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LINEARIZED FINITE DEFORMATION THEORY IN SUSPENSION BRIDGES

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1. INTRODUCTION

The deflection theory of suspension bridges was originated by W. Ritter in 1877, Müller-Breslau in 1881, and J. Melan[1] in 1889. In order to solve Melan's equation, a full treatment was given by many researchers including S. Timoshenko[2], D. B. Steinman[5], D. J. Peery, and others.

When developing Melan's equation for suspension bridges, the following assumptions were used.

Assumption 1: The distance between hangers is so small compared with the span length that hangers may be replaced by a screen.

Assumption 2: The cable curve is parabolic under dead load.

Assumption 3: The initial dead load is carried by the cable without causing any stress in the stiffening frame.

Assumption 4: The horizontal displacement of the cable may be neglected.

Assumption 5: The elongation of hangers may be neglected.

Assumption 6: The inclination of hangers may be neglected so that hangers remain vertical in the strained configuration.

Assumption 7: The shear deformation of girders may be neglected.

Since Timoshenko's work many studies have been done to investigate the secondary effects such as horizontal cable displacement, hanger elongation, hanger inclination, shear deformation of girders, finite spacing of girders, and so forth. Notable work was done by H. H. Rode[3] on the effect of horizontal cable displacement in 1930. A. Selberg[6] derived an equation taking the inclination of hangers into consideration. It had been known that the cable equation determining the additional cable tension caused by live load or temperature change was based on the small displacement theory. The auther appreciates Selberg's work[9] in which he took the effect of cable tension in the cable equation into account. For the lateral bending of a suspension bridge, L. S. Moisseiff and F. Lienhard[4] proposed the elastic distribution method in 1933. C. Z. Erzen[8] showed a workable method to solve the equation derived by Moisseiff's theory.

The recent advance of digital computers and matrix methods in structural mechanics has made possible the analysis of a suspension bridge in the form of a framed structure. D. M. Brotton and G. Arnold presented the distribution matrix method. T. Poskitt developed a method of analyzing suspension bridges by the Newton-Raphson Method. In 1966, S. A. Saafan[10] showed his finite deformation theory in which he took the flexural shortening and the bowing effect of a beam-column into account and applied it to a suspension bridge. Since then, the analysis of suspension bridges has become merely a part of the finite deformation method or the matrix method in structural mechanics. During this stage of development, several authors in Japan including S. Goto[13),14] and T. Fukda[12] contributed to the advance of suspension bridge analysis.

Nowadays the finite deformation theory is extensively utilized for the static and dynamic problems of the longspanned suspension bridges which will be constructed between Shikoku and the main island of Japan. This author has devoted himself to that work and has undertaken to derive the formulas appearing in the deflection theory from those of the linearized finite deformation theory and to give the theoretical background to the linearized finite deformation theory.

It has been hitherto said that in the deflection theory the change in configuration caused by live load is taken into account. The author has shown in this paper that this description may not be correct and should be replaced by the one that the deflection theory is based on the small displacement theory in which the effect of cable tension

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is taken into account. As it is not the primary purpose to solve the reduced new differential equations, the effect of temperature change is not taken into consideration in this paper. The notations used in Sec. 3 are listed in the Appendix.

2. RIGIDITY OF THE BAR MEMBER AND ITS PHYSICAL MEANING

Considering a bar member in three dimensional Cartesian coordinates (x, y, z) subject to an axial force T, with a sectional area A and modulus of elasticity E as shown in Fig. 1 (for simplicity, it

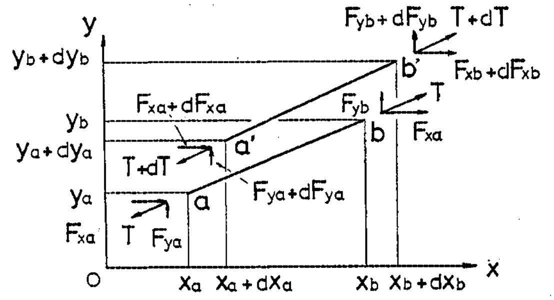

Fig. 1 Glabal coordinates

is shown two dimensionally), its stiffness equation is expressed as follows:

$$dF^g = k^g dx \quad \cdots\cdots(1)$$

where

$$\left.\begin{aligned}
&dF^g = [dF_{xa}, dF_{ya}, dF_{za}, \\
&\qquad dF_{xb}, dF_{yb}, dF_{zb}]^T \\
&dx = [dx_a, dy_a, dz_a, \\
&\qquad dx_b, dy_b, dz_b]^T \\
&k^g = \begin{bmatrix} k^{gg} & -k^{gg} \\ -k^{gg} & k^{gg} \end{bmatrix} \\
&k^{gg} = \frac{AE}{L_0} l_1 + \frac{T}{L} l_2 \\
&l_1 = \begin{bmatrix} l^2 & & \text{sym.} \\ lm & m^2 & \\ ln & mn & n^2 \end{bmatrix} \\
&l_2 = \begin{bmatrix} 1-l^2 & & \text{sym.} \\ -lm & 1-m^2 & \\ -ln & -mn & 1-n^2 \end{bmatrix}
\end{aligned}\right\} \cdots\cdots(2)$$

a, b: Subscripts to distinguish the two ends of a member

L_0, L: unstrained length and strained length, respectively

$dx_a, \cdots, dz_b$: displacement increments of the first order

$dF_{xa}, \cdots, dF_{zb}$: force increments of the first order

l, m, n: direction cosines of the member axis

If a system of Cartesian coordinates (x', y', z') is taken with the origin $0'$ at one end of the member and the axes so directed that $0'x'$ coincides with the member axis, and axes $0'y'$ and $0'z'$ are perpendicular with one another, (the latter ones may be set in any position as shown in Fig. 2), the following stiffness equation is easily derived for the new coordinates.

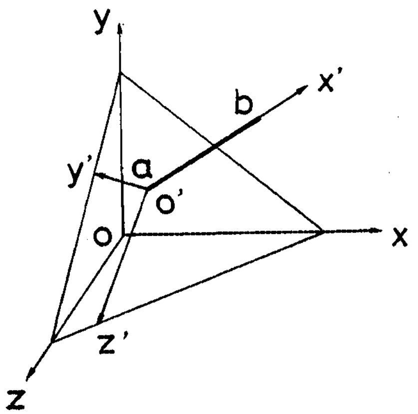

Fig. 2 Local coordinates

$$dF^m = k^m dx^m \quad \cdots\cdots(3)$$

where

$$\left.\begin{aligned}
&dF^m = [dF_{x'a}, dF_{y'a}, dF_{z'a}, \\
&\qquad dF_{x'b}, dF_{y'b}, dF_{z'b}]^T \\
&dx^m = [dx'_a, dy'_a, dz'_a, \\
&\qquad dx'_b, dy'_b, dz'_b]^T \\
&k^m = \begin{bmatrix} k^{mm} & -k^{mm} \\ -k^{mm} & k^{mm} \end{bmatrix} \\
&k^{mm} = \frac{AE}{L_0} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{T}{L} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}\right\} \cdots\cdots(4)$$

Equation (3) can also be derived from the principle of minimum potential energy[11]. In this case, L_0 is replaced by L in Eq. (4). It should be noted that Eq. (1) or (3) is a linear equation concerning forces and displacements so that the influence of deformation of structure is not taken into consideration in equations derived from Eq. (1) or (3).

Equation (3) shows that a bar member subjected to a tension T holds a longitudinal rigidity AE/L_0 and a lateral rigidity T/L. The meaning of the longitudinal rigidity AE/L_0 is apparent in view of the small displacement theory. The physical meaning of the lateral rigidity T/L may be explained in the following example of free vibration of a chord with a tension T and a linear density ρ shown in Fig. 3.

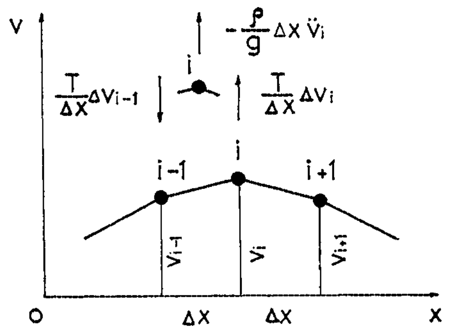

Fig. 3 Vibration of chord

Using the lateral rigidity in Eq. (3), and considering the equilibrium about point i, we have

$$-\frac{\rho}{g}\Delta x\frac{d^2v_i}{dt^2}+\frac{T}{\Delta x}\Delta v_i-\frac{T}{\Delta x}\Delta v_{i-1}=0 \quad \cdots\cdots(5a)$$

where v is a lateral displacement and Δ is a difference sign namely, $\Delta v_i=v_{i+1}-v_i$. Revising Eq. (5a) by difference, and dividing by Δx, we obtain

$$\frac{\rho}{g}\frac{d^2v_i}{dt^2}-T\frac{\Delta}{\Delta x}\frac{\Delta v_{i-1}}{\Delta x}=0 \quad \cdots\cdots\cdots\cdots\cdots\cdots(5b)$$

When Δx is small enough, this can be written as

$$\frac{\rho}{g}\frac{d^2v}{dt^2}-T\frac{d^2v}{dx^2}=0 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(5c)$$

This is the equation of the free vibration of a chord.

Next will be considered the problem of a simple pendulum expressing a mass and an arm length, with m and L respectively as shown in Fig. 4. Denoting a lateral displacement by u, from Eq. (3), we have

$$-m\frac{d^2u}{dt^2}-\frac{mg}{L}u=0 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(6a)$$

because the tension of the arm is mg. Divided by m, Eq. (6a) becomes

$$\frac{d^2u}{dt^2}+\frac{g}{L}u=0 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(6b)$$

As is well known, Eq. (6b) is the equation of the vibration of a simple pendulum. These examples show that the lateral ridigity T/L is associated with the restoring force of a strained chord[11] or a simple pendulum. The lateral ridigity T/L is positive when the axial force is tension, and it is negative in compression.

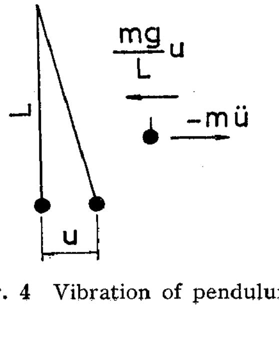

Fig. 4 Vibration of pendulum

3. BASIC EQUATIONS OF SUSPENSION BRIDGES DERIVED FROM THE STIFFNESS EQUATION

In this section the method of deriving the cable equation is fundamentally in agreement with Goto's method shown in Paper (14), and this method is also applied to derive the vertical equilibrium equation of the deflection theory.

Since a cable behaves like a bar member when it is vertically suspended and like a chord member when it is horizontally strained, it may be supposed that a cable diagonally strained holds both properties of a bar member and a chord member, and the equation of equilibrium may be derived from Eq. (1) or (3). Considering the configuration of a cable shown in Fig. 5, the differential equation of equilibrium will be derived by using the principle of passage from discontinuity to continuity. For simplicity, the assumptions in Section 1 are held, except 1, 2, and 4 in the discrete state, and Assumptions 1 and 2 are requested when Δx approaches zero.

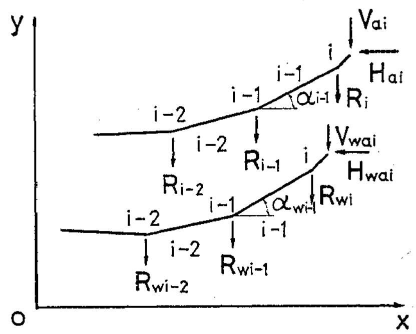

Fig. 5 Configuration of cable

If the effect of hanger inclinations is neglected for simplicity, the condition of equilibrium about a cable point i in an unloaded position shown in Fig. 6 gives

$$H_{wbi-1}+H_{wai}=0 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(7a)$$

$$V_{wbi-1}+V_{wai}+W_{ci}+R_{wi}=0 \quad \cdots\cdots\cdots\cdots\cdots(7b)$$

In a loaded position we have similarly

$$H_{bi-1}+H_{ai}=0 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(8a)$$

$$V_{bi-1}+V_{ai}+W_{ci}+R_i=0 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots(8b)$$

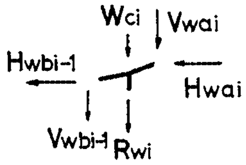

Fig. 6 Equilibrium about cable point i

Applying Taylor expansions to Eqs. (8a) and (8b) in an unloaded position, we obtain

$$H_{wbi-1}+dH_{bi-1}+H_{wai}+dH_{ai}+o(2)=0 \cdots (9a)$$

$$V_{wbi-1}+dV_{bi-1}+V_{wai}+dV_{ai}+W_{ci}+R_{wi}+dR_i+o(2)=0 \quad \cdots\cdots (9b)$$

in which $o(2)$ denotes the small quantities of the second order, and from Eq. (2) we get

$$d\boldsymbol{F}_i=\boldsymbol{k}_i d\boldsymbol{x}_i \quad \cdots\cdots (10)$$

where

$$d\boldsymbol{F}_i=[dH_a, dV_a, dH_b, dV_b]_i^T$$

$$d\boldsymbol{x}_i=[dx_a, dy_a, dx_b, dy_b]_i^T$$

$$\boldsymbol{k}_i=\begin{bmatrix} \boldsymbol{k}^c & -\boldsymbol{k}^c \\ -\boldsymbol{k}^c & \boldsymbol{k}^c \end{bmatrix}$$

$$\boldsymbol{k}^c=\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

$$k_{11}=\frac{A_cE_c}{L_0}\cos^2\alpha_w+\frac{T_w}{L}\sin^2\alpha_w$$

$$k_{21}=k_{12}=\left(\frac{A_cE_c}{L_0}-\frac{T_w}{L}\right)\sin\alpha_w\cos\alpha_w$$

$$k_{22}=\frac{A_cE_c}{L_0}\sin^2\alpha_w+\frac{T_w}{L}\cos^2\alpha_w$$

Here the formulation of the problem is best developed by adding the following basic assumption.

Assumption 8: The second order of small quantities in Taylor expansions may be neglected.

Using Assumption 8, Eqs. (8a), (8b), (9a) and (9b) can be written as

$$dH_{bi-1}+dH_{ai}=0 \quad \cdots\cdots (11a)$$

$$dV_{bi-1}+dV_{ai}+dR_i=0 \quad \cdots\cdots (11b)$$

The cable equation will first be derived. From Eqs. (11a) and (10), we find

$$dH_{bi}=\text{const.}\ (=dH) = -k_{11i}(dx_{ai}-dx_{bi})-k_{12i}(dy_{ai}-dy_{bi}) \quad \cdots\cdots (12)$$

If both sides of Eq. (12) are divided by k_{11i}, and summed up over the entire cable span, with boundary conditions, we have

$$dH\sum_{i=1}^{n-1}\frac{1}{k_{11i}}-\sum_{i=2}^{n-1}\left(\frac{k_{12i}}{k_{11i}}-\frac{k_{12i-1}}{k_{11i-1}}\right)dy_i=0 \quad \cdots\cdots (13)$$

Substituting k_{11i} etc. in Eq. (10) into Eq. (13), using $\Delta x \simeq L_0\cos\alpha_w$, and neglecting the small quantities of the second order of $\delta=(H_w\sec\alpha_{wi}\cdot\tan^2\alpha_{wi})/A_cE_c$ through the calculating process and letting $\Delta x\to 0$, we obtain

$$\frac{H_p}{A_cE_c}\int_c^{c'}\sec^3\alpha_w dx+\frac{d^2y}{dx^2}\int_c^{c'}\eta dx - \frac{H_w}{A_cE_c}\int_c^{c'}\eta\frac{d}{dx}(\tan\alpha_w\sec^3\alpha_w)dx=0 \quad \cdots\cdots (14)$$

in which c and c' denote the fixed points of the cable and the conventional notations, η and H_p of the deflection theory instead of dy and dH have been used. As for the deflection theory, it is convenient that the positive direction of the y axis is taken as downward, and even in that case the form of Eq. (14) remains unaltered.

Next, the equation of vertical equilibruim for suspension bridges will be derived. Eliminating the term $(dx_{ai}-dx_{bi})$ from between Eqs. (10) and (12) gives

$$\begin{aligned} dV_{ai} &= -\frac{k_{12i}}{k_{11i}}dH-\frac{k_{12i}^2-k_{11i}k_{22i}}{k_{11i}}(dy_i-dy_{i+1}) \\ &= -dH\left(\tan\alpha_{wi}-\frac{H_w}{A_cE_c}\tan\alpha_{wi}\sec^3\alpha_{wi}\right) \\ &\quad -H_w\left(\sec^2\alpha_{wi}-\frac{H_w}{A_cE_c}\tan^2\alpha_{wi}\cdot\sec^3\alpha_{wi}\right)\frac{\Delta dy_i}{\Delta x} \quad \cdots\cdots (15) \end{aligned}$$

Since $dV_{bi-1}=-dV_{ai-1}$ by Eq. (10), Eq. (11b) becomes

$$dV_{ai}-dV_{ai-1}=\Delta dV_{ai-1}=-dR_i \quad \cdots\cdots (16)$$

Dividing both sides of Eq. (16) by Δx, substituting Eq. (15), using

$$\frac{dR_i}{\Delta x}=\frac{\Delta^2}{\Delta x^2}\left(E_sI_v\frac{d^2\eta_{i-1}}{dx^2}\right) \quad \cdots\cdots (17)$$

neglecting the small quantities of the second order of δ and letting $\Delta x\to 0$, we arrive at

$$\begin{aligned} &E_sI_v\frac{d^4\eta}{dx^4}-H_w\frac{d}{dx}\left\{\left(\sec^2\alpha_w-\frac{H_w}{A_cE_c}\cdot\tan^2\alpha_w\sec^3\alpha_w\right)\frac{d\eta}{dx}\right\} \\ &\quad -H_p\left\{\frac{d^2y}{dx^2}-\frac{H_w}{A_cE_c}\frac{d}{dx}\cdot\left(\frac{dy}{dx}\sec^3\alpha_w\right)\right\}=p \quad \cdots\cdots (18) \end{aligned}$$

in which the positive direction of the y axis has been taken as downward. Further details concerning the above derivations will be refered to paper (18).

Hereupon, a basic equation different from Eq. (18) will be derived for a suspension bridge with two hinged stiffening girders. Adding again Eq. (16) with respect to all nodal points until i, we obtain

$$dV_{ai}=-S_i+S_{Bi} \quad \cdots\cdots(19)$$

where S_B is the shearing force due to loading on a simple beam. Multiplying both sides by Δx and summing up from 1 to i, the bending moment is

$$M_i=M_{Bi}-\sum_{j=1}^{i} dV_{aj}\Delta x \quad \cdots\cdots(20)$$

where M_B is the bending moment due to loading on a simple beam. Change the positive direction of the y axis downward and the notations with those conventionally used in the deflection theory, and let $\Delta x \to 0$ in the equation obtained from Eqs. (15) and (20), we get

$$M=M_B-H_p\int_0^x\left(1-\frac{H_w}{A_cE_c}\sec^3\alpha_w\right)\frac{dy}{dx}dx \\ -H_w\int_0^x\left(1-\frac{H_w}{A_cE_c}\tan^2\alpha_w\sec\alpha_w\right) \\ \cdot\sec^2\alpha_w\frac{d\eta}{dx}dx \quad \cdots\cdots(21)$$

Finally a differential equation of equilibrium for suspension bridges applying lateral loads will be derived. Make two equilibriums about cable point i among the right and left side cable forces, hanger force dR_{hi} and external load $P_{ci}\Delta x$, and about girder point i among the hanger force, the right and left side shearing forces of the girder, and the external force $P_{si}\Delta x$ as shown in Fig. 7. If we use the following relation:

$$T_{wi}=H_w\sec\alpha_{wi} \quad \cdots\cdots(22)$$

$$L_i=\Delta x\sec\alpha_{wi} \quad \cdots\cdots(23)$$

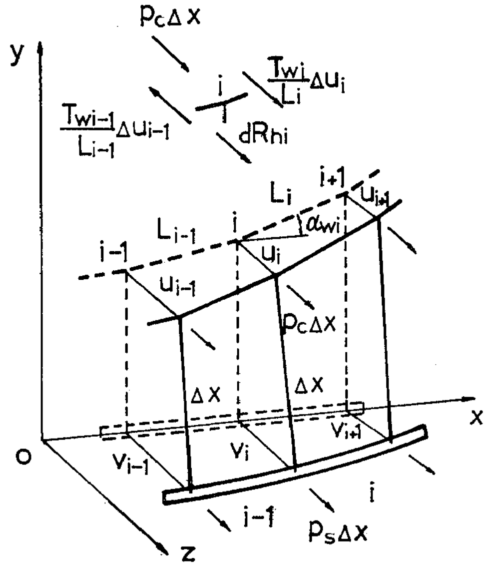

Fig. 7 Deformation due to lateral forces

$$r_h=\lim_{\Delta x\to 0}\frac{dR_{hi}}{\Delta x}=\frac{w_s}{h_x}(v-u) \quad \cdots\cdots(24)$$

we can easily derive the following Eqs. (25) and (26)

$$E_sI_h\frac{d^4v}{dx^4}+\frac{w_s}{h_x}(v-u)=p_s \quad \cdots\cdots(25)$$

$$-H_w\frac{d^2u}{dx^2}-\frac{w_s}{h_x}(v-u)=p_c \quad \cdots\cdots(26)$$

which have been developed by L. S. Moisseiff and F. Lienhard[4)] and reformed by C. Z. Erzen[8)].

4. CONSIDERATION OF BASIC EQUATIONS

Equations (14), (18) and (21) will be investigated more deeply here. If the following approximation is taken

$$\sec^3\alpha_w=(1+\tan^2\alpha_w)^{3/2}\simeq 1 \quad \cdots\cdots(27)$$

Eq. (14) becomes

$$\frac{H_p}{A_cE_c}\int_c^{c'}\sec^3\alpha_w dx \\ +\frac{d^2y}{dx^2}\left(1-\frac{H_w}{A_cE_c}\right)\int_c^{c'}\eta dx=0 \quad \cdots\cdots(28)$$

This expression shows a good agreement with Selberg's equation[9)]. He derived it by the following procedure. In an underformed state, the relation between the element of cable is

$$ds^2=dx^2+dy^2 \quad \cdots\cdots(29)$$

in which ds denotes the length of the cable element, and dx and dy are its components. In a deflected state, the relation will be

$$(ds+de)^2=(dx+d\xi)^2+(dy+d\eta)^2 \quad \cdots\cdots(30)$$

in which de denotes the elongation of ds. From Eqs. (29) and (30) we find

$$\frac{d\xi}{dx}=\frac{ds}{dx}\frac{de}{dx}-\frac{dy}{dx}\frac{d\eta}{dx}-\frac{1}{2}\left(\frac{d\eta}{dx}\right)^2 \\ +\frac{1}{2}\left(\frac{de}{dx}\right)^2-\frac{1}{2}\left(\frac{d\xi}{dx}\right)^2 \quad \cdots\cdots(31)$$

Neglecting the last two terms on the right hand side and using $ds/dx=\sec\alpha_w$, we can write

$$d\xi=\frac{de}{dx}\sec\alpha_w dx-\frac{dy}{dx}\frac{d\eta}{dx}dx \\ -\frac{1}{2}\left(\frac{d\eta}{dx}\right)^2 dx \quad \cdots\cdots(32)$$

The elongation de of the cable element will be

$$de=\frac{1}{A_cE_c}(T-T_w)ds \\ =\frac{1}{A_cE_c}(H_w+H_p)\sec\alpha_w\sec\alpha dx \\ -\frac{H_w}{A_cE_c}\sec^2\alpha_w dx \quad \cdots\cdots(33)$$

and

$$\left.\begin{aligned}\sec\alpha_w&=(1+\tan^2\alpha_w)^{1/2}\\&\simeq1+\frac{1}{2}\left(\frac{dy}{dx}\right)^2\\\tan\alpha&=\frac{dy+d\eta}{dx+d\xi}\\&\simeq\left(\frac{dy}{dx}+\frac{d\eta}{dx}\right)\left(1-\frac{d\xi}{dx}\right)\\\sec\alpha&=(1+\tan^2\alpha)^{1/2}\\&\simeq1+\frac{1}{2}\left(\frac{dy}{dx}+\frac{d\eta}{dx}\right)^2\end{aligned}\right\}\cdots\cdots(34)$$

Substituting Eqs. (33) and (34) into (32) and integrating from c to c', and making $\sec\alpha_w\simeq\sec\alpha$, we obtain

$$\frac{H_p}{A_cE_c}\int_c^{c'}\sec^3\alpha_w dx+\frac{d^2y}{dx^2}\left(1-\frac{H_w}{A_cE_c}\right)\cdot\int_c^{c'}\eta dx-\frac{1}{2}\int_c^{c'}\left(\frac{d\eta}{dx}\right)^2dx=0\cdots\cdots(35)$$

It is found that Eq. (35) agrees with Eq. (28) except for the third term of the left hand side. The error on the plus-minus sign found in his original paper has already been corrected in Eq. (35). It is noted that the third term in Eq. (35) is a small quantity of the second order by this derivation. S. Goto analyzed this problem but a small error was found in his paper[14].

In Eq. (18), neglecting the small quantities, we can write

$$E_sI_v\frac{d^4\eta}{dx^4}-H_w\frac{d}{dx}\left(\sec^2\alpha_w\frac{d\eta}{dx}\right)-H_p\frac{d^2y}{dx^2}=p\cdots\cdots(36)$$

Concerning the equation of vertical equilibrium for suspension bridges that takes the effect of horizontal displacement of cable into consideration, the following Eq. (37) derived by H. H. Rode[3] is notable.

$$E_sI_v\frac{d^4\eta}{dx^4}-(H_w+H_p)\frac{d}{dx}\left(\sec^2\alpha_w\frac{d\eta}{dx}\right)-H_p\frac{d^2y}{dx^2}=p\cdots\cdots(37)$$

The derivation by Rode is as follows. Using notations in the deflection theory and taking the positive direction of the y axis as downward, the vertical component of cable tension before loading is

$$V_w=H_w\frac{dy}{dx}\cdots\cdots(38)$$

and the one after loading is

$$V=(H_w+H_p)\frac{dy+d\eta}{dx+d\xi}$$

$$\simeq(H_w+H_p)\left\{\frac{dy}{dx}\left(1-\frac{d\xi}{dx}\right)+\frac{d\eta}{dx}\right\}\cdots\cdots(39)$$

If we use the following assumption, Eq. (40) can be obtained.

Assumption 9: The elongation of cable de and the small terms conecrning $d\xi^2$ and $d\eta^2$ in Eq. (31) may be neglected.

$$\frac{d\xi}{dx}+\frac{dy}{dx}\frac{d\eta}{dx}=0\cdots\cdots(40)$$

From Eqs. (39) and (40), we obtain

$$V=(H_w+H_p)\left[\frac{dy}{dx}+\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\frac{d\eta}{dx}\right]$$
$$=(H_w+H_p)\left\{\frac{dy}{dx}+\sec^2\alpha_w\frac{d\eta}{dx}\right\}\cdots\cdots(41)$$

Then from Eq. (38), the equation of equilibrium for cable before loading is

$$H_w\frac{d^2y}{dx^2}=w_c-r_w\cdots\cdots(42)$$

where

r_w: intensity of hanger tension per unit length due to dead load

The equation after loading is, from Eq. (41),

$$(H_w+H_p)\left\{\frac{d^2y}{dx^2}+\frac{d}{dx}\left(\sec^2\alpha_w\frac{d\eta}{dx}\right)\right\}=w_c-(r_w+r_p)\cdots\cdots(43)$$

where

r_p: intensity of hanger tension increment per unit length due to live load.

From Eqs. (42) and (43), we find

$$H_p\frac{d^2y}{dx^2}+(H_w+H_p)\frac{d}{dx}\left(\sec^2\alpha_w\frac{d\eta}{dx}\right)=r_p=-p+E_sI_v\frac{d^4\eta}{dx^4}\cdots\cdots(44)$$

Equation (44) agrees with Eq. (37). It is seen that Eq. (36) agrees with Eq. (37) except for H_p in the second term of Eq. (37).

If the following assumption is used in Eq. (36).

Assumption 10: $\mathrm{Sec}^2\alpha_w=1+\tan^2\alpha_w\simeq1$

then Eq. (36) becomes

$$E_sI_v\frac{d^4\eta}{dx^4}-H_w\frac{d^2\eta}{dx^2}-H_p\frac{d^2y}{dx^2}=p\cdots\cdots(45)$$

This is the equation which puts $H_p=0$ in Melan's epuation:

$$E_sI_v\frac{d^4\eta}{dx^4}-(H_w+H_p)\frac{d^2\eta}{dx^2}-H_p\frac{d^2y}{dx^2}=p\cdots\cdots(46)$$

and the former was termed the linearized equation by Bleich[7]. Neglecting the small quantities in Eq. (21), we find

$$M = M_B - H_p y - H_w \int_0^x \sec^2 \alpha_w \frac{d\eta}{dx} dx \quad \cdots(47)$$

and furthermore by using Assumption 10, Eq. (47) reduces to

$$M = M_B - H_p y - H_w \eta \quad \cdots\cdots(48)$$

This equation has the form obtained by substituting $H_p = 0$ in the following equation:

$$M = M_B - H_p y - (H_w + H_p)\eta \quad \cdots\cdots(49)$$

which is the basic equation of the bending moment in the deflection theory.

As for the effect of hanger inclinations, several investigations have been made. The method of analysis for this effect corresponds fundamentally to the consideration of the horizontal increment of the cable tension due to hanger inclination, that is

$$H_{pi} = \int_i^{i+1} \frac{r_w(x)}{h_x}(\xi - \chi) dx \quad \cdots\cdots(50)$$

in which χ denotes the horizontal displacement of the stiffening girder. From Eq. (3), it is clear that H_p has been caused by the lateral rigidity of hangers.

It will be further investigated in detail the relation between Eq. (18) which has been derived by the tangent stiffness equation of cable, and Rode's linearized equation (36) or Melan's linearized equation (45). Considering a cable link shown in Fig. 8, the first order increment of joint force (called here force increment) occurs in the cable link due to loading. In Fig. 8, $\overrightarrow{ac}$ = the force increment due to axial rigidity, $\overrightarrow{ce}$ = the force increment due to lateral rigidity, and $\overrightarrow{ae}$ = the combined force increment. Representing the relation between the combined force increment $\overrightarrow{ae}$ and the corresponding displacement increment by the global coordinates (x, y), we get Eq. (15). Therefore, both axial and lateral rigidities are taken into consideration in Eq. (18).

If an assumption is made as follows:
Assumption 11: The small terms regarding $H_w/A_c E_c$ can be neglected.
then Eq. (15) becomes

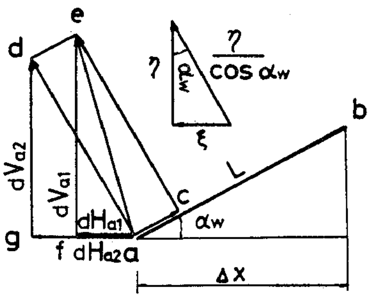

Fig. 8 Member end forces of cable link

$$dV_a = dH_a \tan \alpha_w - H_w \sec^2 \alpha_w \frac{\Delta\eta}{\Delta x} \quad \cdots\cdots(51)$$

Transformation gives

$$[dH_a, dV_a][\sin \alpha_w, -\cos \alpha_w]^T = \frac{T_w}{L} \frac{\Delta\eta}{\cos \alpha_w} \quad \cdots\cdots(52)$$

Since the vector $[\sin \alpha_w, -\cos \alpha_w]$ is a unit vector towards the $\overrightarrow{ad}$ direction and $\Delta\eta/\cos \alpha_w$ is the displacement towards the perpendicular direction of vector $\overrightarrow{ab}$ in Fig. 8, Eq. (52) indicates that the coefficient perpendicular to the cable axis of the combined force increment is a force due to the lateral rigidity of the cable. Assumption 11 is the same as neglecting vector $\overrightarrow{ac}$ and assuming that the combined force is represented by vector $\overrightarrow{ad}$.

Rode's linearized Eq. (36) has been obtained with Assumption 11, and this assumption is also used as Eq. (40) by Rode. Namely, changing Eq. (40) gives

$$[d\xi, d\eta][dx, dy]^T = 0 \quad \cdots\cdots(53)$$

and this equation means that the combined force increment intersects perpendicularly with the cable axis so that it accords with Assumption 11.

Next, the meaning of Assumption 10 will be investigated. Melan's equation was originally obtained by neglecting the horizontal cable displacement $d\xi$ in Eq. (39), thus Rode called his equation the one in which horizontal cable displacement was not neglected. Melan's linearized equation can also be gotten with Assumption 10, when Eq. (51) becomes

$$dV_a = dH_a \tan \alpha_w - H_w \frac{\Delta\eta}{\Delta x} \quad \cdots\cdots(54)$$

which is changed to be

$$[dH_a, dV_a][\sin \alpha_w, -\cos \alpha_w]^T = \frac{T_w}{L} \cos^2 \alpha_w \frac{\Delta\eta}{\cos \alpha_w} \quad \cdots\cdots(55)$$

Comparing Eq. (55) with Eq. (52), we find that Melan's linearized equation uses the lateral rigidity of the cable as $T_w \cos^2 \alpha_w / L$ in place of the true ridigity T_w/L. The equation in Assumption 10 shows a better approximation when the tangent angle of the cable is smaller.

Since Melan's Eq. (46) is included in Rode's Eq. (37), Rode's equation will be regarded as the basic equation for the deflection theory and considered whether the effect of the cable configuration changes, namely whether the cable deflection after loading in the deflection theory is taken into consideration or not. Changing Eq. (37) into the relation of the force increments of a cable link, we can write

$$[dH_a, dV_a][\sin \alpha_w, -\cos \alpha_w]^T = \frac{T_w + T_p}{L} \frac{\Delta\eta}{\cos \alpha_w} \quad \cdots\cdots\cdots\cdots\cdots\cdots(56)$$

where

T_p: the first order increment of the cable link tension.

This equation means that the relation between the force increment and the displacement increment of cable is linear and the lateral ridigity is $(T_w+T_p)/L$, if T_p is already known. These relations are shown in Fig. 9. The point S is the equilibrium point due to dead load and the relation between the true force increment and the displacement increment is expressed as a curved line ST which is nonlinear. Equation (52) which is equivalent to Eq. (36) is expressed by the tangent line SP at the point S and the equilibrium point for external forces is P. Equation (56) which is equivalent to Eq. (37) is expressed by the straight line SR which passes through point S and is parallel to the tangent line at the true point Q (if it is known). It is clear that the true point Q is placed between point P and point R, and we don't know which point is closer to the true point Q. Therefore we can not conclude that Eq. (37) is more accurate than the linearized equation (36) through those equations. A similar discrepancy holds for Melan's equation (46) and its linearized version Eq. (45).

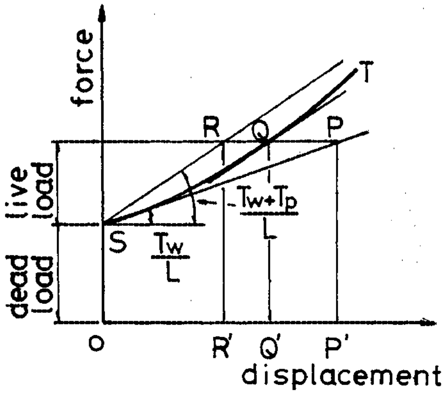

Fig. 9 Relation of equilibrium equations

Since Melan's equation (46) has approximations for the lateral rigidity of cable and the errors are different according to cable inclination angles, it is not easy to estimate the true property, but generally speaking slightly larger displacements may result in Melan's equation than in Rode's equation because of the lower approximation for the lateral rigidity of cable.

It should be noted that the consideration in this section is in the region of such static quantities as the cable displacement and bending moment of the stiffening girder which are mainly related to the static property of cable, and does not include the explanation of such a static quantity closely related to the inclination of hangers as horizontal displacements of the stiffening girder.

By means of the above consideration, it may be concluded that the vertical equilibrium equation in the deflection theory is an essentially linear equation and the equilibrium is taken from the cable configuration under dead load by using the lateral rigidity $(T_w+T_p)/L$ or $(T_w+T_p)\cos^2 \alpha_w/L$. In order to take account of the effect of cable configuration changes, the finite deformation theory should be used with iteration methods[15),16)].

5. CONSIDERATIONS BY NUMERICAL CALCULATIONS

In this section numerical analysis is made for a two hinged suspension bridge which has 300-1,000-300 m span, 6 truck lanes and four railways (only two railways are loaded simultaneously) shown in Fig. 10. Stiffening trusses shown in Fig. 11 are regarded as a beam with bending and axial rigidity. Member properties and dead loads

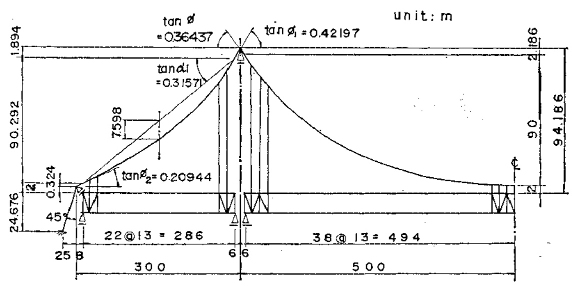

Fig. 10 Two hinged suspension bridge

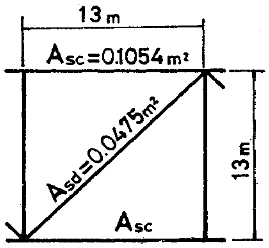

Fig. 11 Stiffening truss

Table 1 Properties of members (per one side)

Member	Terms	Value
Cable	Area	0.75398 m^2
	Modulus of elasticity	2.10×10^7 t/m^2
Hanger	Area (per one hanger)	0.01621 m^2
	Modulus of elasticity	1.4×10^7 t/m^2
Stiffening truss (regarded as a beam)	Area	0.2108 m^2
	Moment of inertia	8.9063 m^4
	Modulus of elasticity	2.1×10^7 t/m^2

are shown in Tables 1 and 2, respectively. The following typical cases are considered: I maximum of the cable tension, II maximum of the bending moment at the center span, III maximum of the deflection at the center span, IV maximum of the tower top displacement,and the loading conditions are shown in Fig. 12. Temperature changes are neglected for simplicity.

The characteristics of each analytical method are shown in Table 3. In Method 7, the hanger

Table 2 Dead loads (per one side)

Member	Center span	Side span	Direction
Cable	6.3102 t/m	6.2995 t/m	Cable direction
	6.4471 t/m	6.6164 t/m	Horizontal direction
Stiffening truss	16.0 t/m	16.0 t/m	Horizontal direction

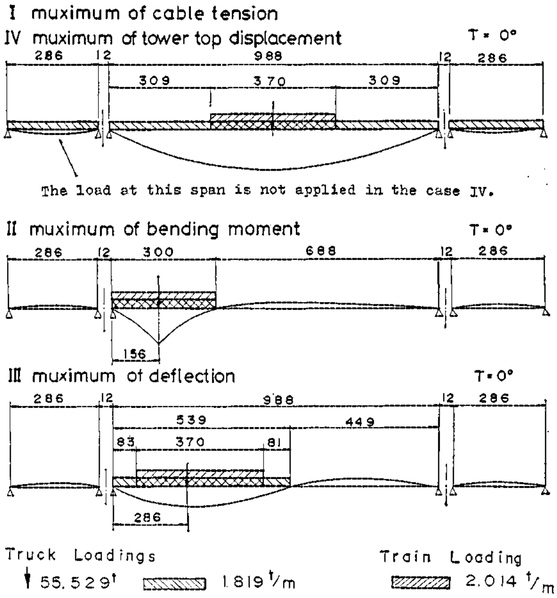

Fig. 12 Loading conditions

area is enlarged 200 times greater than in the other cases. The horizontal cable tension under dead load is 30,320 t according to program A[16] and 30,433 t according to program B[17]. This small difference is due to the programing technique between programs A and B, and is not essential. The calculation results are shown in Table 4, and the following is found.

1) If one considers the values from Method 1 to

Table 3 Distinction of calculation methods

Characteristics	Deform. change	Hanger elong.	Hanger incl.	Cable tension	Program
Method 1 *1	○ *10	○	○	Iteration	A
〃 2 *2	× *11	○	○	T_w	A
〃 3 *3	×	○	○	T_w+T_p	A
〃 4 *4	×	○	○	$T_w \cos^2 \alpha_w$	A
〃 5 *5	×	○	○	$(T_w+T_p)\cos^2 \alpha_w$ *8	A
〃 6 *6	×	○	×	$(T_w+T_p')\cos^2 \alpha_w$ *9	B
〃 7 *7	○	×	○	Iteration	A

*1 Finite deformation theory
*2 Linearized finite deform. theory (similar to Rode's linearized equation)
*3 Similar to Rode's non-linearized equation
*4 Similar to Melan's linearized equation
*5 Similar to Melan's non-linearized equation
*6 Deflection theory: Melan's equation
*7 Finite deformation theory (A_h is 200 times enlarged)
*8 T_p: cable tension increment in Method 2
*9 T_p': cable tension increment which satisfies the cable equation
*10 The effect of the term has been taken into account.
*11 The effect of the term has not been taken into account.

Table 4 Calculated results (two hinged susp. bridge)

Case	Disp. etc.	Method 1	Method 2	Method 3	Method 4	Method 5	Method 6	Method 7
I	T_p (t) *1	3,645 *2 100.0 *3	3,679 100.9	3,641 99.9	3,692 101.3	3,655 100.3	3,650 100.1	3,647 100.1
II	M (tm)	15,532 100.0	15,968 102.8	15,414 99.2	16,773 108.0	16,208 104.4	16,432 105.8	15,421 99.3
III	δ_{max} (m)	−2.02 100.0	−2.15 106.4	−2.03 100.5	−2.22 109.9	−2.10 104.0	−2.12 105.0	−2.01 99.5
IV	ξ_{tower} (cm)	15.8 100.0	16.6 105.1	15.9 100.6	17.1 108.2	16.4 103.8	16.5 105.7	15.7 99.4

*1 At the middle of the center span
*2 Calculated value
*3 % when value of Method 1 is regarded as 100%

be exact, Method 1 gives values between Methods 2 and 3 as was expected.
2) However, the values from Method 3 are very much closed to those from Method 1.
3) Since the lateral rigidity of cable is estimated at a little lower than the true value in Methods 4 and 5, the displacements are a little larger than the other cases as was expected.
4) The effect of hanger elongations is so small that the values among Methods 1-6 scarcely change without consideration of its effect.
5) The values from Method 6 which expectantly resemble those from Method 5 are a little different than those from Method 5. This is because the effect of hanger inclinations is taken into account by the lateral rigidity of hangers in Method 5 but is not taken into account in Method 6, or the boundary conditions of cable near the position of the tower are different from each other in Method 5 and Method 6 in regard to the vertical cable behavior.
6) The values of both the bending moment of the beam and the displacements from Method 5 are 4% larger than those from the exact solution. The reason is that the cable tensions are estimated lower to the degree of $\cos^2 \alpha_w$ times than the true ones.
7) Method 2 has a similar accuracy to Method 6.

Concerning a suspension bridge with a continuous stiffening truss, numerical calculations have been performed for a model which has cable spans of 270-1,100-270 m and almost the same dead and live loads as in the above-mentioned two hinged suspension bridge. The horizontal cable tension under dead load is about 36,000 t per one side of the bridge. The details of the bridge constants, load intensities, and conditions have been omitted here, and the reader should refer to paper[19].

Table 5 shows comparisons among Method 1 (the finite deformation theory), Method 2 (the linearized finite deformation theory) and Method 6 (the deflection theory) for the load cases: I maximum of the cable tension, II maximum of the bending moment of the stiffening truss at the position of the tower, III maximum of the horizontal displacement of the tower top, IV maximum of the horizontal displacement at the end of the stiffening truss.

From the table it is found that the following relation holds among the values of the static

Table 5 Calculated results (susp. bridge with continuous stiff. truss)

Case	Disp. etc.	Method 1 *1	Method 2 *2	Method 3 *3
I	Horizontal cable tension increment	6,019 t 100%	6,048 100	6,111 102
II	Bending moment of stiff. truss at tower position	−62,368 tm 100%	−63,331 102	−65,108 104
III	Horizontal disp. of tower top	25.7 cm 100%	26.3 102	28.2 110
IV	Horizontal disp. of end of stiff. truss	67.9 cm 100%	66.2 97	64.5 95

*1 Finite deformation theory
*2 Linearized finite deformation theory
*3 Deflection theory

quantities.

Method 1<Method 2<Method 6

except for the values of the load case IV. The hanger tensions and the hanger inclinations depend on each load case and the position of the stiffening truss, and the definite tendencies are not found.

Through the numerical calculations for the suspension bridge with a two hinged or a continuous stiffening truss, the foregoing description 7): Method 2 has a similar accuracy to Method 7, has been confirmed.

6. CONCLUSIONS

1) A bar member subjected to a tension T has both a longitudinal rigidity AE/L_0 and a lateral ridigity T/L. The former is related to the small displacement theory and the latter is to the restoring force of a strained chord or a simple pendulum.
2) If we assume that the effects of hanger elongations and inclinations are neglected in a suspension bridge and we take an equilibrium about a cable configuration before loading, the cable equation in which the cable tension is taken into consideration can be obtained by using the stiffness equation composed of the above-mentioned two ridigities, which is shown in Eq. (14).
3) In an analogous way, an equation of vertical equilibrium for the suspension bridge is obtained, which is shown in Eq. (18).
4) After neglecting the small quantities H_w/A_cE_c in Eqs. (14) and (18), both the usual cable equation on the basis of the small displacement theory and Rode's linearized equation (36) are obtained, and the latter becomes Melan's linearized equation (45) when we use Assumption 10.
5) In a similar manner, an expression of the bending moment of the stiffening girder in a two hinged suspension bridge is obtained,which is shown in Eq. (21).
6) Taking an equilibrium in an out-plane direction under dead load for the suspension bridge, transversal equations (25) and (26) are obtained, which have been originally derived by L. S. Moisseiff and F. Lienhard and reformed by C. Z. Erzen.
7) The effect of hanger inclinations can be evaluated by taking the lateral rigidity of hangers into account.
8) Rode's non-linearized equation (37) is obtained by neglecting hanger elongations, hanger inclinations, and the terms regarding H_w/A_cE_c, using the lateral rigidity of the stiffness equation consisting of a cable tension T_w+T_p alone, and also taking the equilibrium toward the perpendicular direciton to the cable axis under dead load.
9) In an analougous way, Melan's non-linearized equation is obtained by using the lateral rigidity corresponding to a cable tension $(T_w+T_p)\cos^2\alpha_w$.
10) In the deflection theory the values of the bending moment of the stiffening girder are 4 to 6% larger than the exact one. In the errors about 2% may be due to the neglection of the effect of hanger inclinations and the remainder may be due to the lower estimation of the cable tension as much as $\cos^2\alpha_w$ times, or the difference of the boundary condition beside it in a two hinged suspension bridge.
11) The linearized finite deformation theory using the lateral rigidities of cable tension under dead load holds almost the same accuracy as Melan's non-linearized equation.
12) As to the deflection theory, it should rather be said that it is not the theory in which the effect of cable deflection is taken into account, but it is the theory in which the effect of lateral rigidity of cable is taken into consideration on the basis of the small displacement theory in the sense that the equilibrium has been taken about the structural configuration under dead load.

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APPENDIX NOTATION

(x, y) = Cartesian coordinates (Fig. 5) and x, y also denote the ordinates of cable before loading
T_w, T = Cable tensions before and after loading, respectively
H_w, H = Horizontal components of cable tension before and after loading, respectively
V_w, V = Vertical components of cable tension before and after loading, respectively
R_w, R = Hanger tensions before and after loading, respectively
dx, dy = Horizontal and vertical displacements due to loading, respectively

dH, dV=Horizontal and vertical components of the first order increment of cable tension due to loading, respectively
dR=The first order increment of hanger tension due to loading
W_c=Cable weight at joint
L_0, L=Unstrained and strained lengths of cable between adjacent hangers respectively
A_c=Cross sectional area of cable
α_w, α=Slopes of cable before and after loading, respectively
Δx=Hanger interval, namely $\Delta x = x_{i+1} - x_i$
S=Shearing force in stiffening girder due to loading
S_B=Shearing force due to loading on a simple beam
M=Bending moment in stiffening girder due to loading
M_B=Bending moment due to loading on a simple beam
p=Live load per unit bridge length
ξ=Horizontal displacement of cable, namely, $\xi = dx$
η=Vertical displacement of cable, namely, $\eta = dy$
H_p=Horizontal component of the first order increment of cable tension due to loading, namely, $H_p = dH$
I_v, I_h=Moments of inertia at the area of stiffening girder in vertical and lateral bending, respectively
χ=Horizontal displacement of stiffening girder
E_s=Modulus of elasticity of stiffening girder
dR_h=Restoring force of hanger for lateral loading
u, v=Lateral displacements of cable and stiffening girder, respectively
w_c, w_s=Weights of cable and stiffening girder per unit bridge length, respectively
p_c, p_s=Lateral, external forces for cable and stiffening girder per unit bridge length, respectively
h, h_x=Hanger lengths
i=Subscript denoting member or joint number
a, b=Subscripts distinguishing member ends
Δ=Difference sign

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