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LATERAL (VERTICAL) VIBLATION OF A CONTINUOUS BEAM-BRIDGE WITH VARIABLE CROSS-SECTIONS

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ABSTRACT

Up today as far as the author knows, only a few paper has been written about the vibration of rod of variable cross-section or about the vibration of conical rod. However these papers are not those which are treating really the vibration of a continuous beam-bridge with variable, prismatical cross-sections. Moreover these papers are treating the motion of cross-sectional part of the beam as a motion of one block and thus they are theoretically regarding the beam only as a line like wire with no depth which, in spite of the matter, has yet finite second moment of cross-section. This is so even in the papers which are taking into consideration the effect of rotatory inertia and the effect of shearing force in the cross-section of a beam.

This paper presents a theoretical method of analysis of vibration of a continuous beam-bridge with variable cross-sections, basing on the acoustical theory about the phase velocity of progressing flexural wave.

1. INTRODUCTION

In the traditional theory of vibration of a beam, it is regarded as if a beam were a wire which has yet a finite second moment of cross-sections, and thus the motion of cross-sectional part of the beam is treated as a motion of one block even in the papers which are taking into consideration the effect of rotatory inertia of cross-sectional part and the effect of shearing force. In such a treatment of vibration of a beam with constant depth, the orthogonality of normal functions always holds true even in the case of continuous beam.

However the aspect of motion of the cross-sectional part of the beam is different from point to point between the vicinity of extreme fiber and that of neutral axis of the beam especially in rotational motion, and thus the orthogonality of normal function does not hold true in a beam with variable cross-sections and in an actual beam which has fairly large depth with variable cross-sections, coupling vibration inevitably take place. Then the traditional theory becomes to have fairly large error in higher frequency of vibration especially in an actual beam with variable cross-section. This is considered due to mainly the effect of coupling vibration of higher harmonics.

It is necessary by all means in an actual beam with variable cross-sections to consider its vibrational motion more precisely basing on the acoustical theory about the phase velocity of progressing flexural wave. This matter is especially necessary if we wish to investigate accurately the higher frequency of vibration.

In the case of a beam with variable cross-sections where the orthogonality of normal functions does not hold true and the coupling of vibration take place, the analysis of vibration must be executed dividing it into two parts. Namely, the one is the analysis of normal functions of the beam which comes out from the constituting standing waves without coupling due to reflection of progressing wave, and the other one is the analysis of coupling vibration between many normal modes of vibration. The above matter seems to have been confused hitherto in traditional theory of vibration of a beam, because there are two kinds of frequency equation in traditional theory, for example in p. 123 and p. 223 in the bibliography 11) in the end of this paper two kinds of frequency equation are introduced, but yet there is no explanation about the relation between them. The author has put clear distinction between the two kinds of frequency equation, i.e. the one as "original (natural) frequency equation" and the another one as

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"coupled frequency equation". The former one is derived from orthogonal normal functions basing on acoustical theory about the phase velocity of progressing wave, and the latter one is derived from Lagrange's equation of coupled vibration.

2. THE ORIGINAL (NATURAL) ANGULAR-FREQUENCY p_r AND THE PROPAGATION CONSTANT γ_r OF PROGRESSING WAVE

If we denote the number of principal modes of vibration as r, the original (natural) angular-frequency as p_r, the propagation constant of progressing wave as γ_r, and the phase velocity of a progressing wave as c, then there is generally a relation as follows

$$c = p_r/\gamma_r \quad \cdots\cdots(1)$$

($r = 1, 2, \cdots\cdots, \infty$: number of normal mode)

where,

$\gamma_r = 2\pi/\lambda_r$, λ_r: the wave-length.

Further, between the phase velocity of various families of elastic waves propagated in an elastic bar and their propagation constant γ, it is already known from the acoustical theory* of guided elastic-waves which was introduced from Navier's equation of motion, that there are relations as expressed as various curves shown in **Fig. 1****. In this figure only the curve of phase velocity of flexural elastic wave in a beam has direct relation for our present problem, and so the author will explain somewhat precisely about the phase velocity of flexural elastic wave in the next article in this paper, but the explanation about the phase velocities of other elastic waves will be omitted here. Anyhow the reader will be able to understand easily that the original (natural) angular frequency p_r is generally expressed by a propagation constant γ_r, only if we are knowing the guided phase velocity c.

In general, the displacement u_i in the vibration of a linear elastic body can be expressed by generalized coordinate displacement component q_r as a linear equation as follows

$$u_i = \alpha_{ir} q_r \quad \text{(Summation convention for } r = 1, 2, \cdots\cdots, \infty) \quad \cdots\cdots(2)$$

where

$\alpha_{ir} = \alpha_{ir}(\gamma_r x_i)$: normal function.

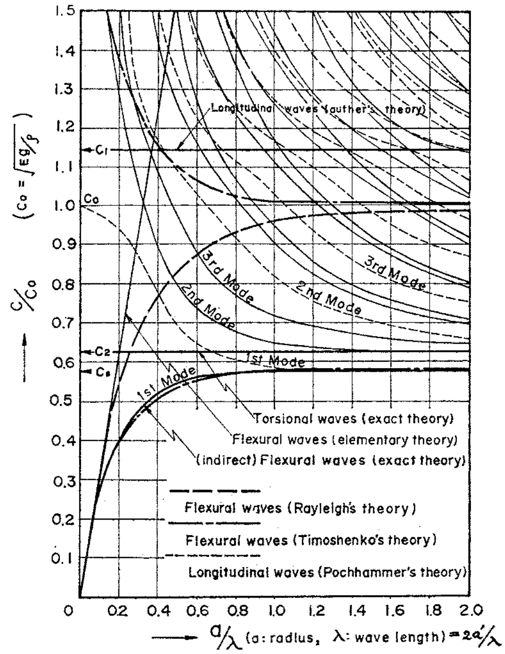

Fig. 1 The relation between the phase velocity c in an elastic bar (Poisson's ratio, $\nu = 0.29$) and the propagation constant $\gamma = 2\pi/\lambda$.
($c_0 = \sqrt{Eg/\rho}$, a': radius of gyration of a cross-section of the bar).

$x_i = i$-component of material-coordinates of a point.
$i = 1, 2, 3$.
$\gamma_r = 2\pi/\lambda_r$, ($\lambda_r$ = wave length): propagation constant
$q_r = q_r(t)$: generalized coordinate displacement-components.
r: number of normal modes

3. THE PROPAGATION CONSTANT γ_r IN THE LATERAL (VERTICAL) VIBRATION OF A CONTINUOUS BEAM-BRIDGE WITH VARIABLE CROSS-SECTION

Let us now consider about the propagation constant γ_r, or the original natural frequency p_r, in the lateral (vertical) vibration of a continuous beam-bridge with variable cross-sections, which is constituted by m-spans as shown in **Fig. 2**.

If we take the origin of coordinate x at the left-end of each span of the continuous beam-bridge, and denote the radius of gyration of a cross-section of each span as $a'_1, a_2', \ldots, a_j', \ldots, a_n'$,

* Ref. Bibliography 12) in the end of this paper.
** Ref. Bibliography 8), (p. 412) in the end of this paper.

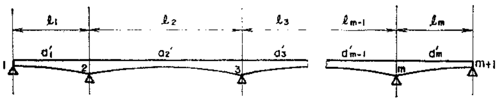

Fig. 2 Continuous beam-bridge with variable cross-section, which is constituted by m–spans.

then by using Hamilton's principle we are able to express the vertical deflection (normal function) of j-th span in the lateral vibration of the continuous beam-bridge generally as follows

$$Y_j(\gamma_r x)\cdot q_r=\{C_{jr}(\cos\gamma_r x+\cosh\gamma_r x)+D_{jr}(\cos\gamma_r x-\cosh\gamma_r x)+E_{jr}\sin(\gamma_r x)+F_{jr}\sinh(\gamma_r x)\}q_r \quad\cdots\cdots(3)$$

where,

- γ_r: propagation constant which is a function of radius of gyration $a_j'(x)$ and of original natural (angular) frequency p_r, common to all spans.
- r: number of normal modes of vibration.
- q_r: original generalized-coordinate corresponding to r-th normal mode of vibration without coupling.

At the intermediate supporting-point j, the following four equations of end-boundary conditions hold true.

$$(Y_{j-1})_{x=l_{j-1}}=(Y_j)_{x=0}=0 \quad\cdots\cdots(4)$$

$$\left\{\frac{\partial Y_{j-1}}{\partial x}\right\}_{x=l_{j-1}}=\left\{\frac{\partial Y_j}{\partial x}\right\}_{x=0} \quad\cdots\cdots(5)$$

$$\left\{\frac{\partial^2 Y_{j-1}}{\partial x^2}\right\}_{x=l_{j-1}}=\left\{\frac{\partial^2 Y_j}{\partial x^2}\right\}_{x=0} \quad\cdots\cdots(6)$$

Further, at the two end-supports of the continuous beam-bridge, the following four equations of end-boundary conditions hold true.

$$(Y_1)_{x=0}=(Y_m)_{x=l_m}=0 \quad\cdots\cdots(7)$$

$$\left\{\frac{\partial^2 Y_1}{\partial x^2}\right\}_{x=0}=\left\{\frac{\partial^2 Y_m}{\partial x^2}\right\}_{x=l_m} \quad\cdots\cdots(8)$$

Hence, the number of end-boundary conditions of all supporting-points in the continuous beam-bridge becomes

$$4(m-1)+4=4m.$$

On the other hand, the number of unknown constants, C_{jr}, D_{jr}, E_{jr}, and F_{jr}, of all spans of the continuous beam-bridge, is equal to 4 m.

Therefore, if we eliminate these unknown constants from the end-boundary conditions as a simultaneous equation, then we are able to obtain a determinant equation concerning to the propagation constant γ_r. This determinant equation is called as original frequency equation or normal frequency equation.

However, the propagation constant γ_r in the original frequency equation is a function of original (natural) angular frequency p_r, and of radius of gyration $a_j'(x)$ of a cross-section at a point x in an optional span j; and it is already known from the acoustical theory of progressing elastic waves that above functional relation between the propagation constant γ_r and the original (natural) angular frequency p_r in the flexural wave in a beam is

$$\begin{vmatrix} \left\{J_1''(ha)+\left(\gamma_1^2-\frac{\gamma_2^2}{2}\right)J_1(ha)\right\} & \left\{\frac{1}{a}J_1'(ka)-\frac{1}{a^2}J_1(ka)+\frac{\gamma^2}{k}J_0'(ka)\right\} & -i\gamma\{J_0'(ka)-J_2'(ka)\} \\ -\frac{2}{a}\left\{J_1'(ha)-\frac{1}{a}J_1(ha)\right\} & -\{2J_1''(ka)+k^2J_1(ka)+\gamma^2J_{-1}(ka)\} & i\gamma k\{J_{-1}(ka)-J_3(ka)\} \\ 2i\gamma J_1'(ha) & \left\{2\frac{i\gamma}{a}J_1(ka)-i\gamma k\left(1-\frac{\gamma^2}{k^2}\right)J_0(ka)\right\} & -(k^2-\gamma^2)\{J_0(ka)-J_2(ka)\} \end{vmatrix}=0 \quad\cdots\cdots(9)^*$$

where in the above equation,

- J_n: Bessel-function of n-th order.
- $J''=\partial^2 J/\partial r^2,\ J'=\partial J/\partial r.$
- $\gamma_1=2\pi/\lambda_1=p/c_1,\ \gamma_2=2\pi/\lambda_2=p/c_2,\ \gamma=p/c.$
- $c_1=\sqrt{2G/\rho(1-\nu/1-2\nu)}$: phase velocity of dilatation-wave in an infinite medium.
- $c_2=\sqrt{G/\rho}$: phase velocity of rotationwave in an infinite medium.
- c: phase velocity of guided flexural wave
- λ_1=wave length of dilatation-wave.
- λ_2: wave length of rotation-wave
- ρ: specific gravity of the maternal
- ν: Poisson's ratio
- $G=E/2(1+\nu)$: modulus of rigidity
- E: Young's modulus
- $h=\gamma_1 l_1,\ k=\gamma_2 l_2,\ l_1=\cos\alpha,\ l_2=\cos\beta.$
- α: incident angle of dilatation wave to free surface boundary.

* Ref. Bibliography 8), (p. 405) in the end of this paper.

β: reflecting angle of rotation wave produced by incident dilatation wave.
a: radius of cylindrical bar.

The above equation (9) is the one which has been derived theoretically in acoustics for a cylindrical bar with various radii of cross-sections. If we show the above relation expressed by the equation (9) it becomes as a curve which is shown in **Fig. 1** with an entry of "(indirect) Flexural waves (exact theory)". It has been examined in the bibliography 1) in the end of this paper that the phase velocity of flexural wave in a beam of prismatical section also coincides with this curve shown in **Fig. 1** with enough accuracy, while the phase velocity of flexural waves in elementary theory has very large error in a case when a/λ (a: radius of the bar, λ: wave length) is larger than 0.1, and even the phase velocity of flexural wave due to Rayleigh's theory has large error in a case when a/λ is larger than 0.18, as shown in **Fig. 1**. In the theory excepting that of acoustics, only the phase velocity of flexural waves due to Timoshenko's theory which has taken into consideration the rotatory inertia effect and shearing force effect, coincides with the actual result of original frequency with fairly good accuracy, but by such a theory as this we cannot make clear the coupled frequency of vibration.

If we approximate the curve of phase velocity of flexural wave as shown in **Fig. 1** by a curve of algebraic function, we obtain the curve as shown in **Fig. 3** The actual beam of length about 30 m or so has generally the depth of 1 m~2 m and the width about 0.5 m or so, and the radius of gyration of cross-section in such a beam is in almost cases about 0.3 m~0.6 m. In the case when we consider the wave-length of flexural wave about 0.6 m~1.2 m which corresponds to the wave-length of higher modes of 25-th~50-th order, the value of $2a'/\lambda$ (a': radius of gyration of a cross-section of the bar, λ: flexural wave length) is in almost cases equal to approximately 0.5. In the general analysis of coupled frequency it is almost enough to take into consideration the higher normal mode below the 25-th order.

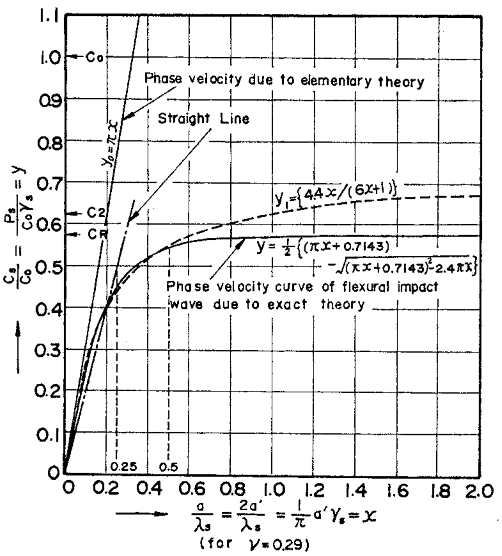

Fig. 3 The functional relation between the propagation constant γ_r and the original natural (angular) frequency p_r. ($\nu=0.29$).

In **Fig. 1** the author has shown many phase velocities of various families of elastic waves, but for the present problem in question in this paper, the curves of phase velocities except the one for the flexural wave have no direct relation, and the explanation of them will be omitted here as mentioned in the preceding article.

As mentioned above there is a functional relation (9) between the propagation constant γ_r and the original (natural) angular frequency p_r in the flexural wave, and it can be approximated by the curve as shown in **Fig. 3**, i.e.

$$2\frac{p_r}{c_0\gamma_r}=(a_j'\gamma_r+0.7143)-\sqrt{(a_j'\gamma_r+0.7143)^2-2.4a_j'\gamma_r} \quad \cdots\cdots(10)^*$$

where

$c_0=\sqrt{Eg/\rho}$ ρ: unit mass of the material,
g: acceleration of gravity
a_j': radius of gyration of a cross-section in j-th span.

The radius of gyration of a cross-section a_j' in the above eq. (10) varies with each span and also with the coordinate x in each span, but the original (natural) angular frequency p_r has a same value throughout all spans of the continuous beam bridge. We are able to express γ_r, which has different value from point to point by only one value, p_r, which is common throughout all spans, from the eq. (10). Namely we obtain from the eq. (10).

$$0.6c_0^2a_j'\gamma_r^3-p_rc_0a_j'\gamma_r^2-0.7143p_rc_0\gamma_r+p_r^2=0$$

* Ref. Bibliography 7), (p. 111) in the end of this paper.

$$\therefore\quad X^3-\left(\frac{p_r}{1.08c_0}+\frac{1.19}{a_j'}\right)\frac{p_r}{c_0}X$$
$$-\left(\frac{p_r}{2.916c_0}-\frac{1}{a_j'}\right)\frac{p_r^2}{c_0^2}=0 \quad\cdots\cdots\cdots(11)$$

where,

$$X=\gamma_r-\frac{p_r}{1.8c_0}\quad\left(\text{or,}\quad \gamma_r=X+\frac{p_r}{1.8c_0}\right)$$

If we solve the equation (11) with respect to X, then we shall obtain

$$X=\sqrt{A}+\sqrt{B}\quad\cdots\cdots\cdots(12)$$

where,

$$\left.\begin{aligned}
A&=\frac{-\eta+\sqrt{\eta^2+4\xi^3}}{2}\\
B&=\frac{-\eta-\sqrt{\eta^2+4\xi^3}}{2}\\
\xi&=-\frac{1}{3}\left(\frac{p_r}{1.08c_0}+\frac{1.19}{a_j'}\right)\frac{p_r}{c_0}\\
\eta&=-\left(\frac{p_r}{2.916c_0}-\frac{1}{a_j'}\right)\left(\frac{p_r}{c_0}\right)^2\\
c_0&=\sqrt{\frac{Eg}{\rho}}
\end{aligned}\right\}$$

The above method to require the original (natural) angular frequency p_r, is the method which can be applied to the cases covering a considerably wide range of $(2a_j'/\lambda_r)$. But in almost practical cases, the value of $(2a_j'/\lambda_r=a_j'\gamma_r/\pi)$ lies in the range between the value of 0∼0.5 as mentioned before. In such a case as this it is permitted that we may estimate the value of the propagation constant γ_r in a beam according to the following equation as shown in **Fig. 3**.

$$\frac{p_r}{c_0\gamma_r}=\frac{4.4a_j'\gamma_r}{6a_r'\gamma_r+\pi}$$
$$\therefore\quad \gamma_r=\frac{6_j'p_r\pm\sqrt{3.6a_j'^2p_r^2+17.6\pi a_j'c_0p_r}}{8.8a_j'c_0}\quad\cdots\cdots\cdots(13)$$

where

$$c_0=\sqrt{\frac{Eg}{\rho}}$$

Further, if it is an actual case where the value of $(2a_j'/\lambda_r=a_j'\gamma_r/\pi)$ lies in a fairly narrow range of 0∼0.25, then we may calculate γ_r more simply by the following equation as shown in **Fig. 3**. Namely,

$$\frac{p_r}{c_0\gamma_r}=\frac{2}{\pi}a_j'\gamma_r$$
$$\therefore\quad \gamma_r=1.25\sqrt{\frac{p_r}{c_0a_j'}}\quad\cdots\cdots\cdots(14)$$

In order to estimate in practical case in what range the value of $(2a_j'/\lambda_r)$ lies, it is almost enough to consider the wave-lengths, λ_r, up to 25-th higher harmonics as mentioned before; and the wave-length λ_r of 25-th higher harmonics can be easily estimated by the span-length and its end-boundary conditions, and on the other hand the radius of gyration at a cross-section of maximum depth in j-th span is already known in practical case.

Dealing with the problem as above, and after we have required the original (natural) angular frequency p_r and, as a result, after we have required the propagation constant $\gamma_r(x)$, we shall be able to obtain the ratio of the constant C_{jr}, D_{jr}, E_{jr} and F_{jr} by substituting the value of $\gamma_r(x)$ into the previous end-boundary conditions. Thus we are able to require the normal function in lateral vibration of the beam bridge and thus clarify the first step of vibration analysis without coupling.

4. COUPLING VIBRATION OF A CONTINUOUS BEAM-BRIDGE WITH VARIABLE CROSS-SECTION AND ITS MAGNIFICATION FACTOR

If we denote the coupling displacement components of the generalized-coordinate in lateral vibration as $\bar{q}_r$, or $\bar{q}_s$, then the total kinetic energy T of the continuous beam-bridge becomes

$$2T=\rho\dot{\bar{q}}_r\dot{\bar{q}}_s\sum_{j=1}^{m}\int_0^{l_j}A\cdot Y_j(\gamma_r x)\cdot Y_j(\gamma_s x)dx\quad\cdots\cdots\cdots(15)$$

and the total strain energy U of the continuous beam-bridge becomes

$$2U=E\bar{q}_r\bar{q}_s\sum_{j=1}^{m}\int_0^{l_j}I\frac{19.36}{(6a'\gamma_r+\pi)\cdot(6a'\gamma_s+\pi)}\cdot\left\{\frac{\partial^2Y_j(\gamma_r x)}{\partial x^2}\right\}\left\{\frac{\partial^2Y_j(\gamma_s x)}{\partial x^2}\right\}dx\quad\cdots\cdots(16)$$

where in the above,

- A: cross-sectional area of the beam.
- I: moment of inertia of a cross-section of the beam.
- a': radius of gyration of a cross-section of the beam.
- ρ: unit mass of the material
- E: Young's modulus

In the above equation (16), the coefficient

$$\frac{19.36}{(6a'\gamma_r+\pi)(6a'\gamma_s+\pi)}$$

inside of the integral sign is the compensating coefficient for the elementary theory (linear theory) of flexural wave in a slender beam.

The Lagranges' equation of motion for the coupled vibration of the beam is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\bar{q}}_r}\right)-\frac{\partial T}{\partial \bar{q}_r}+2\zeta\frac{\partial T}{\partial \dot{\bar{q}}_r}+\frac{\partial U}{\partial \bar{q}_r}=\boldsymbol{F}_\nu\frac{\partial \boldsymbol{r}_\nu}{\partial \bar{q}_r} \quad \cdots\cdots(17)$$

where

- $\boldsymbol{F}_\nu$: external force vector at a point ν
- $\boldsymbol{r}_\nu$: position vector of a point ν from the origin of coordinate in actual three dimensional space.
- ζ: damping coefficient of the vibration.

Substituting the values of T and U which were obtained by the equations (15) and (16) into the Lagranges' equation (17), we obtain the equation of the following form, since the second term of the above equation $\partial T/\partial \bar{q}_r$ is equal to zero by the previous linear relation (2).

$$b_{rs}\ddot{\bar{q}}_s+2\zeta b_{rs}\dot{\bar{q}}_s+K_{rs}\bar{q}_s=Q_r \quad \cdots\cdots(18)$$

where,

$$\left.\begin{aligned} b_{rs}&=\rho\sum_{j=1}^{m}\int_0^{l_j}A\cdot Y_j(\gamma_r x)\cdot Y_j(\gamma_s x)dx\\ K_{rs}&=E\sum_{j=1}^{m}\int_0^{l_j}I\frac{19.36}{(6a'\gamma_r+\pi)(6a'\gamma_s+\pi)}\cdot\left\{\frac{\partial^2 Y_j(\gamma_r x)}{\partial x^2}\right\}\cdot\left\{\frac{\partial^2 Y_j(\gamma_s x)}{\partial x^2}\right\}dx\\ Q_r&=\boldsymbol{F}_\nu\frac{\partial \boldsymbol{r}_\nu}{\partial \bar{q}_r}\end{aligned}\right\}$$

For the practical calculation of the values of coefficients, b_{rs} and K_{rs}, in the above equation (18), we may execute the above integral calculation for the beam with variable cross-section as a composite body constituted by many segment-beams, each of which can be regarded as a small beam of constant depth in each interval of Δx.

If we expand the Langrange's equation (18) with respect to the summation convention for s, then we have

$$\begin{bmatrix} b_{11} & b_{12}\cdots\cdots b_{1n}\\ b_{21} & b_{22}\cdots\cdots b_{2n}\\ \cdots & \cdots\\ b_{n1} & b_{n2}\cdots\cdots b_{nn}\end{bmatrix}\cdot\begin{Bmatrix}\ddot{\bar{q}}_1\\ \ddot{\bar{q}}_2\\ \vdots\\ \ddot{\bar{q}}_n\end{Bmatrix}+2\zeta\begin{bmatrix} b_{11} & b_{12}\cdots\cdots b_{1n}\\ b_{21} & b_{22}\cdots\cdots b_{2n}\\ \cdots & \cdots\\ b_{n1} & b_{n2}\cdots\cdots b_{nn}\end{bmatrix}\cdot\begin{Bmatrix}\dot{\bar{q}}_1\\ \dot{\bar{q}}_2\\ \vdots\\ \dot{\bar{q}}_n\end{Bmatrix}+\begin{bmatrix} K_{11} & K_{12}\cdots\cdots K_{1n}\\ K_{21} & K_{22}\cdots\cdots K_{2n}\\ \cdots & \cdots\\ K_{n1} & K_{n2}\cdots\cdots K_{nn}\end{bmatrix}\cdot\begin{Bmatrix}\bar{q}_1\\ \bar{q}_2\\ \vdots\\ \bar{q}_3\end{Bmatrix}=\begin{Bmatrix}Q_1\\ Q_2\\ \vdots\\ Q_n\end{Bmatrix} \quad \cdots\cdots(19)$$

In a general vibration problem, the external force component of generalized coordinate, Q_r, is a periodic function of time t and thus can be expanded as a Fourier series. If we put a component of this Fourier series as

$$(Q_r)_m=B_r\exp(imt) \quad \cdots\cdots(20)$$

then we have from the equation (18)

$$b_{rs}\ddot{\bar{q}}_s+2\zeta b_{rs}\dot{\bar{q}}_s+K_{rs}\bar{q}_s=B_r\exp(imt) \quad \cdots\cdots(21)$$

Since the above equation (21) is a linear differential equation, then putting its particular solution (solution fo forced vibration) as

$$\bar{q}_s=A_s\exp(imt), \quad \cdots\cdots(22)$$

we have the following equation

$$\{-(m)^2 b_{rs}+2i\zeta m b_{rs}+K_{rs}\}A_s=B_r$$

i.e.

$$\begin{bmatrix} (-(m)^2b_{11}+2i\zeta mb_{11}+K_{11}) & (-(m)^2b_{12}+2i\zeta mb_{12}+K_{12}) & \cdots\cdots(-(m)^2b_{1n}+2i\zeta mb_{1n}+K_{1n})\\ (-(m)^2b_{21}+2i\zeta mb_{21}+K_{21}) & (-(m)^2b_{22}+2i\zeta mb_{22}+K_{22}) & \cdots\cdots(-(m)^2b_{2n}+2i\zeta mb_{2n}+K_{2n})\\ \vdots & \vdots & \vdots\\ (-(m)^2b_{n1}+2i\zeta mb_{n1}+K_{n1}) & (-(m)^2b_{n2}+2i\zeta mb_{n2}+K_{n2}) & \cdots\cdots(-(m)^2b_{nn}+2i\zeta mb_{nn}+K_{nn})\end{bmatrix}\cdot\begin{Bmatrix}A_1\\ A_2\\ \vdots\\ A_n\end{Bmatrix}=\begin{Bmatrix}B_1\\ B_2\\ \vdots\\ B_n\end{Bmatrix}$$

$$\begin{bmatrix} (w^2b_{11}+2\zeta wb_{11}+K_{11}) & (w^2b_{12}+2\zeta wb_{12}+K_{12}) & \cdots\cdots (w^2b_{1n}+2\zeta wb_{1n}+K_{1n}) \\ (w^2b_{21}+2\zeta wb_{21}+K_{21}) & (w^2b_{22}+2\zeta wb_{22}+K_{22}) & \cdots\cdots (w^2b_{2n}+2\zeta wb_{2n}+K_{2n}) \\ \\ (w^2b_{n1}+2\zeta wb_{n1}+K_{n1}) & (w^2b_{n2}+2\zeta wb_{n2}+K_{n2}) & \cdots\cdots (w^2b_{nn}+2\zeta wb_{nn}+K_{nn}) \end{bmatrix} \cdot \begin{Bmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \vdots \\ \bar{A}_n \end{Bmatrix} = 0 \cdots\cdots (25)$$

$$\therefore \begin{vmatrix} (w^2b_{11}+2\zeta wb_{11}+K_{11}) & (w^2b_{12}+2\zeta wb_{12}+K_{12}) & \cdots\cdots (w^2b_{1n}+2\zeta wb_{1n}+K_{1n}) \\ (w^2b_{21}+2\zeta wb_{21}+K_{21}) & (w^2b_{22}+2\zeta wb_{22}+K_{22}) & \cdots\cdots (w^2b_{2n}+2\zeta wb_{2n}+K_{2n}) \\ \\ (w^2b_{n1}+2\zeta wb_{n1}+K_{n1}) & (w^2b_{n2}+2\zeta wb_{n2}+K_{n2}) & \cdots\cdots (w^2b_{nn}+2\zeta wb_{nn}+K_{nn}) \end{vmatrix} \equiv \Delta(w)=0 \cdots\cdots (26)$$

The above equation (26) is an algebraic equation with respect to (ω) of $2n$ degree which has real coefficients. Therefore it is expected that it must have n pairs of conjugate complex roots (including even number of real roots). If we put the conjugate complex roots as

$$\left.\begin{aligned} w_j&=\alpha_j+i\beta_j \\ w_j'&=\alpha_j-i\beta_j \end{aligned}\right\} \quad (j=1, 2, \cdots\cdots, n) \cdots\cdots (27)$$

then we may require from the equation (26) the values of α_j and β_j. Namely, the equation (26) is a frequency equation of coupled natural vibration.

Substituting the solution (27) obtained from the above equation (26) into the equation (25), we are able to require the ratios, ($\bar{A}_{1j}$: $\bar{A}_{2j}$: . . . : $\bar{A}_{nj}$) and ($\bar{A}'_{1j}$: $\bar{A}'_{2j}$: . . . : $\bar{A}'_{nj}$), where $\bar{A}_{sj}$ and $\bar{A}'_{sj}$ show the coefficients in the equation,

$$\bar{q}_{sj}=\bar{A}_{sj}\exp(w_jt)+\bar{A}'_{sj}\exp(w_j't)\ .$$

Namely, we obtain the ratios as shown in the following equation

$$\left.\begin{aligned} \frac{\bar{A}_{1j}}{a_{1j}}&=\frac{\bar{A}_{2j}}{a_{2j}}=\cdots\cdots=\frac{\bar{A}_{sj}}{a_{sj}}=\cdots\cdots=\frac{\bar{A}_{nj}}{a_{nj}}=G_j \\ \frac{\bar{A}'_{1j}}{a'_{1j}}&=\frac{\bar{A}'_{2j}}{a'_{2j}}=\cdots\cdots=\frac{\bar{A}'_{sj}}{a'_{sj}}=\cdots\cdots=\frac{\bar{A}'_{nj}}{a'_{nj}}=H_j \end{aligned}\right\} \cdots\cdots (28)$$

where

- s: number of coordinate axis in the generalized coordinate system.
- $j=1,2$ . . . , n: number of a solution w in the equation (26).
- a_{sj}, a'_{sj}: the minor of s-th column element in a certain definite row (for example, in the n-th row) of the determinant $\Delta(w_j)$ or $\Delta(w_j')$, which is multiplied by $(-1)^{n-s}$ respectively (these are generally complex numbers independent of time t)
- G_j, H_j: arbitrary constants (positive real numbers) for each j, but the same for all number of generalized corodinate s respectively (these arbitrary constants are determined by initial conditions).

It can be proved that the value of $(w^2+2\zeta w)$ is always real negative, since the equation (26) can also be regarded as an algebraic equation concerning to $(\omega^2+2\zeta w)$ of n-th degree with real coefficients.* Therefore if we denote the root by $-c_j^2$ ($j=1,2,\ .\ .\ .\ ,n$), then

$$(w)^2+2\zeta(w)+c_j^2=0 \quad (c_j\text{: real number}) \cdots\cdots (29)$$

$$\therefore\ w_j=-\zeta\pm\sqrt{\zeta^2-c_j^2}$$

$$\therefore\ \alpha_j=-\zeta,\quad \beta_j=\sqrt{c_j^2-\zeta^2} \cdots\cdots (30)$$

And we can see also from the above relation

$$a_{sj}=a'_{sj} \cdots\cdots (31)$$

Substituting the values obtained from the equation (28) into the equation (24) and adding by j, we may require the generalized coordinate displacement-component in the coupled natural vibration as follows.

$$\begin{aligned} \bar{q}_s&=\bar{A}_{s1}\exp[(\alpha_1+i\beta_1)t]+\bar{A}_{s2}\exp[(\alpha_2+i\beta_2)t] \\ &\quad+\cdots+\bar{A}_{sn}\exp[(\alpha_n+i\beta_n)t] \\ &\quad+\bar{A}'_{s1}\exp[(\alpha_1-i\beta_1)t] \\ &\quad+\bar{A}'_{s2}\exp[(\alpha_2-i\beta_2)t] \\ &\quad+\cdots+\bar{A}'_{sn}\exp[(\alpha_n-i\beta_n)t] \\ &=a_{sj}G_j\exp[(\alpha_j+i\beta_j)t] \\ &\quad+a'_{sj}H_j\exp[(\alpha_j-i\beta_j)t] \\ &\cong e^{-\zeta t}b_{sj}\bar{A}_j\cos(\beta_jt+\phi_j) \cdots\cdots (32) \end{aligned}$$

where b_{sj} and $\bar{A}_j$ are respectively real constants independent of time t.

As shown in the above equation, generally many values of j exist for only one normal mode s. In a special case where $b_{rs}=K_{rs}=0$ in the equation (25) or (26), i.e., the left hand side tensor in these equation is diagonalized (this is the case of constant beam-depth over all spans where the normal functions α_{ir} and α_{is} are all orthogonal functions), the generalized $\bar{q}_s$ in the equation (32) becomes to have only one term of $j=s$, viz. $e^{-\zeta t}\bar{A}_s\cos(\beta_st+\phi_s)$, and at the same time becomes to have zero value in all other terms. We call such a case as an orthogonal vibration.

In general, if we denote the number of solution, w, of the equation (26) as j and k, the follwing relation always holds true

* See Bibliography 2) (p. 257) in the end of this paper.

$$j \neq k: \quad b_{rs}a_{rj}a_{sk} = K_{rs}a_{rj}a_{sk} = 0$$
$$j = k: \quad b_{rs}a_{rj}a_{sk} > 0 \text{ and } K_{rs}a_{rj}a_{sk} > 0 \quad * \qquad \cdots\cdots(33)$$

However, in the case of variable cross-section where the beam-depth varies, the orthogonal relation between the normal function themselves in a span of the continuous beam-bridge does not hold true. Namely, in a continuous beam-bridge of variable cross-sections,

$$b_{rs} \neq 0 \text{ and } K_{rs} \neq 0, \qquad \cdots\cdots(34)$$

when the number $r \neq s$.

But even in such a case as above where the orthogonal relation between the normal function themselves of a beam does not hold true, the amplitude $|a_{JJ}|$, which has the number of $j=s$, is generally largest prominently among the amplitudes $|a_{sj}|$ $(j=1,2 \ldots, n)$ in the equation (28) which correspond to a certain angular frequency w_j obtained by the equation (26). Thus from the standing point of view in which we want to understand the approximate aspect of the vibration phenomenon, it is recomended that one adopts only one term $j=s$ in the equation (32) and forsakes all other terms. However, this treatment is different from the operation in which one assumes $b_{rs}=K_{rs}=0$ $(r \neq s)$ in the equation (25) and calculate w_j from the determinant equation (26) which was reduced to a diagonal determinant equation; i.e., it is especially to be noticed that the value w_j is conspicuously different in higher mode from that in the case where the orthogonality holds true.

Putting together the equation (22) and (31), the generalized coordinate displacement-component $\bar{q}_s$ becomes

$$\bar{q}_s = e^{-st}b_{sj}\bar{A}_j \cos(\beta_j t+\phi_j) + A_s \exp(imt) \cdots(35)$$

The first term (natural vibration) in the above equation (35) decays with time and only the second term (forced vibration) remains finally. Though the amplitude $b_{sj}\bar{A}_j$ is a real constant as mentioned before in the equation (32), the amplitude A_s of the second term (forced vibration) is generally a complex value (See equation (23)). If we denote the absolute value of A_s as A'_s, then the amplitude A_s can generally be expressed as $A_s = A'_s \exp(i\varphi)$. And as a consequence, the equation (35) becomes to be expressed as

$$\bar{q}_s = e^{-st}b_{sj}\bar{A}_j \cos(\beta_j t+\phi_j) + A_s' \cos(mt+\varphi) \qquad \cdots\cdots(36)$$

Let's consider now the case where the angular frequency, m, of the external force approaches to c_j (See equation (29)) and finally becomes equal to c_j. Then in this case, it becomes

$$\begin{aligned} &-m^2 b_{rs} + 2i\zeta m b_{rs} + K_{rs} \\ &= -c_j^2 b_{rs} + K_{rs} + 2i\zeta m b_{rs} \\ &= (w^2 b_{rs} + 2\zeta w b_{rs} + K_{rs}) + 2i\zeta m b_{rs} \\ &\qquad (\because \text{ cf. equation (29)}) \end{aligned}$$

If we express the matrix which is constituted by the above elements corresponding to r, $s=1, 2, 3, \ldots, n$, simply as

$$[(w^2 b_{rs} + 2\zeta w b_{rs} + K_{rs}) + 2i\zeta m b_{rs}] \cdots\cdots(37)$$

then the determinant of the above matrix (37) becomes equal to zero by the equation (26) when $\zeta=0$. Namely, in the case where $m \rightarrow = c_j$, we can see that the denominator of the inverse matrix shown in the equation (23) degenerates when $\zeta=0$. Therefore, we can understand that the phenomenon of resonance takes place in the case where the angular frequency, m, of an external force becomes equal to c_j.

Since the vibration of a continuous beam-bridge possesses multiple degree of freedom and the determinant of the matrix, (37), becomes to a complex value, which is not equal to a pure imaginary such as in a vibration with one degree of freedom, then the phase-lag of the forced vibration of a continuous beam-bridge does not become equal to $\pi/2$, as it becomes equal to $\pi/2$ for the phase angle of the external force in a case of vibration with one degree of freedom.

Now, we can generally devide the engineering purpose of the vibration-analysis into the following three items, *i.e.*,

(i) To find out and to design the natural vibration-frequencies of the engineering structure so as to separate from the practical vibration-frequencies of probable external force as possible as we can, and thus to prevent the actual appearance of the phenomenon of resonance as possible as we can.

(ii) To calculate the magnification factor which is the ratio between the amplitude, A'_s, of the forced vibration produced by a pulsating external force, and the fundamental sustaining deflection *i. e.*, the statical deflection, (δ_{st}); and then to examine the safety factor of the structure after requiring how much multiple of statical stress will be produced as dynamical stress due to dynamical pulsating external force.

(iii) To devise an apparatus which has a large damping factor ζ, and thus to decrease the magnification factor mentioned in the above item (ii) until to a desirable value.

We are able to calculate the statical deflection

* See Bibliography 2) (pp. 256~257) in the end of this paper

(δ_{st}) mentioned in the above item (ii) as follows.

If we exclude the dynamical terms, *i. e.*, the first term $b_{rs}\ddot{q}_s$ and the second term $2\zeta b_{rs}\dot{q}_s$, in the left hand side of the equation (21), and if we consider the problem as a statical problem, then we obtain for the fundamental normal function, $Y(\gamma_1 x)$, the following coefficient, (δ_{st}), of statical deflection-curve as

$$(\delta_{st})=(A_1')_{st}=\begin{bmatrix} K_{11} & K_{12}\cdots\cdots K_{1n} \\ K_{21} & K_{22}\cdots\cdots K_{2n} \\ \cdots\cdots & \cdots\cdots\cdots\cdots \\ K_{n1} & K_{n2}\cdots\cdots K_{nn} \end{bmatrix}^{-1} \cdot \begin{Bmatrix} B_1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \qquad \cdots\cdots(38)$$

Therefore the magnification factor in this case becomes

$$\frac{|A_s|}{(\delta_{st})}=\frac{A_s'}{(A_1')_{st}}=\frac{|[(-m^2 b_{rs}+2i\zeta m b_{rs}+K_{rs})]^{-1}\cdot B_r|}{(A_1')_{st}} \qquad \cdots\cdots(39)$$

where

$[\cdots\cdots]^{-1}$ in the numerator of the above euation: denotes the inverse matrix in equation (23).

$|\cdots\cdots|$ in the numerator of the above equation: denotes absolute value.

Indicial notation r, s in the numerator of the right hand side in the above equation (39): denotes the s-th vector-component which is obtained by taking summation convention for the index r in the numerator of the right hand side in the above equation (39).

We may always calculate the magnification factor from the above equation (39) by substituting the equation (38) into the equation (39).

5. CONCLUSION

The lateral vibration of a continuous beam-bridge with variable cross-section has normal functions, $Y_j(\gamma_r x)$, as shown in the equation (3), the propagation constant $\gamma_r(x)$ of which can be required by the equation (10) or (13) or (14), and end boundary conditions; and it has also (coupled) generalized coordinate displacement-component, $\bar{q}_r$, as shown in the equation (36), the values of α_j, β_j, $\tilde{A}_j$, ϕ_j, A_s' and φ in which can be required by the equations (30), (32), (23), (35) and (36) respectively.

The magnification factor which is the ratio between the dynamical stresses due to pulsating external force and the statical stress, can be obtained by the equation (39).

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