PROC. OF JSCE,
No. 260, APRIL 1977

PHASE VELOCITY OF LONGITUDINAL IMPACT WAVE IN SOLID ELASTIC BAR

By Torao TAYA*

SYNOPSIS

The problem of phase velocity of longitudinal impact wave in a solid elastic bar has a very important significance as a basic theory to determine the natural frequency of vibrations in various engineering frame-warks.

The exact theory as to the phase velocity of longitudinal impact wave in an elastic bar which has been hitherto approved is the Pochhammer-Chree theory.

However the author considers that the traditional Pochhammer-Chree theory is not correct and that a large error not tolerable even in engineering sense will be inevitably included in the analytical result of natural frequency obtained by the application of traditional theory.

This paper is the summary of author's presented paper to the 14th IUTAM Congress 1976 (Congress 1976 of the International Union of Theoretical and Applied Mechanics) in Delft, the Netherlands. However the author has paid all the effort not to omit as far as possible the essential parts necessary for readers to understand the outline of his theory.

1. INTRODUCTION: DISPLACEMENT VECTOR FIELD AND THE SOLUTION OF NAVIER'S EQUATION

The displacement vector field $\boldsymbol{u}$ is classified into two parts, one of which has no rotation and the other has rotation, namely

$$\boldsymbol{u}=(\boldsymbol{u}_0)+(\boldsymbol{u}') \quad \cdots(1)$$

The former displacement $(\boldsymbol{u}_0)$ with no rotation has a scalar potential ϕ, namely

$$\text{rot.}(\boldsymbol{u}_0)=0 \quad \cdots(2)$$

$$\therefore \quad (\boldsymbol{u}_0)=\text{grad.}\,\phi \quad \cdots(3)$$

We may take $(\boldsymbol{u}_0)$ so as to satisfy one more condition

$$\text{div.}(\boldsymbol{u}_0)=\text{div.}\,\boldsymbol{u} \quad \cdots(4)$$

If we denote the residual displacement obtained by subtracting $(\boldsymbol{u}_0)$ from $\boldsymbol{u}$ as $(\boldsymbol{u}')$, then we have

$$\text{div.}(\boldsymbol{u}')=0 \quad \cdots(5)$$

$$\text{rot.}(\boldsymbol{u}')=\text{rot.}\,\boldsymbol{u} \quad \cdots(6)$$

$$\boldsymbol{u}=(\boldsymbol{u}_0)+(\boldsymbol{u}')$$

This latter displacement $(\boldsymbol{u}')$ has a vector potential $\boldsymbol{\Psi}$, namely

$$(\boldsymbol{u}')=\text{rot.}\,\boldsymbol{\Psi} \quad \cdots(7)$$

Therefore we can see finally that any displacement vector field $\boldsymbol{u}$ is always classified into two parts and expressed as

$$\boldsymbol{u}=\text{grad.}\,\phi+\text{rot.}\,\boldsymbol{\Psi} \quad \cdots(8)$$

It is to be noticed that there is one displacement-vector $\boldsymbol{u}_0'$ which has both natures of $(\boldsymbol{u}_0)$ and of $(\boldsymbol{u}')$, namely

$$\left.\begin{array}{l}\text{rot.}\,\boldsymbol{u}_0'=0\\ \text{div.}\,\boldsymbol{u}_0'=0\\ \boldsymbol{u}_0'=\text{grad.}\,\phi'=\text{rot.}\,\boldsymbol{\Psi}'\end{array}\right\} \quad \cdots(9)$$

This displacement $\boldsymbol{u}_0'$ is included in both grad. ϕ and rot. $\boldsymbol{\Psi}$. We denote the displacement obtained by subtracting one part of $\boldsymbol{u}_0'$, i.e. $(\boldsymbol{u}_0')_1$, from $(\boldsymbol{u}_0)$ as *pure* $\boldsymbol{u}_0$ and also the displacement obtained by subtracting $(\boldsymbol{u}_0')_2=\boldsymbol{u}_0'-(\boldsymbol{u}_0')_1$ from $(\boldsymbol{u}')$ as *pure* $\boldsymbol{u}'$. The displacement of *pure* $\boldsymbol{u}_0$ is the solution which is included *only* in *grad.* ϕ and not included in rot. $\boldsymbol{\Psi}$. On the other hand, the displacement of *pure* $\boldsymbol{u}'$ is the solution which is included *only* in *rot.* $\boldsymbol{\Psi}$ and not included in *grad.* ϕ.

If we substitute $\boldsymbol{u}_0'$, $(\boldsymbol{u}_0')_1$, and $(\boldsymbol{u}_0')_2$ instead of $\boldsymbol{u}$, $(\boldsymbol{u}_0)$, and $(\boldsymbol{u}')$ respectively in the governing equation of wave motion (Navier's equation),

$$\rho\frac{\partial^2}{\partial t^2}\boldsymbol{u}=2G\left(\frac{1-\nu}{1-2\nu}\right)\nabla^2(\boldsymbol{u}_0)+G\nabla^2(\boldsymbol{u}') \quad \cdots(10)$$

then we have

$$\rho\frac{\partial^2}{\partial t^2}\boldsymbol{u}_0'=2G\left(\frac{1-\nu}{1-2\nu}\right)\nabla^2(\boldsymbol{u}_0')_1+G\nabla^2(\boldsymbol{u}_0')_2$$

* Dr. Eng., Professor, Department Civil Engineering, Tohoku University.

The right hand side of the above equation becomes always equal to zero; because, denoting any vector as $\boldsymbol{V}$, the following equation always holds true

$$\nabla^2 \boldsymbol{V} = \text{grad. div. } \boldsymbol{V} - \text{rot. rot. } \boldsymbol{V} \quad \cdots\cdots\cdots\cdots(11)$$

and further in the special case of $(\boldsymbol{u}_0')_1$ or $(\boldsymbol{u}_0')_2$, both div. $(\boldsymbol{u}_0')_1$ and rot. $(\boldsymbol{u}_0')_1$ or both div. $(\boldsymbol{u}_0')_2$ and rot. $(\boldsymbol{u}_0')_2$ are equal to zero.

Thus we can see

$$\frac{\partial^2}{\partial t^2}\boldsymbol{u}_0' = 0 \quad \cdots\cdots\cdots\cdots(12)$$

Namely we can see that $\boldsymbol{u}_0'$ is a displacement with no acceleration and thus it exists generally as a statical displacement. In other words, the displacement $\boldsymbol{u}_0'$ exists only as a Fourier element-wave of zero frequency or of infinite wavelength. It has not actual meaning to consider the phase velocity of D.C. element-wave, and thus we must exclude the displacement $\boldsymbol{u}_0'$ from the resultant displacement $\boldsymbol{u}$.

The original Navier's equation (10) can be rewritten in the following form:

$$\left\{\frac{\partial^2}{\partial t^2}(\text{grad. }\phi) - c_1^2\nabla^2(\text{grad. }\phi)\right\} + \left\{\frac{\partial^2}{\partial t^2}(\text{rot. }\boldsymbol{\Psi}) - c_2^2\nabla^2(\text{rot. }\boldsymbol{\Psi})\right\} = 0 \quad \cdots\cdots(13)$$

where

$$c_1^2 = \frac{2G}{\rho}\left(\frac{1-\nu}{1-2\nu}\right), \qquad c_2^2 = \frac{G}{\rho}$$

G: modulus of rigidity of the material

ρ: specific gravity of the material

ν: Poisson's ratio

∇^2: Laplacian

If we eliminate $\boldsymbol{\Psi}$ from the above equation (13) by taking divergence of both sides of the equation, we obtain

$$\frac{\partial^2}{\partial t^2}\nabla^2\phi - c_1^2\nabla^4\phi = 0 \quad \cdots\cdots\cdots\cdots(14)$$

and if we eliminate ϕ from the equation (13) by taking rotation of both sides of the equation, then we have

$$\frac{\partial^2}{\partial t^2}\nabla^2\boldsymbol{\Psi} - c_2^2\nabla^4\boldsymbol{\Psi} = 0 \quad \cdots\cdots\cdots\cdots(15)$$

where we adopt the vector potential $\boldsymbol{\Psi}$ so as to satisfy one more condition

$$\text{div. }\boldsymbol{\Psi} = 0 \quad \cdots\cdots\cdots\cdots(16)$$

The above equations (14), (15) and (16) are the necessary equations but not necessarily the sufficient equations. We must generally exclude the displacement $(\boldsymbol{u}_0')_1$ and $(\boldsymbol{u}_0')_2$ from the solutions of the equtions (14), (15) and (16) as already mentioned above.

2. ANALYSIS OF THE PHASE VELOCITY OF LONGITUDINAL IMPACT WAVE IN SOLID PRISMATICAL BAR

Taking the longitudinal center-line of a rectangular prismatical bar as z-axis and also taking the center-lines of a rectangular cross-section as x- and y-axes respectively in cartesian coordinate system as shown in Fig. 1, we may assume the initial displacement vectors in triangular regions in Fig. 2 as follows.

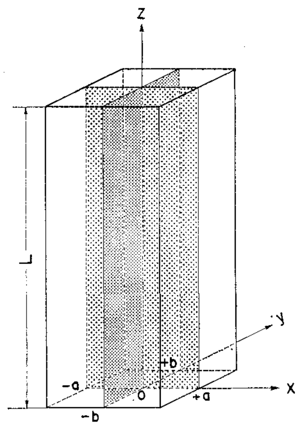

Fig. 1 The coordinate system in a rectangular prismatical bar.

Initial dilatation-wave, $\boldsymbol{u}_0^1$, *in* $x-z$ *plane*:

$(-a \leqq x \leqq 0)$

$$\boldsymbol{u}_0^1 = \left(\begin{Bmatrix}-l_1\\0\\m_1\end{Bmatrix} D_1 \exp[-i\gamma_1 l_1(x+a)] + \begin{Bmatrix}l_1\\0\\m_1\end{Bmatrix} E_1 \exp[i\gamma_1 l_1(x+a)]\right) \cdot i\gamma_1 \exp[i(\gamma z - pt)]$$

$(0 \leqq x \leqq a)$

$$\boldsymbol{u}_0^1 = \left(\begin{Bmatrix}l_1\\0\\m_1\end{Bmatrix} D_1 \exp[i\gamma_1 l_1(x-a)] + \begin{Bmatrix}-l_1\\0\\m_1\end{Bmatrix} E_1 \exp[-i\gamma_1 l_1(x-a)]\right) \cdot i\gamma_1 \exp[i(\gamma z - pt)]$$

$$\cdots\cdots\cdots\cdots(17)$$

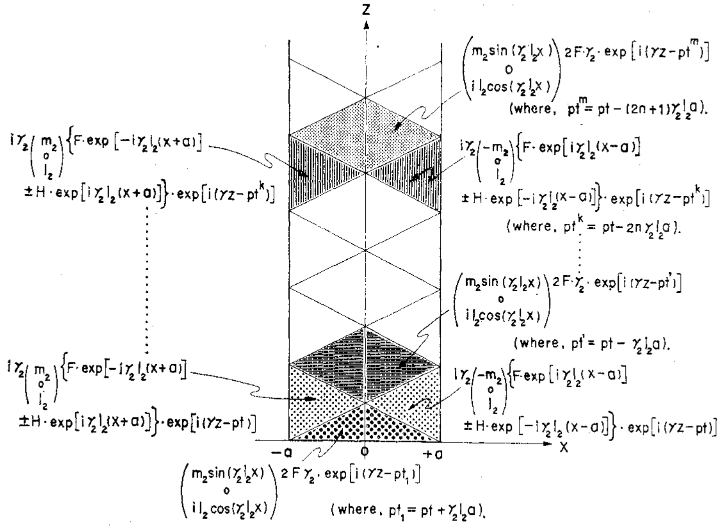

Fig. 2 Illustration showing the regions of various resultant SV rotation-waves in symmetrical case in the interior of a prismatical bar. (The signs ± show x- and z-components respectively.)

Initial dilatation-wave, $\boldsymbol{u}_0^2$, in $y-z$ plane:

$(-b \leqq y \leqq 0)$

$$\boldsymbol{u}_0^2 = \left(\begin{Bmatrix} 0 \\ -l_1 \\ m_1 \end{Bmatrix} D_2 \exp[-i\gamma_1 l_1(y+b)] + \begin{Bmatrix} 0 \\ l_1 \\ m_1 \end{Bmatrix} E_2 \exp[i\gamma_1 l_1(y+b)] \right) \cdot i\gamma_1 \exp[i(\gamma z - pt)]$$

$(0 \leqq y \leqq b)$

$$\boldsymbol{u}_0^2 = \left(\begin{Bmatrix} 0 \\ l_1 \\ m_1 \end{Bmatrix} D_2 \exp[i\gamma_1 l_1(y-b)] + \begin{Bmatrix} 0 \\ -l_1 \\ m_1 \end{Bmatrix} E_2 \exp[-i\gamma_1 l_1(y-b)] \right) \cdot i\gamma_1 \exp[i(\gamma z - pt)] \qquad \cdots\cdots(18)$$

Initial rotation-wave, $\boldsymbol{u}_1'$, in $x-z$ plane:

$(-a \leqq x \leqq 0)$

$$\boldsymbol{u}_1' = \left(\begin{Bmatrix} m_2 \\ 0 \\ l_2 \end{Bmatrix} F_1 \exp[-i\gamma_2 l_2(x+a)] + \begin{Bmatrix} m_2 \\ 0 \\ -l_2 \end{Bmatrix} H_1 \exp[i\gamma_2 l_2(x+a)] \right) \cdot i\gamma_2 \exp[i(\gamma z - pt)]$$

$(0 \leqq x \leqq a)$

$$\boldsymbol{u}_1' = \left(\begin{Bmatrix} -m_2 \\ 0 \\ l_2 \end{Bmatrix} F_2 \exp[i\gamma_2 l_2(x-a)] + \begin{Bmatrix} -m_2 \\ 0 \\ -l_2 \end{Bmatrix} H_1 \exp[-i\gamma_2 l_2(x-a)] \right) \cdot i\gamma_2 \exp[i(\gamma z - pt)] \qquad \cdots\cdots(19)$$

Initial rotation-wave, $\boldsymbol{u}_2'$, in $y-z$ plane:

$(-b \leqq y \leqq 0)$

$$\boldsymbol{u}_2' = \left(\begin{Bmatrix} 0 \\ m_2 \\ l_2 \end{Bmatrix} F_2 \exp[-i\gamma_2 l_2(y+b)] + \begin{Bmatrix} 0 \\ m_2 \\ -l_2 \end{Bmatrix} H_2 \exp[i\gamma_2 l_2(y+b)] \right) \cdot i\gamma_2 \exp[i(\gamma z - pt)]$$

$(0 \leqq y \leqq +b)$

$$\boldsymbol{u}_2' = \left(\begin{Bmatrix} 0 \\ -m_2 \\ l_2 \end{Bmatrix} F_2 \exp[i\gamma_2 l_2(y-b)] + \begin{Bmatrix} 0 \\ -m_2 \\ -l_2 \end{Bmatrix} H_2 \exp[-i\gamma_2 l_2(y-b)] \right) \cdot i\gamma_2 \exp[i(\gamma z - pt)] \qquad \cdots\cdots(20)$$

$$u=u_0+u'=(u_0^1+u_0^2)+(u_1'+u_2') \cdots\cdots\cdots\cdots(20)$$

The above displacements always satisfy the Navier's equation.

The combinations of the vectors (17), and the second terms in the eq. (19); or (18), and the second terms in eq. (20); satisfy the boundary conditions at the surface planes, $x\mp a$ and $y\mp b$, in longitudinal meridian sections, so long as the constants D_1, E_1, and H_1; or D_2, E_2, and H_2; satisfy the condition

$$\left.\begin{aligned}\frac{E}{D}&=\frac{-(l_2^2-m_2^2)^2m_1+4l_1l_2m_2^3}{(l_2^2-m_2^2)m_1+4l_1l_2m_2^3}\\ \frac{H}{D}&=\frac{-4(l_2^2-m_2^2)l_1m_2^2}{(l_2^2-m_2^2)^2m_1+4l_1l_2m_2^3}\end{aligned}\right\}\cdots\cdots\cdots(22)$$

where D, E and H, denote respectively D_1, E_1, H_1 or D_2, E_2, H_2 or the combinations of the vectors (19), and the second terms in eq. (17); or (20), and the second terms in eq. (18); satisfy the boundary conditions at the surface planes, $x=\mp a$ and $y=\mp b$, in longitudinal meridian sections, so long as the constants F_1, H_1 and E_1; or F_2, H_2 and E_2; satisfy the condition

$$\left.\begin{aligned}\frac{H}{F}&=-\frac{m_1(l_2^2-m_2^2)^2-4l_1l_2m_2^3}{m_1(l_2^2-m_2^2)^2+4l_1l_2m_2^3}\\ \frac{E}{F}&=\frac{4l_2m_1m_2(l_2^2-m_2^2)}{m_1(l_2^2-m_2^2)^2+4l_1l_2m_2^3}\end{aligned}\right\}\cdots\cdots\cdots(23)$$

where F, H, and E, denote respectively F_1, H_1, E_1 or F_2, H_2, E_2, and

$$l_1=\cos\alpha,\quad m_1=\sin\alpha,\quad l_2=\cos\beta,$$
$$m_2=\sin\beta,\quad m_1\gamma_1=m_2\gamma_2=\gamma,$$
$$\frac{m_1}{m_2}=\frac{\gamma_2}{\gamma_1}=\frac{c_1}{c_2}\sqrt{2\left(\frac{1-\nu}{1-2\nu}\right)}.$$

In order that the boundary conditions that the normal stresses

$$\sigma_x=2G\left[\left\{\left(\frac{1-\nu}{1-2\nu}\right)\frac{\partial u_0^1}{\partial x}+\frac{\partial u_1'}{\partial x}+\left(\frac{\nu}{1-2\nu}\right)\frac{\partial w_0^1}{\partial z}\right\}+\frac{\nu}{1-2\nu}\left(\frac{\partial v_0^2}{\partial y}+\frac{\partial w_0^2}{\partial z}\right)\right] \cdots\cdots(24)$$

$$\sigma_y=2G\left[\left\{\left(\frac{1-\nu}{1-2\nu}\right)\frac{\partial v_0^2}{\partial y}+\frac{\partial v_2'}{\partial y}+\left(\frac{\nu}{1-2\nu}\right)\frac{\partial w_0^2}{\partial z}\right\}+\frac{\nu}{1-2\nu}\left(\frac{\partial u_0^1}{\partial x}+\frac{\partial w_0^1}{\partial z}\right)\right] \cdots\cdots(25)$$

are equal to zero at the boundary $x=\pm a$ and $y\pm b$ independently of y and x respectively, it is necessary and sufficient that the following condition holds true,

$$l_1=\cos\alpha=0 \qquad (\because\ m_1=\sin\alpha=1) \cdots\cdots(26)$$

Then in this case it becomes naturally

$$2E_1=2E_2=D/2,$$

and it becomes also naturally

$$u_0=u_0^1+u_0^2=\begin{Bmatrix}u_0\\ v_0\\ w_0\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ 1\end{Bmatrix}i\gamma_1D\exp[i(\gamma_1z-pt)]\cdots\cdots(27)$$

The above dilatation-wave (27) and the rotation-waves (19) and (20), always satisfy the boundary conditions at the surfaces $x=\pm a$ and $y=\pm b$, so long as the constants D, F_1, and H_1; or D, F_2, and H_2; satisfy the condition (23) where $l_1=0$ and $m_1=1$. In this case the condition corresponding to the eq. (23) becomes (when $\nu=0.29$)

$$\left.\begin{aligned}&\frac{H_1}{F_1}=\frac{H_2}{F_2}=-1\\ &\frac{F_1}{D}=\frac{F_2}{D}=\frac{(l_2^2-m_2^2)}{4l_2m_2}=\frac{1}{4.474}=0.2235\\ &\qquad(\nu=0.29 \text{ in steel})\\ &m_2\gamma_2=\gamma_1,\quad \beta=32°57',\quad l_2=0.8391,\\ &m_2=0.5439\\ &\therefore\ F_1=F_2,\quad H_1=H_2\end{aligned}\right\}\cdots\cdots(28)$$

The reflecting rotation-wave at the point of 1st incidence on surface boundary, for instance the reflecting rotation-wave at a point on the left-side surface boundary $x=-a$,

$$\begin{Bmatrix}m_2\\ 0\\ -l_2\end{Bmatrix}H_1\exp[i\gamma_2l_2(x+a)]\cdot i\gamma_2\exp[i(\gamma_1z-pt)]$$

impinges to the right-side surface boundary $x=a$ and becomes to be expressed as

$$\begin{Bmatrix}m_2\\ 0\\ -l_2\end{Bmatrix}H_1\exp[i\gamma_2l_2(x-a)]\cdot i\gamma_2\exp[i(\gamma_1z-pt+2\gamma_2l_2a)]$$
$$=\begin{Bmatrix}m_2\\ 0\\ -l_2\end{Bmatrix}H_1\exp[i\gamma_2l_2(x-a)]\cdot l\gamma_2\exp[i(\gamma_1z-pt')]$$

where

$$pt'=pt-2\gamma_2l_2a,$$

and further at the point of 2nd incidence a new dilatation-wave

$$u_0=\begin{Bmatrix}0\\ 0\\ 1\end{Bmatrix}i\gamma_1D\exp[i(\gamma_1-pt')]$$

and a new reflecting rotation-wave

$$\boldsymbol{u}'=\begin{Bmatrix} -m_2 \\ 0 \\ -l_2 \end{Bmatrix} H_1 \exp[-i\gamma_2 l_2(x-a)] \cdot i\gamma_2 \exp[i(\gamma_1 z-pt)]$$

are produced, and thus the right-side surface-boundary-conditions comes to be satisfied as in the same manner as in the case of 1st incidence of rotation-wave. The similar phenomena as above will successively take place at all points of successive incidence of reflecting rotation-wave, and therefore the surface boundary conditions will be always satisfied by the combination of initial dilatation-wave (27) and initial rotation-waves, (19) and (20).

3. ANALYSIS OF THE PHASE VELOCITY OF LONGITUDINAL IMPACT WAVE IN SOLID CYLINDRICAL BAR

Let's consider a cylindrical coordinate system, the z-axis of which is taken to an optional generating-line of the cylindrical surface of a solid cylindrical bar as shown in Fig. 3. (We shall call hereafter this cylindrical coordinate system as a new cylindrical coordinate system.)

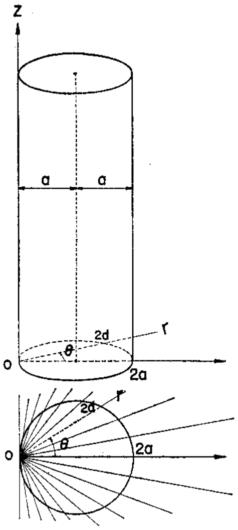

Fig. 3 The coordinate system in the case of zero-order waves of SV family in a circular cylindrical bar.

The scalar potential ϕ and the vector potential $\boldsymbol{\Psi}$ are generally assumed as

$$\phi=U_0(r)\exp[i\{\gamma_1(l_1 r+m_1 z)-pt\}] \quad \cdots\cdots(29)$$

$$\boldsymbol{\Psi}=\begin{Bmatrix} \psi_r \\ \psi_\theta \\ \psi_z \end{Bmatrix}=\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} V(r)\exp[i\{\gamma_2(l_2 r+m_2 z)-pt\}]$$

Substituting the above equations into the Navier's equations (14), (15), and (16) and rejecting the solution coorresponding to the displacement $\boldsymbol{u}_0'$, we obtain the following finite solutions.

$$\phi=K_0 J_0(\gamma_1 l_1 r)\exp[i\{\gamma_1(l_1 r+m_1 z)-pt\}] \quad \cdots\cdots(31)$$

$$\boldsymbol{\Psi}=\begin{Bmatrix} 0 \\ -J_1(\gamma_2 l_2 r) \\ 0 \end{Bmatrix} K\exp[i(\gamma z-pt)] \quad \cdots\cdots(32)$$

where K_0, K: constants

$$\gamma_1=\frac{2\pi}{\lambda_1},\quad \gamma_2=\frac{2\pi}{\lambda_2},\quad \gamma=\frac{2\pi}{\lambda}$$

λ_1, λ_2: wavelengths of dilatation-wave and rotation-wave respectively.
λ: wavelength in z-direction
p: $2\pi f$, f=frequency.

$l_1=\cos\alpha$, $l_2=\cos\beta$, $m_1=\sin\alpha$, $m_2=\sin\beta$

$m_1\gamma_1=m_2\gamma_2=\gamma$

α, β: incident (or reflecting) angle of dilatation-wave and rotation-wave at the surface boundary respectively.

$\frac{p}{\gamma}=c$: (outward) phase velocity in z-direction

J_0, J_1: Bessel function of 0-order and 1st-order respectively.

It is considered naturally from the analytical result in prismatical bar that the propagation-direction of dilatation-wave is actually the z-direc tion as similarly as in prismatical bar. Namely

$$l_1=\cos\frac{\pi}{2}=0,\quad m_1=\sin\frac{\pi}{2}=1$$

$$l_2=\cos 32°57'=0.8391 \qquad (\nu=0.29 \text{ in steel})$$

$$m_2\gamma_2=m_1\gamma_1=\gamma_1=\gamma$$

Then the eqs. (31) and (32) become

$$\phi=K_0\exp[i(\gamma_1 z-pt)] \quad \cdots\cdots(33)$$

$$\boldsymbol{\Psi}=\begin{Bmatrix} \psi_r \\ \psi_\theta \\ \psi_z \end{Bmatrix}=\begin{Bmatrix} 0 \\ -J_1(\gamma_2 l_2 r) \\ 0 \end{Bmatrix} K\exp[i(\gamma_1 z-pt)] \quad \cdots\cdots(34)$$

and the displacement $\boldsymbol{u}$ in this case become

$$\boldsymbol{u}=\left(\begin{Bmatrix} 0 \\ 0 \\ i\gamma_1 \end{Bmatrix} K_0+\begin{Bmatrix} i\gamma_1 J_1(\gamma_2 l_2 r) \\ 0 \\ -\gamma_2 l_2 J_0(\gamma_2 l_2 r) \end{Bmatrix} K\right) \cdot \exp[i(\gamma_1 z-pt)] \quad \cdots\cdots(35)$$

The above displacement vector $\boldsymbol{u}$ was only obtained as a function which satisfies Navier's equation, and thus it is still necessary that the displacement vector is to satisfy the traction-free surface boundary conditions.

$$(\sigma_r)_{r=0}=2G\left\{\frac{\partial u_r}{\partial r}+\left(\frac{\nu}{1-2\nu}\right)\Delta\right\}_{r=0}$$
$$=2G\left\{i\gamma_1 K\frac{\gamma_2 l_2}{2}+\left(\gamma_1^2-\frac{\gamma_2^2}{2}\right)K_0\right\}$$
$$\cdot\exp[i(\gamma_1 z-pt)]$$
$$(\tau_{r\theta})_{r=0}=G\left\{\frac{1}{r}\frac{\partial u_r}{\partial\theta}+r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right\}_{r=0}=0$$
$$(\tau_{rz})_{r=0}=G\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right)_{r=0}=0$$
$$\left[(\tau_{\theta z})_{r=0}=G\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial\theta}\right)_{r=0}=0\right]$$

where in the above

$$\Delta=\text{div.}\,\boldsymbol{u}=\text{div.}\,\boldsymbol{u}_0=\frac{\partial(u_0)_r}{\partial r}+\frac{\partial(u_0)_r}{r}$$
$$+\frac{1}{r}\frac{(u_0)_\theta}{\partial\theta}+\frac{\partial(u_0)_z}{\partial z}$$
$$=-\gamma_1^2 K_0\exp[i(\gamma_1 z-pt)]$$

The above latter three equations are naturally satisfied, and thus the traction-free surface boundary conditions become only the one equation.

$$i\gamma_1 K\frac{\gamma_2 l_2}{2}+\left(\gamma_1^2-\frac{\gamma_2^2}{2}\right)K_0=0$$
$$\therefore\quad\frac{K}{K_0}=\frac{\gamma_2^2-2\gamma_1^2}{i\gamma_1\gamma_2 l_2}$$
$$=\left\{\left(\frac{c_1}{c_2}\right)^2-2\right\}\Big/ i\sqrt{\left(\frac{c_1}{c_2}\right)^2-1}\qquad\cdots\cdots(36)$$

It is to be noticed that $(\sigma_\theta)_{r=0}$, $\theta=\pi/2$, is also the normal stress at the surface boundary which must be equal to zero from the traction-free surface boundary conditions. But since

$$\sigma_\theta=2G\left\{\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)+\frac{\nu}{1-2\nu}\Delta\right\}$$
$$=2G\left\{i\gamma_1 K\frac{1}{r}J_1(\gamma_2 l_2 r)+\frac{\nu}{1-2\nu}(-K_0\gamma_1^2)\right\}$$
$$\cdot\exp[i(\gamma_1 z-pt)]$$
$$\therefore\quad(\sigma_\theta)_{r=0}=\lim_{r=0}(\sigma_\theta)$$
$$=2G\left\{i\gamma_1 K\frac{\gamma_2 l_2}{2}+\left(\gamma_1^2-\frac{\gamma_2^2}{2}\right)K_0\right\}$$
$$\cdot\exp[i(\gamma_1 z-pt)]$$
$$=(\sigma_r)_{r=0}$$

then we obtain the same conditional equation as eq. (36).

We are always able to determine the ratio K/K_0 from the above equation (36), and as a fundamental principle, there is generally no restriction for the absolute value of γ_1 or γ_2.

Therefore, we can understand that the wave expressed by the eq. (35) always satisfies not only the Navier's equation but also the surface boundary conditions at a point on an optional generating-line of the cylindrical surface of a bar, only if the ratio of integral constants, K and K_0, satisfies the condition (36).

But it is further desirable to examine whether the surface boundary conditions at the point of second incidence are surely satisfied or not, when a reflecting-wave at a point on a generating-line of the cylindrical surface of a bar reaches to another point of the cylindrical surface and impinges against that point.

Now, if we consider the Fourier transform of the function $J_0(\gamma_2 l_2 r)\exp[i(\gamma_1 z-pt)]$, then we have

$$A_0(i\gamma_2 l)=\int_{-\infty}^{\infty}J_0(\gamma_2 l_2 r)\cdot\exp[i(\gamma_1 z-pt)]e^{-i\gamma_2 lr}dr$$
$$=\frac{2}{\gamma_2\sqrt{l_2^2-l^2}}\exp[i(\gamma_1 z-pt)]\qquad(l_2^2>l^2)$$
$$\text{or}\qquad 0\qquad\qquad(l_2^2<l^2)$$

where in the above $A_0(i\gamma_2 l)$ denotes the Fourier transform corresponding to the function $J_0(\gamma_2 l_2 r)\cdot\exp[i(\gamma_1 z-pt)]$.

Thus the inverse transform of $A_0(i\gamma_2 l)$ becomes

$$J_0(\gamma_2 l_2 r)\exp[i(\gamma_1 z-pt)]$$
$$=\int_{-\infty}^{\infty}A_0(i\gamma_2 l)e^{i\gamma_2 lr}\frac{\gamma_2}{2\pi}dl$$
$$=\frac{1}{\pi}\int_0^{l_2-0}\frac{1}{\sqrt{l_2^2-l^2}}\{\exp[i(\gamma_2'(l_2'r+m_2'z)-pt)]$$
$$+\exp[i(\gamma_2'(-l_2'r+m_2'z)-pt)]\}dl\cdots\cdots(37)$$

where

$$\gamma_2'=\gamma_2\sqrt{l^2+m_2^2},\quad l_2'=\frac{l}{\sqrt{l^2+m_2^2}},$$
$$m_2'=\frac{m_2}{\sqrt{l^2+m_2^2}}$$

In the same manner as above, the Fourier transform of the function $J_1(\gamma_2 l_2 r)\exp[i(\gamma_1 z-pt)]$ becomes

$$A_1(i\gamma_2 l)=\int_{-\infty}^{\infty}J_1(\gamma_2 l_2 r)\exp[i(\gamma_1 z-pt)]e^{-i\gamma_2 lr}\cdot dt$$
$$=\frac{-2il}{\gamma_2 l_2\sqrt{l_2^2-l^2}}\exp[i(\gamma_1 z-pt)]\qquad(l_2^2>l^2)$$
$$\text{or}\qquad 0\qquad\qquad(l_2^2<l^2)$$

where in the above $A_1(i\gamma_2 l)$ denotes the Fourier transform corresponding to the function $J_1(\gamma_2 l_2 r)\cdot\exp[i(\gamma_1 z-pt)]$.

The inverse transform of $A_1(i\gamma_2 l)$ becomes

$$J_1(\gamma_2 l_2 r)\exp[i(\gamma_1 z-pt)]$$
$$=\int_{-\infty}^{\infty}A_1(i\gamma_2 l)e^{i\gamma_2 lr}\frac{\gamma_2}{2\pi}dl$$

$$=\frac{-i}{l_2\pi}\int_0^{l_2-0}\frac{l}{\sqrt{l_2^2-l^2}}\{\exp[i(\gamma_2'(l_2'r+m_2'z)-pt)]$$
$$-\exp[i(\gamma_2'(-l_2'r+m_2'z)-pt)]\}\,dl \cdots\cdots(38)$$

where γ_2', l_2', and m_2', denote the same quantities as in the eq. (37).

Namely for any value of r, we are always able to express the functions, $J_0(\gamma_2 l_2 r)\exp[i(\gamma_1 z-pt)]$ and $J_1(\gamma_2 l_2 r)\exp[i(\gamma_1 z-pt)]$, as the representative equations, (37) and (38), in the range of $l=(0)\sim(l_2-0)$ respectively.

The first term in the equations, (37) and (38), i.e.,

$$\exp[i\{\gamma_2'(l_2'r+m_2'z)-pt\}],$$

and the second term in the same equations, i.e.,

$$\exp[i\{\gamma_2'(-l_2'r+m_2'z)-pt\}],$$

express respectively the reflecting-wave and the incident-wave at the point $(r=0)$ on a generating line of cylindrical surface of the bar; and at that point $(r=0)$ on surface boundary only the wave

$$\exp[i\{\gamma_2'(l_2'r+m_2'z)-pt\}]$$

reflects and reaches to another point on surface boundary.

If we take the point of second incidence on surface boundary as the origin of new cylindrical coordinate system, and also if we denote the azimuth and length of the circular secant between the two points of 1st incidence and of 2nd incidence in r-direction on surface boundary as θ and $2d$ respectively then l can be expressed as

$$l=l_2\cos\theta \qquad (\theta=90°\sim 0°+\varepsilon)$$

and the reflecting wave at the point of 1st incidence becomes to be expressed in the new cylindrical coordinate system, which has its origin at the point of second incidence on surface boundary, as follows

$$\exp[i\{\gamma_2'(l_2'(2d-r)+m_2'z)-pt\}]$$
$$=\exp[i\{\gamma_2'(-l_2'r+m_2'z)-pt'\}]$$

where $pt'=pt-2\gamma_2'l_2'd=pt-2\gamma_2 ld=pt-2\gamma_2 l_2 d\cos\theta=pt-2\gamma_2 l_2 a\cos^2\theta$.

Then at the point of second incidence on surface boundary, a new reflecting-wave as expressed in the new cylindrical coordinate system as

$$\exp[i\{\gamma_2'(l_2'r+m_2'z)-pt'\}]$$

will be produced.

As a result, the resultant rotation-wave of displacement at the neighbourhood of a point of second incidence on surface boundary becomes to be expressed as follows in the new cylindrical coordinate system.

$$\left\{\begin{matrix} i\gamma_1 J_1(\gamma_2 l_2 r) \\ 0 \\ -\gamma_2 l_2 J_0(\gamma_2 l_2 r)\end{matrix}\right\} K\exp[i(\gamma_1 z-pt')] \cdots\cdots\cdots(39)$$

where $pt'=pt-2\gamma_2 ld=pt-2\gamma_2 l_2 d\cos\theta=pt-2\gamma_2 l_2 a\cdot\cos^2\theta$.

The above eq. (39) is the same form as the rotation-wave in the eq. (35).

Thus we can easily see that, both the rotation-wave of displacement, (39), and the dilatation-wave of displacement, which is also produced from the incident rotation-wave at the point of 2nd incidence on surface boundary and is expressed as

$$\left\{\begin{matrix} 0 \\ 0 \\ i\gamma_1\end{matrix}\right\} K_0\exp[i(\gamma_1 z-pt')] \cdots\cdots\cdots\cdots\cdots\cdots\cdots(40)$$

satisfy not only the Navier's equation but also the combination of these waves satisfies the boundary conditions at the point of 2nd incidence of reflecting rotation-wave in the same manner as in the case of 1st incidence to a point on surface boundary.

In the analysis shown above, the component-propagation-constant in z-direction is

$$m_2'\gamma_2'=\frac{m_2}{\sqrt{l^2+m_2^2}}(\gamma_2\sqrt{l^2+m_2^2})=m_2\gamma_2=\gamma_1$$

and this quantity is independent of the value of l, but the component-propagation-constant in r-direction

$$l_2'\gamma_2'=\frac{l}{\sqrt{l^2+m_2^2}}(\gamma_2\sqrt{l^2+m_2^2})=l\gamma_2 \qquad (l<l_2)$$

is dependent of the value of l; thus the resultant-propagation-constant γ_2' in the direction of reflecting rotation-wave is dependent of the value of l, namely the component-propagation-constant in r-direction, $l_2'\gamma_2'$, is a virtual component-propagation-constant considered only as an analytical means.

However, if we denote $r\cos\theta$ as x, then we have

$$\gamma_2 lr=\gamma_2(l_2\cos\theta)r=\gamma_2 l_2(r\cos\theta)=\gamma_2 l_2 x,$$

and the representative equations, (37) and (38), of the functions, $J_0(\gamma_2 l_2 r)\exp[i(\gamma_1 z-pt)]$ and $J_1(\gamma_2 l_2 r)\exp[i(\gamma_1 z-pt)]$, become to be expressed as

$$J_0(\gamma_2 l_2 r)\exp[i(\gamma_1 z-pt)]$$
$$=\frac{1}{\pi}\int_{\theta=o+\varepsilon}^{\pi/2}\{\exp[i(\gamma_2(l_2x+m_2z)-pt)]$$
$$+\exp[i(\gamma_2(-l_2x+m_2z)-pt)]\}\,d\theta \cdots(41)$$

$$J_1(\gamma_2 l_2 r)\exp[i(\gamma_1 z-pt)]$$
$$=-\frac{i}{\pi}\int_{\theta=o+\varepsilon}^{\pi/2}\cos\theta\{\exp i(\gamma_2(l_2x+m_2z)-pt)]$$
$$-\exp[i(\gamma_2(-l_2x+m_2z)-pt)]\}\,d\theta \cdots(42)$$

where $x=r\cos\theta$.

The above representative equations, (41) and (42), mean that the rotation-wave in new cylindrical coordinate system,

$$\left.\begin{Bmatrix} i\gamma_1 J_1(\gamma_2 l_2 r) \\ 0 \\ -\gamma_2 l_2 J_0(\gamma_2 l_2 r) \end{Bmatrix}\right. K \exp\,[i(\gamma_1 z - pt)]\,,$$

is not essentially conical-wave and can be regarded as actual plane-waves which are emanating from a point on surface boundary of cylindrical bar. Namely, the rotation-wave in new cylindrical coordinate system, the component-amplitudes of which are expressed by Bessel functions, is reconsidered only as a representation of an actual plane-wave with marking variable r, which originally should be expressed by the marking variable x as the terms inside the integral signs in the right-hand sides of the eqs. (41) and (42).

Thus it is generally wrong to differentiate the expression of displacement of longitudinal wave in solid cylinder by the variable r in order to get the stress component for boundary conditions. However at the origin of coordinates itself, even such a manner of treatment by the cylindrical coordinate, gives the same result as in the orthodox treatment by the Cartesian coordinate and as a result such a treatment by new cylindrical coordinate system becomes also correct.

The rotation-waves considered above are basically different from the rotation-waves as considered in the traditional Pochhammer's theory which emanate from a point on center axis of the cylindrical bar. Namely, the rotation-waves considered in author's theory are emanating in a reflecting-plane from a point of incidence on surface boundary.

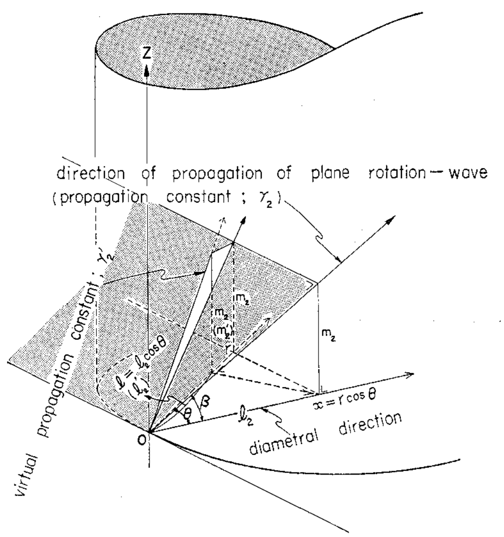

Fig. 4 Illustration of the relation between real and virtual propagation-directions, and of their direction-cosines.

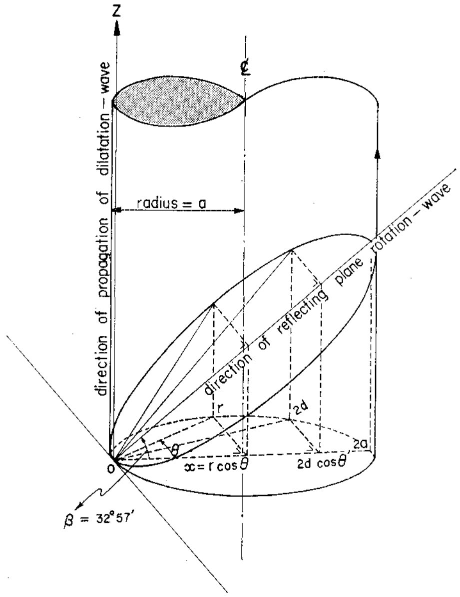

Fig. 5 Illustration of the plane-wave which is emanating in a plane from a point on surface boundary of a cylindrical bar.

The time-lag $(2\gamma_2 l_2 a \cos^2\theta)/p$ of an infinitesimal element of rotation-wave varies from zero to $2\gamma_2 l_2 a/p$, when θ varies from 90° to zero; but in actual experimental results, only the wave with time-lag of $2\gamma_2 l_2 a/p$ can be observed and the waves with other time-lags cannot almost be observed as will be reported later. It is supposed that this matter is due to the fact that the reflecting-waves at the neighbourhood of a diametral direction, $(x)_{\theta=0}=(r\cos\theta)_{\theta=0}$, of the circular cross-section are so strong as not comparable with those in other directions.

If the initial impact-wave is not an isolated pulse but a continuous wave, it is necessary that the following relation holds true as similarly as in the case of longitudinal impact wave in a prismatical bar, in order that the wave-motion should be continuous along the border-line between the regions of triangles and parallelograms in the neighbourhood of a diametral section. (Ref. Fig. 2 in Sec.2.) Namely,

$$(\gamma_2 l_2 x)_{x=a} = \mp\pi,\ \mp 2\pi, \cdots \qquad (43)$$

On the other hand, if we assume $\gamma=\gamma_1$ (i.e. $c=c_1$) in Pochhammer's theory, we have as a frequency equation

$$J_1(\gamma_2 l_2 a)=0, \quad \text{i.e. } \{J_1(\gamma_2 l_2 r)\}_{r=a}=0 \cdots\cdots(44)$$

and thus we obtain also in this case the discrete numerical value of γ_2 as zero-points of Bessel function of 1st order, $J_1(\gamma_2 l_2 r)$. However the eq. (43) is clearly different from the frequency equation (44), and we can see also from the above results that the author's theory is based on a different concept from that of the traditional Pochhammer's theory.

Fig. 6 shows phase velocities of various impact waves in cylindrical bars (for $\nu=0.29$) which were calculated by electronic computor according to traditional theory and author's theory for longitudinal impact wave.

Fig. 7 shows a photograph of the configuration of secondary peaks in echoes-envelope of longitudinal ultra-sonic wave, which are delayed by the time-interval $(2\gamma_2 l_2 a)/p$ from the primary peaks. The delay-time of secondary peaks from the primary peaks in echoes-envelope, well coincides with the time necessary for the reflecting rotation-wave with reflecting angle 32°57′ ($\nu=0.29$) at one-side surface boundary to cross obliquely the diametral section of the bar as shown in this figure. (We could not observe such a phenomenon in comparatively massive bars. This is supposed due to the attenuation of reflecting rotation wave in crossing the diametral section of comparatively massive bars.) This actual fact is considered to be a powerful proof that the author's opinions as to the phase velocity of longitudinal impact wave in a bar, will be surely correct. On the other hand, the traditional Pochhammer's theory cannot explain the reason of this actual fact.

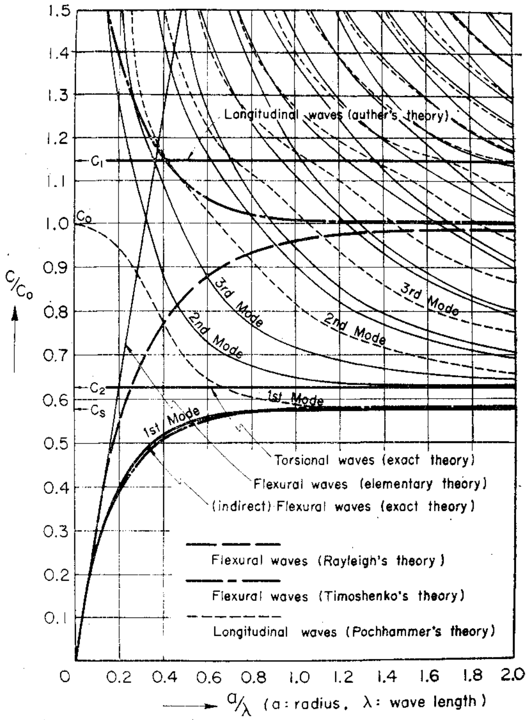

Fig. 6 Phase velocity of various impact waves in cylindrical bars (for $\nu=0.29$).

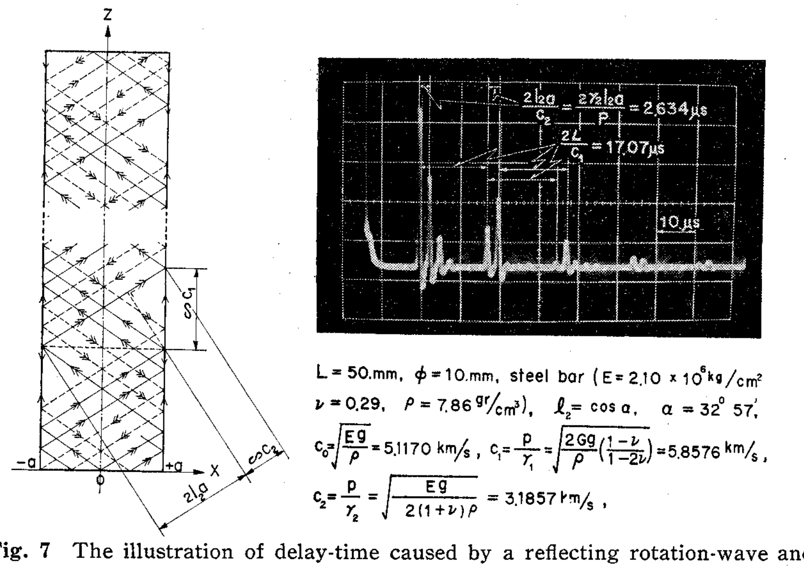

Fig. 7 The illustration of delay-time caused by a reflecting rotation-wave and the photograph of delayed pulse which has appeared in an experiment executed by using a pulse of ultra-sonic wave of 10 MHz.

4. CONCLUSION

It was verified both theoretically and experimentally that the phase velocity of longitudinal impact wave in solid elastic bar is always equal to

$$c_1=\sqrt{\frac{2Gg}{\rho}\left(\frac{1-\nu}{1-2\nu}\right)}$$

which is definite value in a material. Further the phase velocity of longitudinal impact wave in solid elastic bar is non-dispersive.

BIBLIOGRAPHY

1) Pochhammer, L.: Uber die Fortpflanzungsgeschwindigkeiten Schiwingungen in einem Umbergrenzten Isotropen Kreiszylinder, J. fvr Math. 81, pp. 324-336 (1876).
2) Lamb, H.: On the Propagation of Tremors over the Surface of an Elastic Solid, Phil. Trans. Roy. Soc. London, A203, pp. 1-42 (1904).
3) Lamb, H.: Proc. Roy. Soc., A93, pp. 114-128 (1917).
4) Love, A. E. H.: A Treatise on the Mathematical Theory of Elasticity, Cambridge Univ. Press, Fourth Edition, pp. 289-292 (1934).
5) Kolsky, H.: Stress Wave in Solids, Dover Publications, Inc., New York, (1952).
6) Kolsky, H.: The Propagation of Longitudinal Elastic Waves along Cylindrical Bars, Phil. Mag. Series 7, Vol. 45, pp. 712-726 (1954).
7) Mason, Warren P.: Physical Acoustics, vol. 1-Part A, pp. 55-109, pp. 112-139 (R. N. Thurston, T. R. Meeker and A. H. Meitzler), Academic Press Inc., New York and London, December, (1963).
8) Nigul, U. K.: The Application of the Three Dimensional Theory of Elasticity to the Analysis of Flexural Waves in a Semi-Infinite Plate Acted on by a Short-Time Boundary Loading, J. Appl. Math. Mech. (P.M.M.), 27, pp. 1602-1620 (1963-1964).
9) Taya, T.: Tensor Wave Theory in Solid Elastic Body of Three Dimension, Part 3, pp. 81-90, pp. 101-105, The Tech. Repts. of the Tohoku Univ., Japan, vol. 38, No. 1, June, (1973).
10) Taya, T.: Tensor Wave Theory in Solid Elastic Body of Three Dimension, Part 5, pp. 1-76, The Tech. Repts. of the Tohoku Univ., Japan, vol. 39, No. 1, June, (1974).
11) Taya, T.: Phase Velocity of Longitudinal Impact Wave in Solid Prismatical Bar, The Tech. Repts. of the Tohoku Univ., Japan, vol. 40, No. 2, pp. 383-426, December, (1975).
12) Taya, T.: Phase Velocity of Longitudinal Impact Wave in Solid Cylindrical Bar, The Tech. Repts. of the Tohoku Univ., Japan, vol. 40, No. 2, pp. 427-454, December, (1975).
13) Taya, T.: Miscellaneous Notes to the Experiments of Phase Velocity of Longitudinal Impact Wave in an Elastic Bar, The Tech. Repts. of the Tohoku Univ., Japan, vol. 40, No. 2, pp. 455-487, December, (1975).
14) Taya, T.: Experimental Proof of the Phase Velocity, c_1, of Longitudinal Impact Wave in an Elastic Bar, The Tech. Repts. of the Tohoku Univ., Japan, vol. 40, No. 2, pp. 489-519, December, (1975).

(*Received Oct. 29, 1975*)