the analysis of a circularly curved box girder bridge subjected to the following boundary and loading conditions: The bridge is simply supported in a manner shown in **Fig. 3** and submitted to the external intermediate loads acting vertically downward and to the external end bending moments about a horizontal axis and end warping moments.

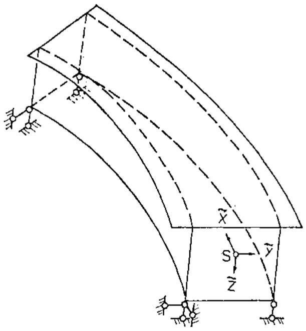

Fig. 3 Simply supported curved box girder bridge.

The analytical model considered herein may be practically important.

The external intermediate loads can be decomposed into the only two components, p_z^* and m_x^*. It is then apparent from the combination of the first three in Eqs. (60) and the last in Eqs. (61) that N_x^*, Q_y^*, and M_z^* vanish when the boundary and loading conditions are applied. Thus, eliminating Q_z^* and T_x^* from the last in Eqs. (60) and the first two in Eqs. (61) yields the governing differential equation for M_y^*:

$$\frac{d^2M_y^*}{d\theta^2}+M_y^*=-R_s(R_sp_z^*+m_x^*) \quad \cdots\cdots\cdots(68)$$

We combine Eq. (28), the fourth in Eqs. (64), and Eq. $(67)_2$ to get

$$\psi_x^*=\kappa^{*2}\chi^*+(1-\kappa^{*2})\frac{T_x^*}{G_sJ_T^*} \quad \cdots\cdots\cdots\cdots\cdots(69)$$

where κ^* is called the "warping shear parameter" and is defined by the equation

$$\kappa^{*2}=\frac{\nu^*}{1+\nu^*} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(70)$$

In a special case where the cross-sectional property ν^* is permitted to approach infinity, we find from Eqs. (69) and (70) that the intensity χ^* of warping coincides with the rate of twist ψ_x^*. Consequently, the conventional theory of torsion bending may be regarded as a special case of the modified theory.

Combining the last in Eqs. (12), the first in Eqs. (61), and Eqs. (62), $(67)_2$, and (69), we arrive at the governing differential equation for M_ω^*:

$$\frac{d^2M_\omega^*}{d\theta^2}-\kappa^{*2}\mu^{*2}M_\omega^*=-\kappa^{*2}R_s(M_y^*+R_sm_x^*) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(71)$$

where

$$\mu^{*2}=R_s^2\frac{G_sJ_T^*}{E_sC_\omega^*} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(72)$$

Eliminating w^* from the second in Eqs. (5) and Eq. (8) gives

$$\frac{d\psi_x}{d\theta}-\psi_y^*=\frac{1}{R_s}\left(\frac{d^2\phi}{d\theta}+\phi\right) \quad \cdots\cdots\cdots\cdots\cdots\cdots(73)$$

Recalling now that $N_x^*=M_z^*=0$, we find from the first and last two in Eqs. (64) that $M_y=M_y^*$ and $M_z=0$. With these in mind, we substitute the second in Eqs. (12) and the first derivative of Eq. (69) with respect to θ, along with the first in Eqs. (61), into Eq. (73) to get the governing differential equation for ϕ:

$$\frac{d^2\phi}{d\theta}+\phi=\frac{1}{G_sJ_T^*}\{(\kappa^{*2}-\beta-1)R_sM_y^* -\kappa^{*2}\mu^{*2}M_\omega^*+(\kappa^{*2}-1)R_s^2m_x^*\} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(74)$$

where

$$\beta=\frac{G_s}{E_s}\frac{R_0}{R_s}\frac{J_zJ_T^*}{J_yJ_z-J_{yz}^2} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(75)$$

Combining both the second equations in Eqs. (5) and (12) yields the differential equation for determining w^*:

$$\frac{d^2w^*}{d\theta^2}=-\frac{R_0R_sJ_z}{E_s(J_yJ_z-J_{yz}^2)}M_y^*-R_s\phi \quad \cdots\cdots(76)$$

Now, from an inspection of **Fig. 3** and in view of the condition of end loadings we find that the boundary conditions are given as follows:

$$\left.\begin{array}{l} M_y^*=M_{y1}^*, \quad M_\omega^*=M_{\omega1}^*, \quad \phi=w^*=0 \\ \qquad\qquad\qquad\qquad \text{at } \theta=0 \\ M_y^*=M_{y2}^*, \quad M_\omega^*=M_{\omega2}^*, \quad \phi=w^*=0 \\ \qquad\qquad\qquad\qquad \text{at } \theta=\Theta \end{array}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(77)$$

We can determine M_y^*, M_ω^*, ϕ, and w^* without difficulty by solving Eqs. (68), (71), (74),

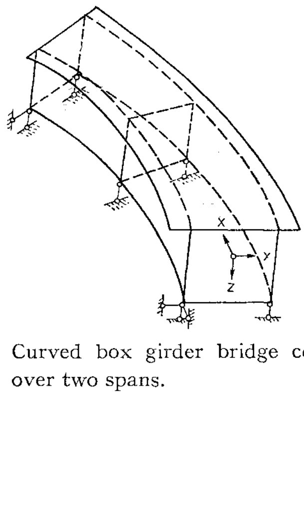

Fig. 4 Curved box girder bridge continuous over two spans.

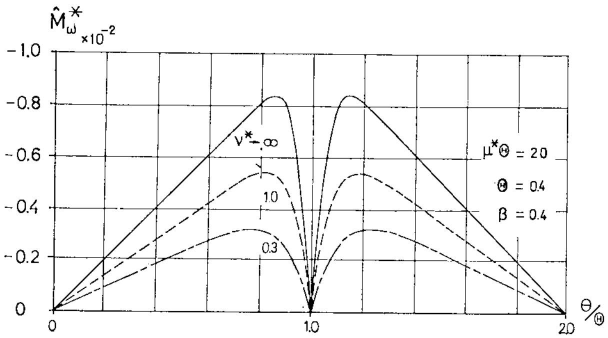

Fig. 5 Influence-lines for the warping moment $\hat{M}_\omega^*$ on the cross section located at the intermediate support.

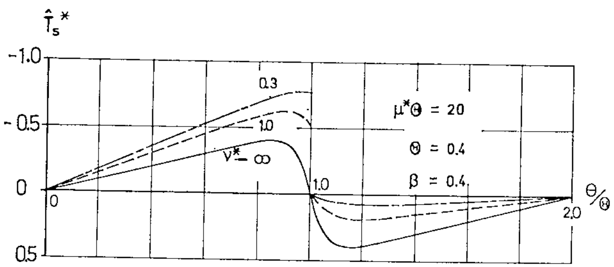

Fig. 6 Influence-lines for the primary torsional moment $\bar{T}_s^*$ on the cross section just to the right of the intermediate support.

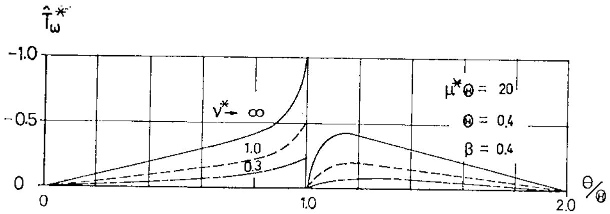

Fig. 7 Influence-lines for the secondary torsional moment $\bar{T}_\omega^*$ on the cross section just to the right of the intermediate support.

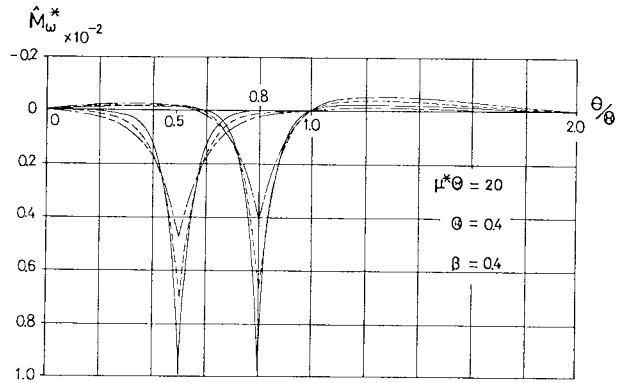

Fig. 8 Influence-lines for the warping moment $\hat{M}_\omega^*$ on the cross sections located at the middle and four-fifth points of the first span.

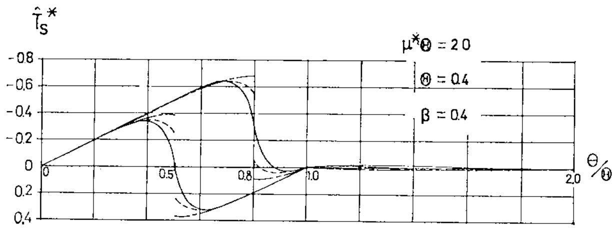

Fig. 9 Influence-lines for the primary torsional moment $\bar{T}_s^*$ on the cross sections located at the middle and four-fifth points of the first span.

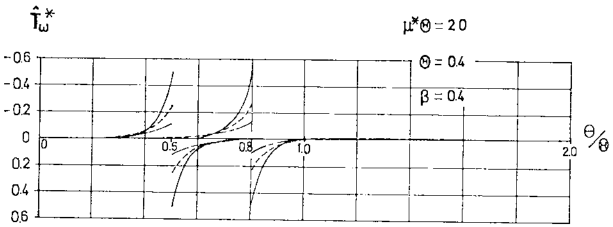

Fig. 10 Influence-lines for the secondary torsional moment $\bar{T}_\omega^*$ on the cross sections located at the middle and four-fifth points of the first span.

and (76) for these quantities, under the boundary conditions (77). This is because the right side of these differential equations become known quantities when the equations are solved in this sequence. Once the solutions for ϕ and w^* are obtained, ψ_x^* is determined from Eq. (8). Therefore, we can determine T_s^* using Eq. (28). Furthermore, T_ω^* can be determined from the introduction of M_ω^* into Eq. (49), and χ^* is then evaluated from Eq. $(67)_2$ by using ψ_x^* and T_ω^* thus obtained.

In **Appendix**, the solutions for these quantities are given for the case in which the concentrated torque $\bar{T}$ and force $\bar{P}$ applied at the section $\theta=\Phi$ are taken as the external intermediate loads. In the list of the solutions, we avoid the unnecessary complexity of obtaining an exact description of the solutions by introducing the assumption that $R_S=R_O\equiv R$. This assumption may be admissible in the case of the curved box girder bridges encountered in practical use.

3. NUMERICAL EXAMPLES

We will investigate to what extent the warping-shear correction factor $1/\nu^*$ affects the influence-line ordinates for the stress couples related to restrained torsion, M_ω^*, T_s^*, and T_ω^*. For this purpose, let us consider a curved box girder bridge continuous over two spans, supported in the manner shown in **Fig. 4**, as a simple example. The bridge has two equal developed span lengths and constant curvature throughout its axis. The structure is statically indeterminate to the second degree. For the analysis of this indeterminate system, it is convenient to treat M_y and M_ω^* developed on the cross section at the intermediate support as the redundants. Let us denote these stress couples by X_1 and X_2, respectively. Then, the equations of consistent deformations are written in the form:

$$\left.\begin{aligned}\delta_{11}X_1+\delta_{12}X_2+\delta_{10}=0\\ \delta_{21}X_1+\delta_{22}X_2+\delta_{20}=0\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots(78)$$

The first (second) equation indicates the compatibility condition physically stating that the deflection angle $(1/R)(dw^*/d\theta)$ (warping U—which amounts to the same as the intensity of warping χ^*) of the left side relative to the right side at the cross section on the intermediate support must be zero. The flexibility coefficients $\delta_{ij}(i,j=1,2)$ and $\delta_{i0}(i=1,2)$ can be obtained from the expressions for w^* and χ^* given in Appendix.

The influence-line ordinates for the stress couples related to restrained torsion at some cross sections were computed using Eqs. (78) only for the loading case of unit concentrated torque $\bar{T}$ moving across the structure. The results are shown in **Fig. 5** through **Fig. 10**. For the sake of generality, the influence-line diagrams are represented in a non-dimensional form by the use of the dimensionless quantities defined by $\hat{M}_\omega^*=M_\omega^*/G_SJ_T^*$, $\hat{T}_s^*=RT_s^*/G_SJ_T^*$, $\hat{T}_\omega^*=RT_\omega^*/G_SJ_T^*$, and $\hat{T}=R\bar{T}/G_SJ_T^*$. In the diagrams, the dimensionless stress couples versus θ/Θ are plotted against the three values of the parameter ν^*, namely, 0.3 (dot-dash line), 1.0 (broken line), and infinity (solid line), while $\mu^*\Theta$ is kept to 20, β to 0.4, and Θ to 0.4.

4. CONCLUDING REMARKS

The author has extended the modified theory of torsion bending based on the shearing stress field in equilibrium with warping normal stresses, proposed by R. Heilig, so as to cover a circularly curved box girder bridge. The modified theory developed herein coincides with the conventional one as the parameter ν^* approaches infinity in the limit.

The parameter ν^* serves as a measure for us to evaluate the influence of the secondary shear deformation due to restrained torsion on the magnitudes of the stress couples related to restrained torsion.

It can be seen from the numerical examples that the influence mentioned above appears appreciably at the section closest to the intermediate support of a continuous curved girder where torsion is highly restrained.

APPENDIX

Solutions for Statical and Kinematical Quantities

For the sake of brevity, we introduce the following notation:

$$\bar{\theta}=\Theta-\theta,\qquad \tilde{\Phi}=\Theta-\theta\qquad\cdots\cdots\cdots\cdots(A.1)$$

and

$$\lambda^*=\frac{\kappa^{*2}}{1+\kappa^{*2}\mu^{*2}},\qquad \alpha=\frac{1}{2}(1-\lambda^*+\beta)\cdots\cdots(A.2)$$

Then, the solutions are given as follows:

$$M_y^*(\theta_1)=M_{y1}^*\frac{\sin\tilde{\theta}}{\sin\Theta}+M_{y2}\frac{\sin\theta}{\sin\Theta}+\begin{cases}(\tilde{T}+R\tilde{P})\dfrac{\sin\tilde{\Phi}}{\sin\Theta}\sin\theta & \text{for } 0\leqq\theta\leqq\Phi\\ (\tilde{T}+R\tilde{P})\dfrac{\sin\tilde{\theta}}{\sin\Theta}\sin\Phi & \text{for } \Phi\leqq\theta\leqq\Theta\end{cases}\quad\cdots\cdots\text{(A.3)}$$

$$\begin{aligned}M_\omega^*(\theta)=&RM_{y1}^*\lambda^*\left(\frac{\sin\tilde{\theta}}{\sin\Theta}-\frac{\sinh\kappa^*\mu^*\tilde{\theta}}{\sinh\kappa^*\mu^*\Theta}\right)+RM_{y2}^*\lambda^*\left(\frac{\sin\theta}{\sin\Theta}-\frac{\sinh\kappa^*\mu^*\theta}{\sinh\kappa^*\mu^*\Theta}\right)\\&+M_{\omega1}^*\frac{\sinh\kappa^*\mu^*\tilde{\theta}}{\sinh\kappa^*\mu^*\Theta}+M_{\omega2}^*\frac{\sinh\kappa^*\mu^*\theta}{\sinh\kappa^*\mu^*\Theta}\\&\begin{cases}+\lambda^*\left\{(R\tilde{T}+R^2\tilde{P})\dfrac{\sin\tilde{\Phi}}{\sin\Theta}\sin\theta+\left(\kappa^*\mu^*R\tilde{T}-\dfrac{1}{\kappa^*\mu^*}R^2\tilde{P}\right)\dfrac{\sinh\kappa^*\mu^*\tilde{\Phi}}{\sinh\kappa^*\mu^*\Theta}\sinh\kappa^*\mu^*\theta\right\}\\ \qquad\text{for } 0\leqq\theta\leqq\Phi\\ +\lambda^*\left\{(R\tilde{T}+R^2\tilde{P})\dfrac{\sin\Phi}{\sin\Theta}\sin\tilde{\theta}+\left(\kappa^*\mu^*R\tilde{T}-\dfrac{1}{\kappa^*\mu^*}R^2\tilde{P}\right)\dfrac{\sinh\kappa^*\mu^*\Phi}{\sinh\kappa^*\mu^*\Theta}\sinh\kappa^*\mu^*\tilde{\theta}\right\}\\ \qquad\text{for } \Phi\leqq\theta\leqq\Theta\end{cases}\quad\cdots\cdots\text{(A.4)}\end{aligned}$$

$$\begin{aligned}T_s^*(\theta)=&-M_{y1}^*\left\{(1-\lambda^*)\frac{\cos\tilde{\theta}}{\sin\Theta}+\lambda^*\kappa^*\mu^*\frac{\cosh\kappa^*\mu^*\tilde{\theta}}{\sinh\kappa^*\mu^*\Theta}-\frac{1}{\Theta}\right\}\\&+M_{y2}^*\left\{(1-\lambda^*)\frac{\cos\theta}{\sin\Theta}+\lambda^*\kappa^*\mu^*\frac{\cosh\kappa^*\mu^*\theta}{\sinh\kappa^*\mu^*\Theta}-\frac{1}{\Theta}\right\}\\&+\frac{1}{R}M_{\omega1}^*\left(\kappa^*\mu^*\frac{\cosh\kappa^*\mu^*\tilde{\theta}}{\sinh\kappa^*\mu^*\Theta}-\frac{1}{\Theta}\right)-\frac{1}{R}M_{\omega2}^*\left(\kappa^*\mu^*\frac{\cosh\kappa^*\mu^*\theta}{\sinh\kappa^*\mu^*\Theta}-\frac{1}{\Theta}\right)\\&\begin{cases}+(\tilde{T}+R\tilde{P})(1-\lambda^*)\dfrac{\sin\tilde{\Phi}}{\sin\Theta}\cos\theta-(\tilde{T}-R\tilde{P})\lambda^*\kappa^{*2}\mu^{*2}\dfrac{\sinh\kappa^*\mu^*\tilde{\Phi}}{\sinh\kappa^*\mu^*\Theta}\cosh\kappa^*\mu^*\theta-R\tilde{P}\dfrac{\tilde{\Phi}}{\Theta}\\ \qquad\text{for } 0\leqq\theta\leqq\Phi\\ -(\tilde{T}+R\tilde{P})(1-\lambda^*)\dfrac{\sin\Phi}{\sin\Theta}\cos\tilde{\theta}+(\tilde{T}-R\tilde{P})\lambda^*\kappa^{*2}\mu^{*2}\dfrac{\sinh\kappa^*\mu^*\Phi}{\sinh\kappa^*\mu^*\Theta}\cosh\kappa^*\mu^*\tilde{\theta}+R\tilde{P}\dfrac{\Phi}{\Theta}\\ \qquad\text{for } \Phi\leqq\theta\leqq\Theta\end{cases}\quad\cdots\cdots\text{(A.5)}\end{aligned}$$

$$\begin{aligned}T_\omega^*(\theta)=&-M_{y1}^*\lambda^*\left(\frac{\cos\theta}{\sin\Theta}-\kappa^*\mu^*\frac{\cosh\kappa^*\mu^*\theta}{\cosh\kappa^*\mu^*\Theta}\right)+M_{y2}^*\lambda^*\left(\frac{\cos\theta}{\sin\Theta}-\kappa^*\mu^*\frac{\cosh\kappa^*\mu^*\theta}{\sinh\kappa^*\mu^*\Theta}\right)\\&-\frac{1}{R}M_{\omega1}^*\kappa^*\mu^*\frac{\cosh\kappa^*\mu^*\theta}{\sinh\kappa^*\mu^*\Theta}+\frac{1}{R}M_{\omega2}^*\kappa^*\mu^*\frac{\cosh\kappa^*\mu^*\theta}{\sinh\kappa^*\mu^*\Theta}\\&\begin{cases}+\lambda^*\left\{(\tilde{T}+R\tilde{P})\dfrac{\sin\tilde{\Phi}}{\sin\Theta}\cos\theta+(\kappa^{*2}\mu^{*2}\tilde{T}-R\tilde{P})\dfrac{\sinh\kappa^*\mu^*\tilde{\Phi}}{\sinh\kappa^*\mu^*\Theta}\cosh\kappa^*\mu^*\theta\right\}\\ \qquad\text{for } 0\leqq\theta\leqq\Theta\\ -\lambda^*\left\{(\tilde{T}+R\tilde{P})\dfrac{\sin\Phi}{\sin\Theta}\cos\tilde{\theta}+(\kappa^{*2}\mu^{*2}\tilde{T}-R\tilde{P})\dfrac{\sinh\kappa^*\mu^*\Phi}{\sinh\kappa^*\mu^*\Theta}\cosh\kappa^*\mu^*\tilde{\theta}\right\}\\ \qquad\text{for } \Phi\leqq\theta\leqq\Theta\end{cases}\quad\cdots\cdots\text{(A.5)}\end{aligned}$$

$$\begin{aligned}G_sJ_T^*\chi^*(\theta)=&-M_{y1}^*\left(\lambda^*\mu^{*2}\frac{\cos\tilde{\theta}}{\sin\Theta}+\lambda^*\frac{\mu^*}{\kappa^*}\frac{\cosh\kappa^*\mu^*\tilde{\theta}}{\sinh\kappa^*\mu^*\Theta}-\frac{1}{\Theta}\right)\\&+M_{y2}^*\left(\lambda^*\mu^{*2}\frac{\cos\theta}{\sin\Theta}+\lambda^*\frac{\mu^*}{\kappa^*}\frac{\cosh\kappa^*\mu^*\theta}{\sinh\kappa^*\mu^*\Theta}-\frac{1}{\Theta}\right)\\&+\frac{1}{R}M_{\omega1}^*\left(\frac{\mu^*}{\kappa^*}\frac{\cosh\kappa^*\mu^*\tilde{\theta}}{\sinh\kappa^*\mu^*\Theta}-\frac{1}{\Theta}\right)-\frac{1}{R}M_{\omega2}^*\left(\frac{\mu^*}{\kappa^*}\frac{\cosh\kappa^*\mu^*\theta}{\sinh\kappa^*\mu^*\Theta}-\frac{1}{\Theta}\right)\\&\begin{cases}+\lambda^*\mu^{*2}\left\{(\tilde{T}+R\tilde{P})\dfrac{\sin\tilde{\Phi}}{\sin\Theta}\cos\theta-\left(\tilde{T}-\dfrac{1}{\kappa^{*2}\mu^{*2}}R\tilde{P}\right)\dfrac{\sinh\kappa^*\mu^*\tilde{\Phi}}{\sinh\kappa^*\mu^*\Theta}\cosh\kappa^*\mu^*\theta\right\}-R\tilde{P}\dfrac{\tilde{\Phi}}{\Theta}\\ \qquad\text{for } 0\leqq\theta\leqq\Phi\\ -\lambda^*\mu^{*2}\left\{(\tilde{T}+R\tilde{P})\dfrac{\sin\Phi}{\sin\Theta}\cos\theta-\left(\tilde{T}-\dfrac{1}{\kappa^{*2}\mu^{*2}}R\tilde{P}\right)\dfrac{\sinh\kappa^*\mu^*\Phi}{\sinh\kappa^*\mu^*\Theta}\cosh\kappa^*\mu^*\theta\right\}+R\tilde{P}\dfrac{\Phi}{\Theta}\\ \qquad\text{for } \Phi\leqq\theta\leqq\Theta\end{cases}\end{aligned}$$

$$\cdots\cdots\text{(A.6)}$$

$$
\begin{aligned}
G_sJ_T^*\phi(\theta) = & -RM_{y1}^*\left\{\lambda^{*2}\mu^{*2}\left(\frac{\sin\tilde{\theta}}{\sin\Theta}-\frac{\sinh\kappa^*\mu^*\tilde{\theta}}{\sinh\kappa^*\mu^*\Theta}\right)-\alpha\left(\tilde{\theta}\,\frac{\cos\tilde{\theta}}{\sin\Theta}-\frac{\Theta\cos\Theta}{\sin^2\Theta}\sin\tilde{\theta}\right)\right\} \\
& -RM_{y2}^*\left\{\lambda^{*2}\mu^{*2}\left(\frac{\sin\theta}{\sin\Theta}-\frac{\sinh\kappa^*\mu^*\theta}{\sinh\kappa^*\mu^*\Theta}\right)-\alpha\left(\theta\,\frac{\cos\theta}{\sin\Theta}-\frac{\Theta\cos\Theta}{\sin^2\Theta}\sin\theta\right)\right\} \\
& +M_{\omega1}^*\lambda^*\mu^{*2}\left(\frac{\sin\tilde{\theta}}{\sin\Theta}-\frac{\sinh\kappa^*\mu^*\tilde{\theta}}{\sinh\kappa^*\mu^*\Theta}\right)+M_{\omega2}^*\lambda^*\mu^{*2}\left(\frac{\sin\theta}{\sin\Theta}-\frac{\sinh\kappa^*\mu^*\theta}{\sinh\kappa^*\mu^*\Theta}\right) \\
& \begin{cases}
+(\kappa^{*2}\mu^{*2}R\tilde{T}-R^2\tilde{P})\,\lambda^{*2}\mu^{*2}\left(\dfrac{\sin\tilde{\Phi}}{\sin\Theta}\sin\theta-\dfrac{1}{\kappa^*\mu^*}\dfrac{\sinh\kappa^*\mu^*\tilde{\Phi}}{\sinh\kappa^*\mu^*\Theta}\sinh\kappa^*\mu^*\theta\right) \\
-(R\tilde{T}+R^2\tilde{P})\alpha\left\{\dfrac{\sin\tilde{\Phi}}{\sin\Theta}\left(\Theta\cos\Theta\dfrac{\sin\theta}{\sin\Theta}-\theta\cos\theta\right)+(\sin\tilde{\Phi}-\tilde{\Phi}\cos\tilde{\Phi})\dfrac{\sin\theta}{\sin\Theta}\right\} \\
+R\tilde{T}(1-\kappa^{*2})\dfrac{\sin\tilde{\Phi}}{\sin\Theta}\sin\theta \qquad \text{for} \quad 0\leqq\theta\leqq\Phi \\
+(\kappa^{*2}\mu^{*2}R\tilde{T}-R^2\tilde{P})\,\lambda^{*2}\mu^{*2}\left(\dfrac{\sin\Phi}{\sin\Theta}\sin\tilde{\theta}-\dfrac{1}{\kappa^*\mu^*}\dfrac{\sinh\kappa^*\mu^*\Phi}{\sinh\kappa^*\mu^*\Theta}\sinh\kappa^*\mu^*\tilde{\theta}\right) \\
-(R\tilde{T}+R^2\tilde{P})\alpha\left\{\dfrac{\sin\Phi}{\sin\Theta}\left(\Theta\cos\Theta\dfrac{\sin\tilde{\theta}}{\sin\Theta}-\tilde{\theta}\cos\tilde{\theta}\right)+(\sin\Phi-\Phi\cos\Phi)\dfrac{\sin\tilde{\theta}}{\sin\Theta}\right\} \\
+R\tilde{T}(1-\kappa^{*2})\dfrac{\sin\Phi}{\sin\Theta}\sin\tilde{\theta} \qquad \text{for} \quad \Phi\leqq\theta\leqq\Theta
\end{cases}
\end{aligned}
\qquad\cdots\cdots\cdots\cdots(\text{A.7})
$$

$$
\begin{aligned}
G_sJ_T^*w^*(\theta) = & R^2M_{y1}^*\left\{\left(\frac{\lambda^*}{\kappa^*}\right)^2\left(\frac{\sin\tilde{\theta}}{\sin\Theta}-\frac{\sinh\kappa^*\mu^*\tilde{\theta}}{\sinh\kappa^*\mu^*\Theta}\right)+\alpha\left(\tilde{\theta}\,\frac{\cos\tilde{\theta}}{\sin\Theta}-\frac{\Theta\cos\Theta}{\sin^2\Theta}\sin\tilde{\theta}\right)-\frac{\sin\tilde{\theta}}{\sin\Theta}+\frac{\tilde{\theta}}{\Theta}\right\} \\
& +R^2M_{y2}^*\left\{\left(\frac{\lambda^*}{\kappa^*}\right)^2\left(\frac{\sin\theta}{\sin\Theta}-\frac{\sinh\kappa^*\mu^*\theta}{\sinh\kappa^*\mu^*\Theta}\right)+\alpha\left(\theta\,\frac{\cos\theta}{\sin\Theta}-\frac{\Theta\cos\Theta}{\sin^2\Theta}\sin\theta\right)-\frac{\sin\theta}{\sin\Theta}+\frac{\theta}{\Theta}\right\} \\
& +RM_{\omega1}^*\lambda^*\left(\mu^{*2}\frac{\sin\tilde{\theta}}{\sin\Theta}+\frac{1}{\kappa^{*2}}\frac{\sinh\kappa^*\mu^*\tilde{\theta}}{\sinh\kappa^*\mu^*\Theta}-\frac{\tilde{\theta}}{\Theta}\right) \\
& -RM_{\omega2}^*\lambda^*\left(\mu^{*2}\frac{\sin\theta}{\sin\Theta}+\frac{1}{\kappa^{*2}}\frac{\sinh\kappa^*\mu^*\theta}{\sinh\kappa^*\mu^*\Theta}-\frac{\theta}{\Theta}\right) \\
& \begin{cases}
-(\kappa^{*2}\mu^{*2}R^2\tilde{T}-R^3\tilde{P})\left(\dfrac{\lambda^*}{\kappa^*}\right)^2\left(\dfrac{\sin\tilde{\Phi}}{\sin\Theta}\sin\theta-\dfrac{1}{\kappa^*\mu^*}\dfrac{\sinh\kappa^*\mu^*\tilde{\Phi}}{\sinh\kappa^*\mu^*\Theta}\sinh\kappa^*\mu^*\theta\right) \\
-(R^2\tilde{T}-R^3\tilde{P})\alpha\left\{\dfrac{\sin\tilde{\Phi}}{\sin\Theta}\left(\Theta\cos\Theta\,\dfrac{\sin\theta}{\sin\Theta}-\theta\cos\theta\right)+(\sin\tilde{\Phi}-\tilde{\Phi}\cos\tilde{\Phi})\dfrac{\sin\theta}{\sin\Theta}\right\} \\
-R^3\tilde{P}\left(\dfrac{\sin\tilde{\Phi}}{\sin\Theta}\sin\theta-\dfrac{\tilde{\Phi}}{\Theta}\theta\right) \qquad \text{for} \quad 0\leqq\theta\leqq\Phi \\
-(\kappa^{*2}\mu^{*2}R^2\tilde{T}-R^3\tilde{P})\left(\dfrac{\lambda^*}{\kappa^*}\right)^2\left(\dfrac{\sin\Phi}{\sin\Theta}\sin\tilde{\theta}-\dfrac{1}{\kappa^*\mu^*}\dfrac{\sinh\kappa^*\mu^*\Phi}{\sinh\kappa^*\mu^*\Theta}\sinh\kappa^*\mu^*\tilde{\theta}\right) \\
-(R^2\tilde{T}-R^3\tilde{P})\alpha\left\{\dfrac{\sin\Phi}{\sin\Theta}\left(\Theta\cos\Theta\,\dfrac{\sin\tilde{\theta}}{\sin\Theta}-\tilde{\theta}\cos\tilde{\theta}\right)+(\sin\Phi-\Phi\cos\Phi)\dfrac{\sin\tilde{\theta}}{\sin\Theta}\right\} \\
-R^3\tilde{P}\left(\dfrac{\sin\Phi}{\sin\Theta}\sin\tilde{\theta}-\dfrac{\Phi}{\Theta}\tilde{\theta}\right) \qquad \text{for} \quad \Phi\leqq\theta\leqq\Theta
\end{cases}
\end{aligned}
\qquad\cdots\cdots(\text{A.8})
$$

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