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# COUPLED VIBRATION OF CONTINUOUS TRUSS-BRIDGE ON MANY PIERS

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## SYNOPSIS

In this paper the author has shown the correct method of vibration analysis of solid elastic structure, taking an example in the problem of transversal vibration (in vertical plane) of continuous truss-bridge on many piers. He has also applied in this paper his long-cherished opinion as to the phase velocity of longitudinal progressing-wave in solid elastic bar, which was already reported in his previous paper, 'Phase Velocity of Longitudinal Impact Wave in Solid Elastic Bar' in the Trans. of JSCE, No. 260, April, 1977.

## 1. INTRODUCTION

All the vibration-phenomena in solid elastic bodies are the phenomena of standing-waves which are formed by two progressing waves in two opposite directions, the one is original progressing waves and the another is its reflecting wave at the ends of the elastic body and which have a form $\alpha_{ir}q_{ro}=\varphi_{ir}(x_1, x_2, x_3)\cdot\phi_{ro}(t)$ (The subscript $o$ denotes that it corresponds to normal frequency or normal mode, and does not correspond to coupled frequency or coupled mode.

In order to solve the coupled vibration problem of an elastic structure, it is necessary

(1) to determine the propagation constants $\gamma_r$ of a progressing wave and the constant-ratios in normal functions by the end-fastening conditions of an elastic structure, and

(2) to set up Lagrange's differential equation with normal functions which are composed of the propagation constants $\gamma_r$ and the constant-ratios, and to require the coupled frequency $w_j$ and (coupled) generalized-coordinates $q_r$ by solving this Lagrange's equation.

The former equation to determine the value of $\gamma_r$ and the constant-ratios in normal functions, is derived as a simultaneous equation from the end-fastening conditions of an elastic structure and the determinant equation which is obtained by eliminating the constant-ratios in the simultaneous equation, is called 'normal frequency equation', since the normal angular frequency $p_r$ is determined by the formula $p_r=c\gamma_r$ (where $c$ is a known phase velocity of a progressive wave).

The latter equation to require the coupled frequency $w_j$ and the amplitude-ratios of (coupled) generalized-coordinates $q_r$ in a structure, is derived also as a simultaneous equation from Lagrange's equation and the determinant equation which is obtained by eliminating the amplitude-ratios of (coupled) generalized coordinates $q_r$, is called 'coupled frequency equation' (refer Bibliography 27)).

## 2. TRANSVERSAL VIBRATION OF A CONTINUOUS TRUSS-BRIDGE ON MANY PIERS

### (1) End Conditions, Propagation Constant, Normal Function, and Normal Frequency

Let's consider about the transversal vibration (in vertical plane) of a continuous steel truss-bridge on many concrete piers as shown in Fig. 1.

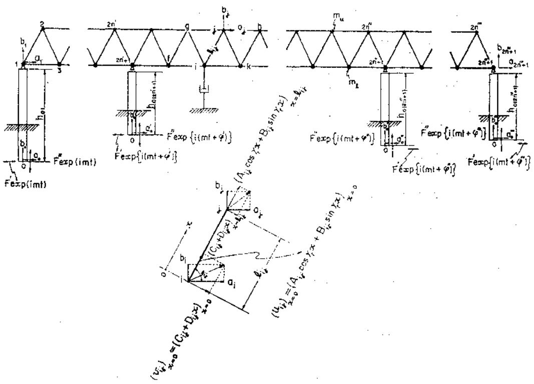

**Fig. 1** The Schematic Illustration of a Truss-Bridge

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We are able to express the normal function (deflection curve) of a member-bar $(ij)$ corresponding to a normal angular frequency $p_r$ in longitudinal vibration as

$$\left.\begin{aligned} u_{ij} &= A_{ij}\cos(\gamma_r x) + B_{ij}\sin(\gamma_r x) \\ v_{ij} &= C_{ij} + D_{ij}x \end{aligned}\right\} \cdots\cdots(1)$$

where

$u_{ij}$: axial displacement of a member-bar $(ij)$.
$v_{ij}$: displacement perpendicular to the axis of a member-bar $(ij)$.
$x$ : axial distance of a point from the origin $i$ in each member-bar.
$r=1, 2, \cdots, \infty$: number of normal mode of vibration.
$\gamma_r$ : propagation constant of $r$-th normal mode of longitudinal vibration in steel member-bars.

It is to be noticed that the above former equation seems outwardly not to include the linear displacement of variable $x$ such as shown in the above latter equation, but it also includes implicitly the linear displacement of variable $x$, because

$$\begin{aligned} & A_{ij}\cos(\gamma_r x) + B_{ij}\sin(\gamma_r x) \\ &= A_{ij}\left\{1 - \frac{(\gamma_r x)^2}{2!} + \frac{(\gamma_r x)^4}{4!} - \cdots\right\} \\ &\quad + B_{ij}\left\{(\gamma_r x) - \frac{(\gamma_r x)^3}{3!} + \frac{(\gamma_r x)^5}{5!} - \cdots\right\} \\ &= A_{ij} + B_{ij}(\gamma_r x) - \cdots \end{aligned}$$

and, when $\gamma_r \to 0$, $B_{ij} \to \infty$, and $B_{ij}\gamma_r \to$ (finite and definite), the above equation becomes almost a linear equation of variable $x$.

The following four equations generally hold true for each member-bar due to the transformation of coordinates.

$$\left.\begin{aligned} & \{A_{ij}\cos(\gamma_r x) + B_{ij}\sin(\gamma_r x)\}_{x=0} \\ &\quad = a_i\cos\theta_{ij} + b_i\sin\theta_{ij} \\ & \{A_{ij}\cos(\gamma_r x) + B_{ij}\sin(\gamma_r x)\}_{x=l_{ij}} \\ &\quad = a_j\cos\theta_{ij} + b_j\sin\theta_{ij} \\ & (C_{ij} + D_{ij}x)_{x=0} = a_i\sin\theta_{ij} - b_i\cos\theta_{ij} \\ & (C_{ij} + D_{ij}x)_{x=l_{ij}} = a_j\sin\theta_{ij} - b_j\cos\theta_{ij} \end{aligned}\right\}$$

$$\therefore \left.\begin{aligned} A_{ij} &= a_i\cos\theta_{ij} + b_i\sin\theta_{ij} \\ B_{ij} &= \{(a_j - a_i\cos\gamma_r l_{ij})\cos\theta_{ij} \\ &\quad + (b_j - b_i\cos\gamma_r l_{ij})\sin\theta_{ij}\}\frac{1}{\sin(\gamma_r l_{ij})} \\ C_{ij} &= a_i\sin\theta_{ij} - b_i\cos\theta_{ij} \\ D_{ij} &= \{(a_j - a_i)\sin\theta_{ij} - (b_j - b_i)\cos\theta_{ij}\}\frac{1}{l_{ij}} \end{aligned}\right\} \cdots\cdots(2)$$

Thus the coefficients, $A_{ij}$, $B_{ij}$, $C_{ij}$, $D_{ij}$, in the normal functions of steel member-bars in truss-bridge can always be expressed by the horizontal and vertical components, $a_i$, $a_j$, $b_i$, $b_j$, of the displacements of connecting-points of member-bars. The number of the unknown horizontal and vertical components, $a_i$, $a_j$, $b_i$, $b_j$, of the displacements of the connecting-points of member-bars of truss-bridge is equal to $2(2n'''+1)$.

We may also express the normal function (deflection curve) of a pier $(oi)$ corresponding to a normal angular frequency $p_r$ as

$$\left.\begin{aligned} u_{oi} &= A_{oi}\cos(\gamma_r' x) + B_{oi}\sin(\gamma_r' x) \\ &= A'_{oi}\cos(\gamma_r' x + \phi) \\ v_{oi} &= C_{oi}\cos(\gamma_r'' x) + D_{oi}\sin(\gamma_r'' x) \\ &\quad + G_{oi}\cosh(\gamma_r'' x) + H_{oi}\sinh(\gamma_r'' x) \\ &= C'_{oi}\cos(\gamma_r'' x + \varphi) + D'_{oi}\cosh(\gamma_r'' x + \Phi) \end{aligned}\right\} \cdots\cdots(3)$$

where

$u_{oi}$ : vertical displacement of a point $x$ in the pier $(oi)$.
$v_{oi}$ : horizontal displacement of a point $x$ in the pier $(oi)$.
$\phi, \varphi$: real phase angles.
$\Phi$ : real or complex phase angle.
$\gamma_r'$ : propagation constant of $r$-th normal mode of longitudinal vibration of concrete piers.
$\gamma_r''$ : propagation constant of $r$-th normal mode of lateral vibration of concrete piers.

If we assume the number of concrete piers as 4, then we see that the number of unknown constants, $A'_{oi}$, $\phi$, $C'_{oi}$, $\varphi$, $D'_{oi}$, and $\Phi$ in the normal functions of concrete piers is equal to $6\times4=24$.

On the other hand we may always obtain as the end-conditions of steel member-bars in truss-bridge the $2(2n'''+1)$ horizontal and vertical equilibrium-equations of the connecting-points of member-bars.

For example, if we take the point $i$ as a typical connecting-point of member-bars, then we have the equilibrium equations for requiring propagation constant $\gamma_r$ approximately as follows.

$$\left.\begin{aligned} & c_1^2\rho\left\{S_{ij}\left(\frac{\partial u_{ij}}{\partial x}\right)_{x=0}\cos\theta_{ij} - S_{gi}\left(\frac{\partial u_{gi}}{\partial x}\right)_{x=l_{gi}}\cos\theta_{gi} \right. \\ &\quad \left. + S_{ik}\left(\frac{\partial u_{ik}}{\partial x}\right)_{x=0} - S_{fi}\left(\frac{\partial u_{fi}}{\partial x}\right)_{x=l_{fi}}\right\} - m_l a_i p_r^2 \\ &\quad = 0 \\ & c_1^2\rho\left\{S_{ij}\left(\frac{\partial u_{ij}}{\partial x}\right)_{x=0}\sin\theta_{ij} + S_{gi}\left(\frac{\partial u_{gi}}{\partial x}\right)_{x=l_{gi}}\sin\theta_{gi}\right\} \\ &\quad - m_l b_i p_r^2 = 0 \end{aligned}\right\} \cdots\cdots(4)$$

where

$c_1^2=\frac{2G}{\rho}\left(\frac{1-\nu}{1-2\nu}\right)$: phase velocity of longitudinal wave in steel member bars.

$S_{ij}$: cross-sectional area of the member-bar $(ij)$.

$m_l$: mass at a lower connecting-point due to floor-beam and stringer.

($m_u$: mass at an upper connecting-point due to lateral member-bar.)

$\gamma_r$ : $r$-th original normal-propagation-constant excluding attenuation factor due to distance. (i.e. real part of complex propagation-constant.)

$p_r=c_1\gamma_r$: original normal-angular-frequency which does not include damping effect and corresponds to $\gamma_r$.

$\rho$ : mass of unit volume of steel material.

It is to be noticed that the damping force due to dash-pot damper can be put aside from our consideration so long as we are considering about the original normal-propagation-constant excluding attenuation factor due to distance and about the original normal-angular-frequency excluding damping effect.

As the special cases of the conditional equation (4) the following equations hold true at the tops of concrete piers.

$$\left.\begin{aligned}
&-c_1'^2\rho' S_{oi}\left(\frac{\partial u_{oi}}{\partial x}\right)_{x=h_{oi}}\\
&\quad+c_1^2\rho\left\{S_{i(i+1)}\left(\frac{\partial u_{i(i+1)}}{\partial x}\right)_{x=0}\sin\theta_{i(i+1)}\right.\\
&\quad\left.+S_{(i-1)i}\left(\frac{\partial u_{(i-1)i}}{\partial x}\right)_{x=l(i-1)i}\sin\theta_{(i-1)i}\right\}\\
&\quad-m_l b_i p_r^2=0,\quad (i=2n'+1,\,2n''+1,\,2n'''+1)\\
&-c_1'^2\rho' S_{01}\left(\frac{\partial u_{01}}{\partial x}\right)_{x=h_{01}}+c_1^2\rho S_{12}\left(\frac{\partial u_{12}}{\partial x}\right)_{x=0}\sin\theta_{12}\\
&\quad-m_l b_1 p_r^2=0,\quad (i=1)\\
&-\left\{\frac{\partial}{\partial x}E'I'\left(\frac{\partial^2 v_{01}}{\partial x^2}\right)\right\}_{x=h_{01}}\\
&\quad+c_1^2\rho\left\{S_{13}\left(\frac{\partial u_{13}}{\partial x}\right)_{x=0}+S_{12}\left(\frac{\partial u_{12}}{\partial x}\right)_{x=0}\cos\theta_{12}\right\}\\
&\quad-m_l a_1 p_r^2=0,\quad (i=1)
\end{aligned}\right\}\quad\cdots\cdots(5)$$

where

$\gamma_r'=\frac{p_r}{c_1'}$: original propagation-constant of $r$-th normal mode of longitudinal vibration of concrete piers.

$\gamma_r''=\frac{p_r}{c}$: original propagation-constant of $r$-th normal mode of lateral vibration of concrete piers.

$c_1'=\sqrt{\frac{2G'}{\rho'}\left(\frac{1-\nu'}{1-2\nu'}\right)}$: phase velocity of longitudinal impact-wave in concrete piers.

$c$ : phase velocity of flexural impact-wave in concrete piers (see Fig. 2).

$G', E', \nu'$: denote respectively the modulus of rigidity, Young's modulus, and Poisson's ratio of concrete material.

$\rho'$ : unit mass of concrete piers.

$I'$ : 2nd moment of cross-section of a concrete pier.

$S_{oi}$: cross sectional area of the concrete pier $(oi)$.

We may also assume as reflecting-conditions of impact-waves at the bottoms of concrete piers that the following phase angles $\phi$, $\varphi$, and $\Phi$, are known. Further we have always the following end-conditions for concrete piers.

$$\left.\begin{aligned}
&A_{oi}'\cos(\gamma_r'h_{oi}+\phi)=b_i,\\
&\qquad (i=1,\,2n'+1,\,2n''+1,\,2n'''+1)\\
&C_{01}'\cos(\gamma_r''h_{01}+\varphi)+D_{01}'\cosh(\gamma_r''h_{01}+\Phi)=a_1,\\
&\qquad (i=1)\\
&\left(\frac{\partial^2 v_{oi}}{\partial x^2}\right)_{x=h_{oi}}=0,\\
&\qquad (i=1,\,2n'+1,\,2n''+1,\,2n'''+1)\\
&\left(\frac{\partial^3 v_{oi}}{\partial x^3}\right)_{x=h_{oi}}=0,\\
&\qquad (i=1,\,2n'+1,\,2n''+1,\,2n'''+1)
\end{aligned}\right\}\quad\cdots\cdots(6)$$

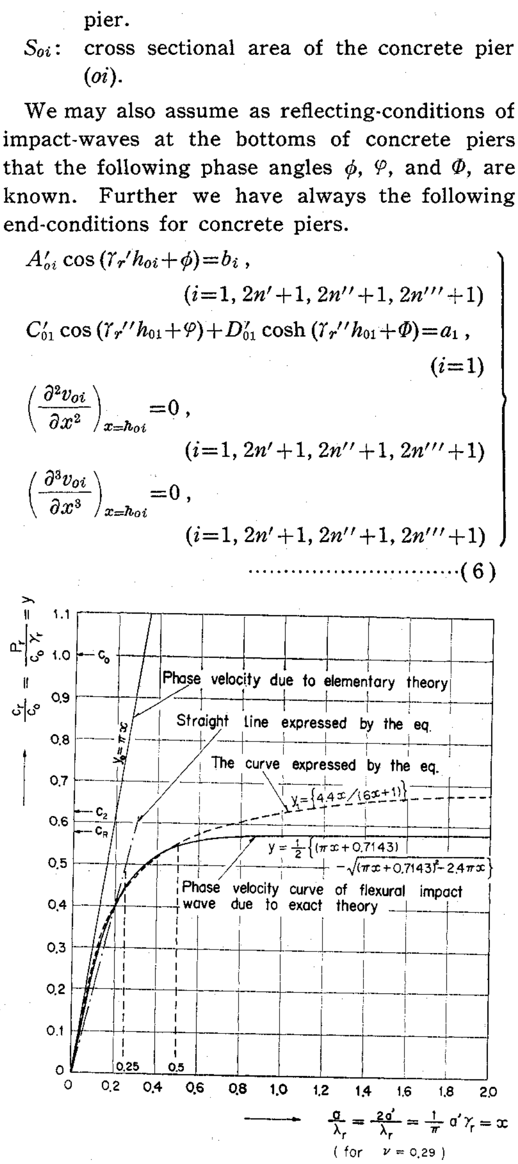


**Fig. 2** Phase Volocity of Flexural Wave and its Approximate Curves (for $\nu=0.29$)

The total number of conditional equations in the Eqs. (4), (5), and (6), is $2(2n'''+1)+3\times 4 = 4n'''+14$, and the number of unknown constants in these equations is $2(2n'''+1)+4(6-3)= 4n'''+14$; namely the number of conditional equations is equal to the number of unknown constants. We can see also that the unknown constants, $a_i$, $b_i$, $A'_{oi}$, $C'_{oi}$, $D'_{oi}$, in the Eqs. (4), (5), and (6), are contained in these equations in linear form.

Thus, eliminating these unknown constants, $a_i$, $b_i$, $A'_{oi}$, $C'_{oi}$, $D'_{oi}$, from the Eqs. (4), (5), and (6), we are able to obtain a determinant equation which can be expressed as an equation regarding to a normal angular frequency $p_r$, and which is thus called as 'normal frequency equation'.

Substituting the value of $p_r$ obtained from the above determinant equation into the $4n'''+13$ equations out of the Eqs. (4), (5), and (6), we are also able to require the ratios, $a_i:b_i:A'_{oi}:C'_{oi}:D'_{oi}:\cdots$.

The values of $p_r$ or $\gamma_r$ and the ratios of the constants, $a_i:b_i:A'_{oi}:C'_{oi}:D'_{oi}:\cdots$, obtained above, are those which correspond only to a normal (an original) natural angular-frequency $p_r$; and the number of $r$ of $p_r$ which are required by the determinant equation, is generally infinity. In practical case, we have to satisfy ourselves by adopting only a first few values out of the infinite number of values of $p_r$ or $\gamma_r$.

The number of values of $p_r$ or $\gamma_r$ to be adopted in the determinant equation, i.e. the number of degree of the determinant equation in practical cases, seems generally to be almost enough with the number which is equal to the number of member-bars in the truss-bridge.

## (2) Lagrange's Equation and Coupled Frequencies

In the next place, we must constitute Lagrange's equation in order to require the actual coupled-angular-frequency (or actual structural-angular-frequency) $w_j$ which is resulted by the coupling vibration of respective normal-angular-frequency $p_r$.

First of all, the kinetic energy per unit volume of this truss-bridge (steel) is equal to

$$\left(\frac{1}{2}\rho\alpha_{ir}\alpha_{is}\dot{q}_r\dot{q}_s\right)_{\text{truss}} = \frac{1}{2}\{(A_{ijr}\cos\gamma_r x+B_{ijr}\sin\gamma_r x)\cdot(A_{ijs}\cos\gamma_s x+B_{ijs}\sin\gamma_s x)+(C_{ijr}+D_{ijr}x)(C_{ijs}+D_{ijs}x)\}\rho\dot{q}_r\dot{q}_s \quad\cdots\cdots(7)$$

where

$A_{ijr}, B_{ijr}, C_{ijr}, D_{ijr}$: denote respectively the values of $A_{ij}$, $B_{ij}$, $C_{ij}$, and $D_{ij}$, which are expressed by the constants, $a_{ir}$, $b_{ir}$, $a_{jr}$, and $b_{jr}$, corresponding to the $r$-th propagation constant $\gamma_r$, or the $r$-th normal angular-frequency $p_r$.

$A_{ijs}, B_{ijs}, C_{ijs}, D_{ijs}$: denote respectively the similar values of $A_{ij}$, $B_{ij}$, $C_{ij}$, and $D_{ij}$, to the above corresponding to the $s$-th propagation constant $\gamma_s$ or the $s$-th normal (original) angular-frequency $p_s$.

The kinetic energy of a mass $m_u$ or of a mass $m_l$ is equal respectively to

$$\left.\begin{aligned}&\frac{1}{2}m_u(a_{jr}a_{js}+b_{jr}b_{js})\dot{q}_r\dot{q}_s\\&\frac{1}{2}m_l(a_{ir}a_{is}+b_{ir}b_{is})\dot{q}_r\dot{q}_s\end{aligned}\right\} \quad\cdots\cdots(8)$$

The kinetic energy per unit volume of a pier $(oi)$ is equal to

$$\frac{1}{2}(\rho'\alpha_{ir}\alpha_{is}\dot{q}_r\dot{q}_s)_{\text{pier }(oi)} = \frac{1}{2}[A'_{oir}\cos(\gamma_r'x+\phi)\cdot A'_{ois}\cos(\gamma_s'x+\phi) + \{C'_{oir}\cos(\gamma_r''x+\varphi)+D'_{ois}\cosh(\gamma_r''x+\Phi)\}\cdot\{C'_{ois}\cos(\gamma_s''x+\varphi)+D'_{ois}\cosh(\gamma_s''x+\Phi)\}]\cdot\rho'\dot{q}_r\dot{q}_s \quad\cdots\cdots(9)$$

Thus the total kinetic energy of the whole structure constituted by truss and piers, is approximately equal to

$$\frac{1}{2}\dot{q}_r\dot{q}_s\Bigg[\rho\sum^{ij}\int_0^{l_{ij}}S_{ij}\{(A_{ijr}\cos\gamma_r x+B_{ijr}\sin\gamma_r x)(A_{ijs}\cos\gamma_s x+B_{ijs}\sin\gamma_s x)+(C_{ijr}+D_{ijr}x)(C_{ijs}+D_{ijs}x)\}dx$$
$$+m_u\sum_{j=2}^{2n'''\,(\text{even})}(a_{jr}a_{js}+b_{jr}b_{js})+m_l\sum_{i=1}^{2n'''+1\,(\text{odd})}(a_{ir}a_{is}+b_{ir}b_{is})$$
$$+\rho'\sum^{i=1,\,2n'+1,\,2n''+1,\,2n'''+1}\int_0^{h_{oi}}S_{oi}\{A'_{oir}\cos(\gamma_r'x+\phi)\cdot A'_{ois}\cos(\gamma_s'x+\phi)$$
$$+(C'_{oir}\cos(\gamma_r''x+\varphi)+D'_{oir}\cosh(\gamma_r''+\Phi))(C'_{ois}\cos(\gamma_s''x+\varphi)+D'_{ois}\cosh(\gamma_s''x+\Phi))\}dx\Bigg]$$

and $b_{rs}$ in the equation (14) becomes

$$b_{rs}=\rho\sum^{ij}\int_0^{l_{ij}}S_{ij}\{(A_{ijr}\cos\gamma_r x+B_{ijr}\sin\gamma_r x)(A_{ijs}\cos\gamma_s x+B_{ijs}\sin\gamma_s x)+(C_{ijr}+D_{ijr}x)(C_{ijs}+D_{ijs}x)\}dx$$
$$+m_u\sum_{j=2}^{2n'''(\text{even})}(a_{jr}a_{js}+b_{jr}b_{js})+m_l\sum_{i=1}^{2n'''+1(\text{odd})}(a_{ir}a_{is}+b_{ir}b_{is})$$
$$+\rho'\sum^{i=1,2n'+1,2n''+1,2n'''+1}\int_0^{h_{oi}}S_{oi}\{A'_{oir}\cos(\gamma_r'x+\phi)\cdot A'_{ois}\cos(\gamma_s'x+\phi)$$
$$+(C'_{oir}\cos(\gamma_r''x+\varphi)+D'_{oir}\cosh(\gamma_r''x+\Phi))(C'_{ois}\cos(\gamma_s''x+\varphi)+D'_{ois}\cosh(\gamma_s''x+\Phi))\}dx \quad \cdots\cdots(10)$$

where

$S_{ij}, S_{oi}$: denote the cross-sectional areas of the member-bar $(ij)$ and of a pier $(oi)$ respectively.

Secondly, the strain energy per unit volume of this truss-bridge (steel) is equal to

$$\frac{1}{2}\rho c_1^2 q_r q_s\left\{\frac{\partial}{\partial x}(A_{ijr}\cos\gamma_r x+B_{ijr}\sin\gamma_r x)\right\}$$
$$\cdot\left\{\frac{\partial}{\partial x}(A_{ijs}\cos\gamma_s x+B_{ijs}\sin\gamma_s x)\right\}$$
$$=\frac{1}{2}\rho c_1^2 q_r q_s\{(A_{ijr}\sin\gamma_r x-B_{ijr}\cos\gamma_r x)$$
$$\cdot(A_{ijs}\sin\gamma_s x-B_{ijs}\cos\gamma_s x)\gamma_r\gamma_s\} \quad \cdots\cdots(11)$$

where

$$c_1^2=\frac{2G}{\rho}\left(\frac{1-\nu}{1-2\nu}\right)$$

The strain energy of a pier $(oi)$ is equal to

$$\frac{1}{2}\rho' c_1'^2 q_r q_s\int_0^{h_{oi}}S_{oi}\left\{\frac{\partial}{\partial x}A'_{oir}\cos(\gamma_r'+\phi)\right.$$
$$\left.\cdot\frac{\partial}{\partial x}A'_{ois}\cos(\gamma_s'x+\phi)\right\}dx$$
$$+\frac{1}{2}E'q_r q_s\int_0^{h_{oi}}I'\frac{y_1(a'\gamma_r'')}{y_0(a'\gamma_r'')}\cdot\frac{y_1(a'\gamma_s'')}{y_0(a'\gamma_s'')}$$
$$\cdot\left\{\frac{\partial^2 Y(\gamma_r''x)}{\partial x^2}\right\}\left\{\frac{\partial^2 Y(\gamma_s''x)}{\partial x^2}\right\}dx$$
$$=\frac{1}{2}q_r q_s\left[\rho' c_1'^2\int_0^{h_{oi}}S_{oi}\left\{A'_{oir}\sin(\gamma_r'x+\phi)\right.\right.$$
$$\left.\cdot A'_{ois}\sin(\gamma_s'x+\phi)\cdot\frac{\partial\gamma_r'}{\partial x}\cdot\frac{\partial\gamma_s'}{\partial x}\right\}dx$$
$$+E'\int_0^{h_{oi}}I'\frac{19.36}{(6a'\gamma_r''+\pi)(6a'\gamma_s''+\pi)}$$
$$\cdot\left\{\frac{\partial^2 Y(\gamma_r''x)}{\partial x^2}\right\}\left\{\frac{\partial^2 Y(\gamma_s''x)}{\partial x^2}\right\}dx \quad \cdots\cdots(12)$$

where

$$Y(\gamma_r''x)=v_{oi}(\gamma_r''x)=C'_{oir}\cos(\gamma_r''x+\varphi)+D'_{oir}\cosh(\gamma_r''x+\Phi)$$
$$Y(\gamma_s''x)=v_{oi}(\gamma_s''x)=C'_{ois}\cos(\gamma_s''x+\varphi)+D'_{ois}\cosh(\gamma_s''x+\Phi)$$

$a'$: radius of gyration of the cross-section of a pier $(oi)$.

Thus $K_{rs}$ in the equation (14) becomes

$$K_{rs}=\rho c_1^2\sum^{ij}\int_0^{l_{ij}}S_{ij}\{(A_{ijr}\sin\gamma_r x-B_{ijr}\cos\gamma_r x)(A_{ijs}\sin\gamma_s x-B_{ijs}\cos\gamma_s x)\gamma_r\gamma_s\}dx$$
$$+\rho' c_1'^2\sum^{i=1,2n'+1,2n''+1,2n'''+1}\int_0^{h_{oi}}S_{oi}\left\{A'_{oir}\sin(\gamma_r'x+\phi)\cdot A'_{ois}\sin(\gamma_s'x+\phi)\frac{\partial\gamma_r'}{\partial x}\cdot\frac{\partial\gamma_s'}{\partial x}\right\}dx$$
$$+E'\sum^{i=1,2n'+1,2n''+1,2n'''+1}\int_0^{h_{oi}}I'\frac{19.36}{(6a'\gamma_r''+\pi)(6a'\gamma_s''+\pi)}\left\{\frac{\partial^2 Y(\gamma_r''x)}{\partial x^2}\right\}\left\{\frac{\partial^2 Y(\gamma_s''x)}{\partial x^2}\right\}dx \quad \cdots\cdots(13)$$

where

$E'$: Young's modulus of concrete-pier.

$I'$: 2nd moment of inertia of the cross-section of a concrete-pier.

We are now able to constitute Lagrange's equation for this whole structure and to require the actual coupled angular-frequency $w_j$, which is resulted by the coupling of respective normal angular-frequency $p_r$. The Lagrange's equation for natural coupled-vibration can be written as

$$\left.\begin{aligned}&b_{rs}\ddot{q}_s+2\zeta b_{rs}\dot{q}_s+K_{rs}q_s=-\eta b_{ir}b_{is}\dot{q}_s\\ \therefore\ &b_{rs}\ddot{q}_s+(2\zeta b_{rs}+\eta b_{ir}b_{is})\dot{q}_s+K_{rs}q_s=0\end{aligned}\right\}\cdots(14)$$

where

$\zeta$: coefficient of internal viscous-damping.

$\eta$: coefficient of viscous friction (force) of dash-pot damper.

$b_{ir}, b_{is}$: vertical displacement-amplitude at the connecting-point $i$ due to $r$-th or $s$-th normal mode of vibration respectively.

It is to be noticed that, if we substitute a mass $m_l$ (or $m_u$) for floor beam and stringer (or for upper lateral beam) as above, then the strain energy in the floor-beam and the stringer (or in the upper lateral beam) will inevitably be neglected and only the lumped mass of their distributed masses will be taken into consideration.

In the case when externel forces as shown in

Fig. 1 act at the bottom of piers, the Lagrange's equation for the case of this forced vibration becomes

$$b_{rs}\ddot{q}_s+(2\zeta b_{rs}+\eta b_{ir}b_{is})\dot{q}_s+K_{rs}q_s=C_r \exp(imt) \quad \cdots\cdots(15)$$

where

$C_r$ : denotes complex number and $C_r \exp(imt)$ denotes

$$\begin{aligned} C_r \exp(imt) &= (a_{or}F'+b_{or}F'')\exp(imt) \\ &+(a'_{or}F'+b'_{or}F'')\exp\{i(mt+\varphi')\} \\ &+(a''_{or}F'+b''_{or}F'')\exp\{i(mt+\varphi'')\} \\ &+(a'''_{os}F'+b'''_{or}F'')\exp\{i(mt+\varphi''')\} \end{aligned}$$

$a'_{or}$ : horizontal displacement-amplitude at the bottom of the pier $(o, 2n'+1)$ due to $r$-th normal mode of vibration.

$b'_{or}$ : vertical displacement-amplitude at the bottom of the pier $(o, 2n'+1)$ due to $r$-th normal mode of vibration.

The method to require the solution of the above equation (15) is as shown in author's paper, Bibliography 27).

## 3. COUPLED TORSIONAL MOTION OF SLAB WITH TRANSVERSAL VIBRATION OF TRUSS

If the phase angles of transversal vibrations of left-side and right-side trusses are different from each other by the phase angle $\pi$ then it is considered that coupled torsional vibrations of slab with transversal vibrations of trusses will be caused.

The torsional vibration of slab may be caused independently by wind fluttering-action and may be also accompanied in most cases with deflective motion of truss-bridge in the lateral direction perpendicular to truss-plane. However we consider here only a coupled torsional motion of slab with transversal vibration which is due to vertical loads with phase difference of $\pi$ in left side and right side of cross-section of slab.

We may suppose that the torsional motion of slab may cause a rotational motion of a frame of cross-section of truss bridge which is constituted by lateral member-bars and the member-bars in truss-plane, and thus the motion of resistive distortion of the frame of cross-section of truss-bridge scarcely may happen. Namely we may suppose that the torsional vibration in this case will be almost due to the resistive torsional-stress of the slab of the truss-bridge.

We are able to express the normal functions of $k$-span of the continuous concrete-slab in torsional vibration and of the pier $(oi)$ in lateral vibration in the direction perpendicular to truss-plane, all of which correspond to a normal angular frequency $p_r$, as

$$\Theta_k(\gamma_r'''z)=A_k\cos(\gamma_r'''z)+B_k\sin(\gamma_r'''z) \quad \cdots\cdots(16)$$

$$w_{oi}=C''_{oi}\cos(\gamma_r'x+\varphi)+D''_{oi}\cosh(\gamma_r'x+\Phi) \quad \cdots\cdots(17)$$

where

$\Theta_k(\gamma_r'''z)$: Angular deflection at a cross-section $z$ in $k$-span of concrete-slab due to torsion.

$k=1, 2, 3$

$\gamma_r'''$: Propagation constant of $r$-th normal of mode torsional vibration of concrete-slab.

$p_r=c_2\gamma_r'''$: Original normal-angular-frequency of $r$-th normal mode of vibration.

$c_2$ : Phase velocity of torsional impact-wave in concrete-slab.

$z$ : Distance from the left support along the length of each span of concrete-slab.

$w_{oi}$ : Lateral deflection of pier $(oi)$ in the direction perpendicular to the palne of truss.

$i=1, 2n'+1, 2n''+1, 2n'''+1$

Then we shall obtain as the end-conditions for the above normal functions, (16) and (17), the following 14 equations.

$$\left.\begin{aligned}
&\Theta_1(0)=\left(\frac{\partial w_{01}}{\partial x}\right)_{x=h_{01}} \\
&\Theta_1(\gamma_r'''L_1)=\Theta_2(0)=\left(\frac{\partial w_{0,2n'+1}}{\partial x}\right)_{x=h_{0,2n'+1}} \\
&\Theta_2(\gamma_r'''L_2)=\Theta_3(0)=\left(\frac{\partial w_{0,2n''+1}}{\partial x}\right)_{x=h_{0,2n''+1}} \\
&\Theta_3(\gamma_r'''L_3)=\left(\frac{\partial w_{0,2n'''+1}}{\partial x}\right)_{x=h_{0,2n'''+1}} \\
&G'I_z\left[\frac{\partial\{\Theta_1(\gamma_r'''z)\}}{\partial z}\right]_{z=0}=E'I'\left(\frac{\partial^2 w_{01}}{\partial x^2}\right)_{x=h_{01}} \\
&G'I_z\left[\frac{\partial\{\Theta_2(\gamma_r'''z)\}}{\partial z}\right]_{z=L_1}-G'I_z\left[\frac{\partial\{\Theta_2(\gamma_r'''z)\}}{\partial z}\right]_{z=0} \\
&\quad+E'I'\left(\frac{\partial^2 w_{0,2n'+1}}{\partial x^2}\right)_{x=h_{0,2n'+1}}=0 \\
&G'I_z\left[\frac{\partial\{\Theta_3(\gamma_r'''z)\}}{\partial z}\right]_{z=L_2}-G'I_z\left[\frac{\partial\{\Theta_3(\gamma_r'''z)\}}{\partial z}\right]_{z=0} \\
&\quad+E'I'\left(\frac{\partial^2 w_{0,2n''+1}}{\partial x^2}\right)_{x=h_{0,2n''+1}}=0 \\
&G'I_z\left[\frac{\partial\{\Theta_3(\gamma_r'''z)\}}{\partial z}\right]_{z=L_3} \\
&\quad=-E'I'\left(\frac{\partial^2 w_{0,2n'''+1}}{\partial x^2}\right)_{x=h_{0,2n'''+1}}
\end{aligned}\right\}$$

$$\left(\frac{\partial^3 w_{01}}{\partial x^3}\right)_{x=h_{01}}=\left(\frac{\partial^3 w_{0,2n'+1}}{\partial x^3}\right)_{x=h_{0,2n'+1}}=\left(\frac{\partial^3 w_{0,2n''+1}}{\partial x^3}\right)_{x=h_{0,2n''+1}}=\left(\frac{\partial^3 w_{0,2n'''+1}}{\partial x^3}\right)_{x=h_{0,2n'''+1}}=0 \quad \cdots\cdots(18)$$

where

$G'$: Modulus of rigidity of concrete material.
$E'$: Young's modulus of concrete material.
$I_z$ : 2nd moment of cross-section of concrete-slab around the $z$-axis of concrete-slab.
$I'$ : 2nd moment of cross-section of concrete pier around the $y$-axis in a plane of cross-section of concrete pier.
$L_1, L_2, L_3$: Lengths of the 1st span, the 2nd span, and the 3rd span, of continuous concrete-slab from left to right in this order respectively.

We can see that the number of unknown constants, $A_k$, $B_k$, $C''_{oi}$, and $D''_{oi}$ ($k=1, 2, 3$; $i=1, 2n'+1, 2n''+1, 2n'''+1$), which has increased by the consideration of torsional motion of slab connected with lateral vibration of truss, is equal to the number of equation (17). Thus we are able to require all the ratio of unknown constants and the value of original angular frequency $p_r$ from the end-conditions in this case when the coupled torsional vibration of concrete-slab was also taken into consideration.

The kinetic energies of the slab (concrete) and of the piers which should be added to Lagrange's equation in this case are

$$\left.\begin{aligned} T_{\text{slab}}&=\frac{1}{2}\rho'\dot{q}_r\dot{q}_s I_z\left[\sum^{k=1,2,3}\int_0^{L_k}\{\Theta_k(\gamma_r'''z)\}\{\Theta_k(\gamma_s'''z)\}dz\right]\\ T_{\text{pier}}&=\frac{1}{2}\rho'\dot{q}_r\dot{q}_s\left[\sum^{i=1,2n'+1,2n''+1,2n'''+1}\int_0^{h_{oi}}S_{oi}\{w_{oi}(\gamma_r'x)\}\{w_{oi}(\gamma_s'x)\}dx\right]\end{aligned}\right\} \quad \cdots\cdots(19)$$

The strain energies of the slab (concrete) and of the piers which should be added also to Lagrange's equation in this case are

$$\left.\begin{aligned} U_{\text{slab}}&=\frac{1}{2}\rho' q_r q_s c_2^2 I_z\left[\sum^{k=1,2,3}\int_0^{L_k}\frac{\partial\{\Theta_k(\gamma_r'''z)\}}{\partial z}\cdot\frac{\partial\{\Theta_k(\gamma_s'''z)\}}{\partial z}\cdot dz\right]\\ U_{\text{pier}}&=\frac{1}{2}E' q_r q_s\left[\sum^{i=1,2n'+1,2n''+1,2n'''+1}\int_0^{h_{oi}}I'\left\{\frac{\partial^2 w_{oi}(\gamma_r'x)}{\partial x^2}\right\}\left\{\frac{\partial^2 w_{oi}(\gamma_s'x)}{\partial x^2}\right\}dx\right]\end{aligned}\right\} \quad \cdots\cdots(20)$$

It is to be noticed in the above that the flexural vibration of a pier in the direction $w_{oi}$ in this case, which is perpendicular to the truss-plane, is a vibration due to originally antisymmetrical longitudinal S.V. impact wave (See, author's paper; "Phase Velocity of Longitudinal Impact Wave in Solid Prismatical Bar", The Tech. Repts of the Tohoku University, Vol. 40, (1975), No. 2, December.), then we have to adopt the propagation constant $\gamma_r'$ instead of $\gamma_r''$ in the normal function $w_{oi}$ in this case.

In the case of forced vibration when couple forces act on the left-side and on the right-side of cross-section of the slab, we have to add a following term to the right hand side of Lagrange's equation.

$$M\exp(im't)\{\Theta_k(\gamma_r'''z)\}_{z=z'} \quad \cdots\cdots(21)$$

where

$M$: Complex amplitude of external moment.
$m'$: Angular frequency of external moment.
$z'$ : Distance of the point from the left support of $k$-span in longitudinal direction of concrete slab where external couple forces of moment, $M\exp(im't)$, act.

and the Lagrange's equation finally becomes, from the eqs. (14), (15), (19), (20), and from the above eq. (21), generally as

$$b_{rs}\ddot{q}_s+(2\zeta b_{rs}+\eta b_{ir}b_{is})\dot{q}_s+K_{rs}q_s=C_r\exp(imt)+M_r\exp(im't) \quad \cdots\cdots(22)$$

where

$$M_r=M\cdot\{\Theta_k(\gamma_r'''z)\}_{z=z'}$$

Thus the particular solution of the Lagrange's equation in this case becomes generally the equation of the following form

$$q_s=A_s\exp(imt)+B_s\exp(im't) \quad \cdots\cdots(23)$$

where

$$\begin{Bmatrix}A_1\\A_2\\\vdots\\A_n\end{Bmatrix}=\begin{bmatrix}\{-m^2b_{11}+(2\zeta b_{11}+\eta b_{i1}b_{i1})im+K_{11}\} & \{-m^2b_{12}+(2\zeta b_{12}+\eta b_{i1}b_{i2})im+K_{12}\} & \cdots\\ \{-m^2b_{21}+(2\zeta b_{21}+\eta b_{i2}b_{i1})im+K_{21}\} & \{-m^2b_{22}+(2\zeta b_{22}+\eta b_{i2}b_{i2})im+K_{22}\} & \cdots\\ \vdots & & \\ \{-m^2b_{n1}+(2\zeta b_{n1}+\eta b_{in}b_{i1})im+K_{n1}\} & \cdots\cdots & \cdots\end{bmatrix}^{-1}\cdot\begin{Bmatrix}C_1\\C_2\\\vdots\\C_n\end{Bmatrix}$$

$$\begin{Bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{Bmatrix} = \begin{bmatrix} \{-m'^2b_{11}+(2\zeta b_{11}+\eta b_{i1}b_{i1})im'+K_{11}\} & \{-m'^2b_{12}+(2\zeta b_{12}+\eta b_{i1}b_{i2})im'+K_{12}\} & \cdots \\ \{-m'^2b_{21}+(2\zeta b_{21}+\eta b_{i2}b_{i1})im'+K_{21}\} & \{-m'^2b_{22}+(2\zeta b_{22}+\eta q_{i2}b_{i2})im'+K_{22}\} & \cdots \\ \vdots & & \\ \{-m'^2b_{n1}+(2\zeta b_{n1}+\eta b_{in}b_{i1})im'+K_{n1}\} & \cdots\cdots & \cdots \end{bmatrix}^{-1} \cdot \begin{Bmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{Bmatrix}$$

## 4. COMPARISON OF THE RESULTS CALCULATED BY AUTHOR'S METHOD AND BY TRADITIONAL METHOD WITH THE EXPERIMETAL RESULTS

For the present purpose as to compare simply the accuracy of author's method with that of traditional methods, it is considered to be enough to compare the results calculated by these methods with the experimental results for a model of simple truss-bridge as shown in Fig. 3 (steel, $\rho=7.85\,\text{kg/cm}^2$), though this is not the example of the vibration of continuous truss-bridge on many piers.

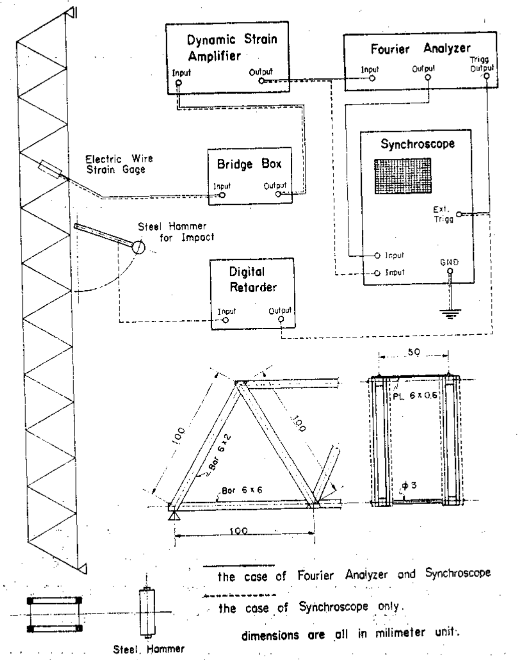


**Fig. 3** Schematic illustration of the truss which was used for numerical calculation of natural frequencies, and its experimental device which was used in the experimental measurement of the frequencies of truss vibration.

The following Table 1 shows only the final results obtained by these methods using electronic computer, while the detail explanation of the process of which was omitted here on account of limited paper-space. The values of 1st column are due to author's method mentioned above in this paper, the values of 2nd column are due to traditional modelling-method to lump up the distributed masses of member-bars to a few connecting points of member-bars, the values of 3rd column are due to the elementary method of lateral vibration assumed as a simple solid-beam which has a mean moment of inertia of cross-sections of the truss-bridge, and the values of 4th column are due to experimental results.

## 5. CONCLUSION

We can conclude from the Table 1 that in the calculation of vibration-frequency of truss-bridge the author's method has far more accuracy than the traditional modelling-method to lump up the distributed masses of member-bars to a few connecting points of member-bars of truss-bridge.

The reason why fairly lower frequency than actual one is obtained by the traditional modelling-method which lumps up the distributed masses, is considered that the distributed masses of diagonal members of truss-bridge are lumped up to the upper and lower connecting points of

**Table 1**

| Coupled natural angular frequency | Author's method | Traditional method due to particle and spring system | Lateral vibration method as a simple solid beam | Experimental results |
|---|---|---|---|---|
| $w_1$ | 2,085 | 1,025 | 1,945 | 2,200 |
| $w_2$ | 6,342 | 4,428 | 7,780 | 6,500 |
| $w_3$ | 7,306 | 6,203 | 17,505 | ... |
| $w_4$ | 15,550 | 9,486 | 31,120 | ... |
| $w_5$ | 21,094 | 13,440 | 48,625 | ... |
| ... | ... | ... | ... | ... |

member bars where the motion in transversal vibration of the truss-bridge is fairly large and thus $b_{rs}$ due to kinematic energy is conspicuously increased but $K_{rs}$ due to strain energy remains constant.

We can see that the traditional modelling method which lumps up the distributed masses gives rather lower accuracy for the first and second coupled-frequencies than the lateral vibration method assumed as a solid-beam, and then it is recommended that it is rather better for low frequency to use the lateral vibration method assumed as a solid-beam than the traditional modelling method to lump up the distributed masses. But the lateral vibration method assumed as a solid-beam gives very large positive error in higher frequency.

This is considered due to the reason that all the coupling modes become to be neglected in the elementary theory of lateral vibration of a solid beam, on account of the orthogonality of normal functions; while the actual configuration in vibration of such a beam becomes, so to speak, much softer in high frequency by couplings between normal modes and between members than that of a solid beam of constant depth.

We can see further through all the range of frequency that the traditional modelling method which lumps up the distributed masses has always a tendency to give lower frequencies than actual ones.

The author makes finally an additional remarks that this method mentioned above is also applicable for any vibration problem of statically indeterminate frame-works.

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