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THERMODYNAMIC THEORY OF INELASTIC MATERIALS AND ITS APPLICATION TO STRESS WAVE PROPAGATION IN COHESIVE SOIL

By Koichi AKAI* *and Fusao* OKA**

1. INTRODUCTION

When the explosive and vibrational loads arising from earthquakes, traffics, construction machines and explosions are applied to the ground, the stress wave propagates through the ground. So, it is important to clarify the dynamical behavior of the ground and the mechanism of dynamic interaction between the ground and structures during vibratory loading to design the earthquake resistant structures. It should be said, however, that the present dynamic analysis of ground motion is not sufficient to be used. One of the reasons is that the constitutive relation of soil material is not sufficiently established. Macroscopically the soil material is a mixture constituting of three phases of solid, fluid and air, and is markedly characterized by inelasticity and nonlinearity. To describe the dynamic behavior of soil accounting for these characters, this study is concerned with the constitutive relation of cohesive soil under dynamic loading and the wave propagation characteristics through it. The wave propagation test is carried out by the special triaxial cell which is connected with the shock tube. Such a stress wave propagation test has been carried out by many researchers; Hampton and Wetzel[1)], Vey and Strauss[2)], Heierli[3)], Seaman[4)], Whitman[5)] and Akai and Hori[6),7)]. They tried the theoretical formulation of the constitutive relation of soil materials by assuming the various models for soil material; viscoelastic model, elasto-plastic model, compacting model and locking model, *etc.* For cohesive soil, it is pointed out by various authors that dynamic behaviors in comparatively wide frequency range are approximately expressed by assuming as a three-parameter viscoelastic model. Generally, it is known that there exist inelastic deformation and energy dissipation depending upon the stress or strain level during stress wave propagation. To these characters, various dissipative models are proposed by Salvadori[8)], Seaman and Heierli *et al.* The theoretical formulation of stress wave propagation problem accompanied with the inelastic deformation of material was developed for the metal bar in various ways. These are roughly divided into two theories; the strain rate dependent theory and the strain rate independent theory. The strain rate independent theory was mainly developed by Karman[9)], Taylor[10)] and Rakhmatulin. On the contrary, Sokolowskii and Malveren[11)] developed the longitudinal wave propagation theory (which is called "SM" theory), accounting for strain rate dependency. This "SM" theory was then generalized by Cristescu[12)] and Lubliner[13)]. Parkin[14)] applied this theory to the stress wave propagation test through sand which was carried out by Whitman[15)]. Perzyna[16),17)] developed the three-dimensional elastic-viscopastic theory. Adachi and Okano[18)] used the Perzyna's theory to formulate the dynamic stress-strain relations of cohesive soil. It should be noted, however, that Perzyna's theory of viscoplasticity is not sufficient to be applied to the dynamic plasticity of clay. The dynamic behavior of clay includes both high strain rate behaviors and comparatively low strain rate ones. So, the constitutive relations of cohesive soil must express the dynamic behavior in a wide range of strain and strain rate. In the present study, based on the internal state variable theory, the authors introduce the more elaborated constitutive relations for cohesive soil. In section 2, the experimental study by means of shock tube technique is presented. In section 3, the constitutive relations for inelastic materials for inelastic material is

* Dr. Eng., Professor of Civil Engineering, Kyoto University.
** M.S.C.E., Doctorial course student, Kyoto University.

proposed. In section 4, the constitutive relations for a normally consolidated clay is introduced, based on the internal state variable theory, and companied with the various experimental results; the undrained creep test, the stress relaxation test and the undrained shear test. In section 5 is discussed the one-dimensional stress wave propagation in a rate dependent material. Numerical results by the method of characteristics are shown and the comparison between experimental results and theoretical study is discussed. Main conclusions are finally noted in section 6.

2. EXPERIMENTAL STUDY

2.1 Introduction

The stress wave propagation test has been carried by using the triaxial cell connected with the air shock tube of which mechanism has been described in some detail elsewhere[7]. The soil specimen is made of Fukakusa dry clay seived by a 400μ net, kneaded with water, and consolidated under the pressure of $2.0\,kg/cm^2$ for 40 days long. Table 1 shows the physical properties of a silty loam used as the specimen. The soil specimen is a cylinder of 130 cm long and 7.5 cm diameter, consisting of four segments. The soil stress gage and accelerometer are embedded in the specimen.

Table 1 Physical properties of silty loam used.

Specific gravity	2.67
L.L	54.5-59.5 %
P.L	30.0-31.9 %
P.I	22.6-29.5
Uniformity coefficient	2.85
Water content	39.1-42.5 %
Bulk density	1.73-1.83 g/cm^3

2.2 Attenuation of Stress Wave

It is observed that the amplitude of stress wave through soils attenuates with distance. Figs. 1(a) and (b) show the attenuation of the peak stress in traveling through the clay specimen. In Figs. 1(a) and (b), the peak stress attenuates 30-50% by the distance 0.32 m from the input end of specimen, the rate of attenuation slowing down thereafter. The experimental results conducted by Akai, Hori and Shimogami[6] are shown in Fig. 2. The rate of attenuation behind the location of first soil stress gage embedded in the

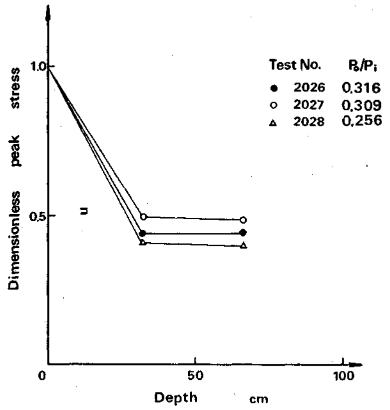

Fig. 1(a) Relationship between peak stress and depth.

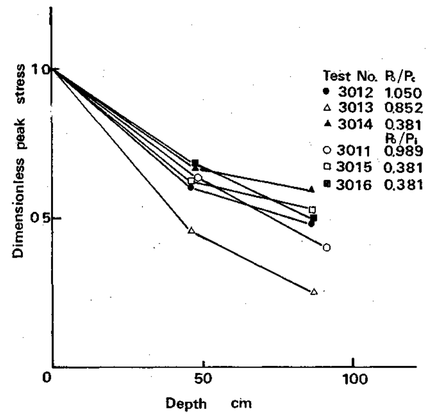

Fig. 1(b) Relationship between peak stress and depth.

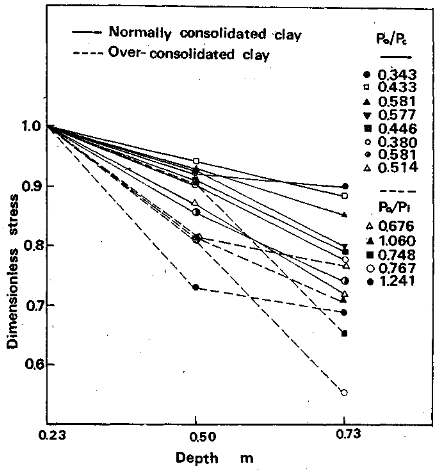

Fig. 2 Attenuation of peak stress in the rod wave propagation test.

specimen depends on the ratio p_0/p_c where p_0 is the maximum input peak pressure and p_c is the consolidation pressure. When tests were carried out under the cell pressure less than p_c with undrained condition, the pre-consolidated pressure p_i was used instead of p_c (see Fig. 2).

2.3 Wave Velocity

The wave velocity is calculated from the arrival time measured by the accelerometer. Fig. 3 shows the relationship between the rod wave

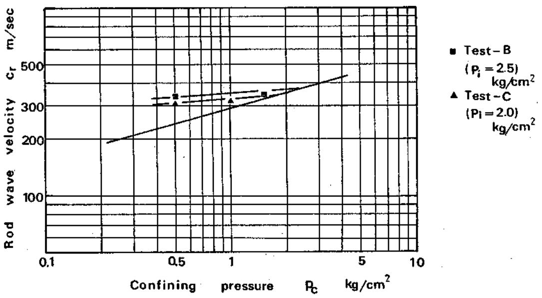

Fig. 3 Relationship between c_r and p_c.

velocity c_r (m/sec) and the confining pressure p_c (kg/cm^2) on the logarithmic paper. It is expressed by

$$c_r = 290 p_c^{0.25} \quad \cdots\cdots (1)$$

It is impossible to vary the void ratio of soil specimen at a constant effective stress, because the void ratio of a normally consolidated clay is a function of effective stress. In orther to investigate the influence of the effective stress on the wave velocity, two kinds of tests were conducted. The first kind of test was carried out after the primary consolidation is completed and the second one was under the cell pressure less than the consolidation pressure with undrained condition. From Fig. 3, it is understood that the wave velocity only depends on the consolidated pressure under the condition in which the void ratio is constant and the cell pressure is less than the consolidation pressure.

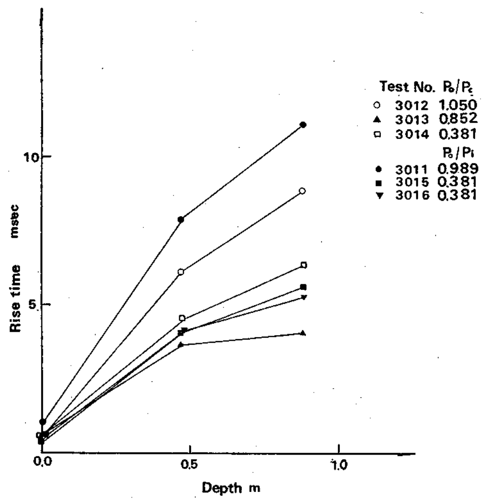

Fig. 4 Relationship between rise time and depth.

2.4 Rise Time

The rise time of stress wave is defined at a distance, as the difference between the arrival time of the wave and the time when stress is maximum. From Fig. 4, it is shown that the rise time increases with distance and finally approaches constant.

3. CONSTITUTIVE EQUATION FOR INELASTIC MATERIAL

3.1 Introduction

Generally, the deformation of material is accompanied with the energy dissipation. There exist several phenomenological approaches which account for the energy dissipation of continua. The most simplest one is to introduce a viscous stress which depends upon the rate of strain. The second one is to assume that the entire history of strain influences the stress in a manner compatible with a principle of fading memory[19]. This method demands the entire past history of material in order to predict a future thermomechanical response. But this seems to be impossible. The last one is to postulate the internal state variables which influence the dissipation effects. The state of the unit system of continua is determined, not only by the quantity which is observed on the surface of the unit system, but also by the quantity which governs the internal structure of the material. We call the quantity, which is not able to be observed on the surface of unit system of the material, the internal state variable. The non-equilibrium thermodynamics which introduces internal state variables have been developed by various researchers, Onsager, Biot[20], Ziegler[21], Coleman and Gurtin[22], Kestin and Rice[23], Lubliner[24], Valanis[25], Nemat-Nasser[26] and other investigators. Coleman and Gurtin developed the internal state variable theory based on the thermodynamics of continuous media developed by Coleman and Noll[27]. They assumed the existance of entropy and internal state variables. After all, they axiomatically assumed the existence of entropy. Valanis and Nemat-Nasser use the generalized or modified Caratheodory's principle as the second law of thermodynamics. Furthermore, assuming the first law of thermodynamics and the existence of temperature, they tried to establish the nonequilibrium thermodynamics involving the internal state variables. They consequently showed that the existence of entropy can be reduced from the postulate of internal variables and the entropy serves the potential at constant value of internal variables. In this study, we do not refer to these points any longer. At the present, there does not exist the exact definition of internal state variables. This seems to be very important for future development. In this section, the constitutive relation for the inelastic material is derived based on the nonequilibrium thermodynamics.

3.2 Constitutive Assumption

Preliminaries and Thermodynamics for inelastic material are shown in the appendix (A). It is assumed that the behavior of material at the point X_K is characterized by six response functions; ϕ, η, E_{KL}, q, P_{KL}, κ.

$$\left.\begin{aligned} \phi &= \hat{\phi}(T_{KL}, \theta, g, P_{KL}, \kappa) \\ \eta &= \hat{\eta}(T_{KL}, \theta, g, P_{KL}, \kappa) \\ E_{KL} &= \hat{E}_{KL}(T_{KL}, \theta, g, P_{KL}, \kappa) \\ q &= \hat{q}(T_{KL}, \theta, g, P_{KL}, \kappa) \\ \dot{P}_{KL} &= \hat{\dot{P}}_{KL}(P_{KL}, \theta, g, P_{KL}, \kappa) \\ \dot{\kappa} &= \hat{\dot{\kappa}}(T_{KL}, \theta, g, P_{KL}, \kappa) \end{aligned}\right\} \quad \cdots\cdots\cdots(2)$$

ϕ is the complementary energy density function, η the entropy density, E_{KL} strain tensor in Lagragian form, P_{KL} Kirchhoff stress tensor, q heat flux vector, g temperature gradient vector, and θ absolute temperature, respectively. P_{KL} and κ are internal state variables. The stability of solution of Eq. (2) is discussed in appendix (B). Generally, the internal state variables are the components of n-th order tensor. Here, in order to simplify the reduction of the theory, we use scalar and second order tensor as internal state variables. The rate equation of ϕ is

$$\dot{\phi} = \frac{\partial\phi}{\partial T_{KL}}\dot{T}_{KL} + \frac{\partial\phi}{\partial P_{KL}}\dot{P}_{KL} + \frac{\partial\phi}{\partial\kappa}\dot{\kappa} + \frac{\partial\phi}{\partial\theta}\dot{\theta} + \frac{\partial\phi}{\partial g}\dot{g} \quad \cdots\cdots(3)$$

The internal dissipation inequality is expressed by

$$\dot{\phi} - \frac{1}{\rho_0}E_{KL}\dot{T}_{KL} - \eta\dot{\theta} \geq 0 \quad \cdots\cdots(4)$$

Substituting Eq. (3) into Eq. (4),

$$\left(\frac{\partial\phi}{\partial T_{KL}} - \frac{1}{\rho_0}E_{KL}\right)\dot{T}_{KL} + \frac{\partial\phi}{\partial P_{KL}}\dot{P}_{KL} + \frac{\partial\phi}{\partial\kappa}\dot{\kappa} + \left(\frac{\partial\phi}{\partial\theta} - \eta\right)\dot{\theta} + \frac{\partial\phi}{\partial g}\dot{g} = 0 \quad \cdots\cdots(5)$$

Following the Coleman's method, we get the relations as follows:

$$\frac{1}{\rho_0}E_{KL} = \frac{\partial\phi}{\partial T_{KL}} \quad \cdots\cdots(6)$$

$$\eta = \frac{\partial\phi}{\partial\theta} \quad \cdots\cdots(7)$$

$$\frac{\partial\phi}{\partial\theta} = 0 \quad \cdots\cdots(8)$$

$$\frac{\partial\phi}{\partial P_{KL}}\dot{P}_{KL} + \frac{\partial\phi}{\partial\kappa}\dot{\kappa} \geqq 0 \quad \cdots\cdots(9)$$

In view of Eq. (8), we must exclude g from the independent variables of ϕ.

3.3 Constitutive Equation for Inelastic Materials

In order to satisfy Eq. (9), we assume following internal equations as the sufficient condition of Eq. (9).

$$P_{KL} = M_1\frac{\partial\phi}{\partial P_{KL}}, \quad M_1 = M_1(T_{KL}, P_{KL}, \kappa, \theta) \geq 0 \quad \cdots\cdots(10)$$

$$\kappa = M_1\frac{\partial\phi}{\partial\kappa}, \quad M_2 = M_2(T_{KL}, P_{KL}, \kappa, \theta) \geq 0 \quad \cdots\cdots(11)$$

From Eqs. (10) and (11), the left hand side of Eq. (9) becomes

$$\frac{\partial\phi}{\partial P_{KL}}\dot{P}_{KL} + \frac{\partial\phi}{\partial\kappa}\dot{\kappa} = M_1\left(\frac{\partial\phi}{\partial P_{KL}}\right)^2 + M_2\left(\frac{\partial\phi}{\partial\kappa}\right)^2 \geq 0$$

Therefore, Eqs. (10) and (11) are sufficient conditions for Eq. (9). The internal state variable κ is the hardening parameter in the classical theory of plasticity. Therefore, on the analogy of hardening rule in the theory of plasticity, we postulate that scalar variable κ is determined by

$$\dot{P}_{KL} = G_{KL}(T_{KL}, P_{KL}, \kappa, \theta)\dot{\kappa} \quad \cdots\cdots(12)$$

Under the condition that P_{IJ} is constant, the function f is defined by

$$f = \int_0^{T_{KL}} M_1\frac{\partial\phi}{\partial P_{KL}}dT_{KL} \quad (P_{IJ} = \text{constant}) \quad \cdots\cdots(13)$$

If $\dfrac{\partial\dot{P}_{KL}}{\partial T_{IJ}} = \dfrac{\partial\dot{P}_{IJ}}{\partial T_{KL}}$, the term in the integrand is exactly differential. Then

$$\dot{P}_{KL} = \frac{\partial f}{\partial T_{KL}} \quad \cdots\cdots(14)$$

From Eqs. (6) and (12) the strain rate may be expressed as follows:

$$\frac{1}{\rho_0}\dot{E}_{KL}={}_1\beta_{KLIJ}\dot{T}_{IJ}+{}_2\beta_{KLIJ}\dot{P}_{IJ} \quad \cdots\cdots\cdots(15)$$

In view of Eq. (14), the stress-strain relation is given by

$$\frac{1}{\rho_0}\dot{E}_{KL}={}_1\beta_{KLIJ}\dot{T}_{IJ}+{}_2\beta_{KLIJ}\frac{\partial f}{\partial T_{IJ}} \quad \cdots\cdots(16)$$

where

$${}_1\beta_{KLIJ}={}_1\beta_{KLIJ}(T_{MN}, P_{MN}, \theta, \kappa)$$
$${}_2\beta_{KLIJ}={}_2\beta_{KLIJ}(T_{MN}, P_{MN}, \theta, \kappa)$$
$$f=f(T_{MN}, P_{MN}, \theta, \kappa)$$

3.4 Relation to the Rate Independent Behavior

If ${}_1\beta_{KLIJ}$, ${}_2\beta_{KLIJ}$ and f in the Eq. (16) are determined, the stress-strain relation can be obtained. One of the methods to determine the f-function in dynamic range is to clear the relation between f-function in equilibrium and that in dynamic range. In the process of reducing Eq. (16), differentiation is done with respect to time t. If differentiation is performed with respect to the other parameter which increases with time and does not related with time explicitly, the stress-strain relation becomes rate independent. In the case of rate independent behavior, f is denoted by f_s. Perzyna defined the F-function in the elastic-viscoplastic material developed by himself and gave the relation between f_d and f_s. f_d is a dynamic loading function and f_s is a static loading function.

$$F=(f_d-f_s)/f_s$$

$F=0$ is a static yield condition. In the present study, we define F-function in the sense of generalization of the concept of that developed by Perzyna. F-function may be arbitrary function if f is equal to f_s only when $F=0$.

$$F=F(T_{KL}, P_{KL}, \theta, \kappa) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(17)$$

Here we must note that f and f_s are not equal to loading functions in the sense of Perzyna's theory. When F and f_s are given, therefore, the dynamic stress-strain relation is given by

$$\frac{1}{\rho_0}\dot{E}_{KL}={}_1\beta_{KLIJ}\dot{T}_{IJ}+{}_2\beta_{KLIJ}\frac{\partial f}{\partial T_{IJ}} \quad \cdots\cdots(18)$$

$$\left.\begin{aligned} &f=f(F, P_{MN}, T_{MN}, \theta, \kappa)\\ &{}_1\beta_{KLIJ}={}_1\beta_{KLIJ}(F, P_{MN}, T_{MN}, \theta, \kappa)\\ &{}_2\beta_{KLIJ}={}_2\beta_{KLIJ}(F, P_{MN}, T_{MN}, \theta, \kappa)\end{aligned}\right\} \cdots(19)$$

3.5 Comparison with Other Theories

Comparing the reduced stress-strain relation Eq. (16) with the other theories, we shall discuss the theoretical feature of Eq. (16). Now, there have been proposed various elastic-viscoplastic theories. We shall begin with the elastic-viscoplastic theory proposed by Perzyna[16]. He proposed the following stress-strain relation in the case of infinitesimal strain field.

$$\dot{\varepsilon}_{ij}=\frac{\dot{s}_{ij}}{2\mu}+\frac{1-2\nu}{E}\dot{\sigma}_{kk}\frac{1}{3}\delta_{ij}+\gamma(\theta)<\Phi(F)>\frac{\partial f_d}{\partial \sigma_{ij}} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(20)$$

where ε_{ij}; strain tensor, σ_{ij}; stress tensor.

$$s_{ij}=\sigma_{ij}-\frac{1}{3}\sigma_{kk}\delta_{ij} \qquad <\Phi(F)>=\Phi(F) \quad F>0$$
$$=0 \quad F\leq 0$$

Static yield condition is given by

$$F=\frac{f_d(\sigma_{ij}, \sigma_{ij}^p)}{\kappa}-1=0$$

where $\kappa=\kappa\left(\int_0^{\varepsilon_{ij}^p}\sigma_{ij}\,d\varepsilon_{ij}^p\right)$, κ; work-hardening parameter.

Starting with the Drucker's postulate of stable inelastic material and adding the assumption of decomposition of strain rate tensor, Perzyna obtained the convexity of the subsequent dynamic loading surfaces and the normality of the inelastic strain rate vector to the yield surface. After that, he thermodynamically formulated the elastic-viscoplastic body by using the themodynamics with internal state variables developed by Coleman and Gurtin[22]. In that study, ε_{ij}^{vp} is regarded as the internal state variable. Perzyna did not refer to the normality of the inelastic strain rate vector to the yield surface. Finally the following relation was obtained.

$$\dot{\varepsilon}_{ij}^{vp}=\gamma(\theta)<\Phi(F)>M_{ij}(\varepsilon_{kl}, \theta, \sigma_{kl}, \kappa)$$

So, the normality of inelastic strain rate vector to the dynamic yield surface must be postulated in order to reduce Eq. (20). In contrast to this, in Eq. (18), the normality of ε_{ij}^{vp} to f is not used, but the exact differentiability is postulated. And function $<\Phi(F)>$ which is used by Perzyna dose not appear. Differentiating with respect to the parameter which involves time implicitly, Eq. (18) becomes rate independent stress-strain relation. In Eq. (18), $\dot{E}_{KL}^{vp}$ (viscoplastic strain rate tensor) is defined by

$$\dot{E}_{KL}^{vp}=\rho_0\,{}_2\beta_{KLIJ}\frac{\partial f}{\partial T_{IJ}}=\rho_0\,{}_2\beta_{KLIJ}\dot{P}_{IJ} \quad \cdots(21)$$

Integrating the Eq. (20) along the path, we get

$$E_{KL}^{vp}=\int_0^{P_{IJ}}\rho_0\,{}_2\beta_{KLIJ}\,dP_{IJ} \quad \cdots\cdots\cdots\cdots\cdots\cdots(22)$$

In equilibrium state, as the stress tensor is used for independent variable, the conception of loading surface in classical plasticity theory must be introduced in order to distinguish between the loading state and unloading state. Considering the loading process, the following relation is reduced from Eqs. (14) and (17).

$$E^p_{KL} = \int_0^{T_{MN}} \rho_0\, {}_2\beta_{KLIJ} \frac{-\dfrac{\partial F}{\partial T_{MN}}\dfrac{\partial f}{\partial T_{IJ}}}{\left(\dfrac{\partial F}{\partial P_{MN}}+\dfrac{\partial F}{\partial \kappa}\dfrac{\partial \kappa}{\partial P_{MN}}\right)\dfrac{\partial f}{\partial T_{MN}}}\, dT_{MN}$$

(along the path) ....(23)

where E^p_{KL} is the plastic strain tensor.

Valanis obtained the second law of thermodynamics by generalizing the Caratheodory's principle and proved the existence of entropy and temperature. Valanis[25] derived the stress-strain relation of viscoelastic material, elastic-viscoplastic material and elastic-plastic material by using the intinsic time scale $Z(\zeta)$ which is a function of time and strain. The equation equal to Eq. (23) is given by

$$-\frac{dP_{ij}}{dZ}\frac{\partial \phi}{\partial P_{ij}} \geq 0 \quad \cdots\cdots(24)$$

Integrating Eq. (24) along the path of $Z(\zeta)$, P_{ij} is obtained. And substituting P_{ij} into the equation which is equal to Eq. (19), stress-strain relation is obtained. So, there does not exist the function that serve as the plastic potential. After all, if free energy function is given, the stress-strain relation is obtained. More interesting theoretical approaches are achieved by Rice[28], Lubliner[29] *et al.* In the present, however, we do not refer to it.

4. CONSTITUTIVE EQUATION FOR NORMALLY CONSOLIDATED CLAY

4.1 Reduction of Constitutive Equation

We will be restricted to infinitesimal strain field and isothermal condition. ε_{ij} is the strain tensor, ε^{vp}_{ij} the viscoplastic strain tensor and σ_{ij} the stress tensor. Deviatoric strain is defined by e_{ij} and deviatoric stress tensor is defined by s_{ij}; $e_{ij}=\varepsilon_{ij}-\frac{1}{3}\varepsilon_{kk}\delta_{ij}$, $s_{ij}=\sigma_{ij}-\frac{1}{3}\sigma_{kk}\delta_{ij}$. The second invariant of deviatoric strain tensor is denoted by $\sqrt{2J_2}=\sqrt{s_{ij}s_{ij}}$. Eq. (18) is rewritten by

$$\dot{\varepsilon}_{ij}={}_1\beta_{ijkl}\dot{\sigma}_{kl}+{}_2\beta\frac{\partial f}{\partial \sigma_{ij}} \quad \cdots\cdots(26)$$

where β_2 is a scalar function. Elastic strain rate is defined by

$$\dot{\varepsilon}^e_{ij}={}_1\beta_{ijkl}\dot{\sigma}_{kl}$$

There does not exist the exact theory for stress-strain relation of fully saturated clay in equilirium state. Roscoe's original theory[30] and Burland's modified energy theory[31] are not sufficient to explain the behavior of fully saturated clay in equilibrium state, but they can explain the behavior of saturated clay in equilibrium to some extent. At the first step, we will use the Roscoe's original theory to construct the constitutive equation of clay, which includes the rate effect. Two following assumptions are set up in order to apply Eq. (26) to normally consolidated clay.

(1) We will use the Roscoe's original energy theory which is extended to three-dimensional case by Adachi *et al.* as the constitutive theory in equilibrium. f-function in equilibrium is approximated by static yield function f_s of extended Roscoe's original theory given by

$$f_s-\kappa_s=\sqrt{2J_2}+M^*\sigma'_m \ln \sigma'_m/\sigma'_{my}=0 \quad \cdots\cdots(27)$$

where σ'_{my} is a hardening parameter shown in Fig. 5.

(2) F-function is given by

$$F=F(\varepsilon^{vp}_{ij}, T_{ij}, \kappa, \theta)$$
$$=f_s-f$$
$$=\alpha(\varepsilon^{vp}_{ij})(\sqrt{2J_2}^{\,d}-\sqrt{2J_2}^{\,s}) \quad \cdots\cdots(28)$$

Therefore, if $F=0$, $f=f_s$.

f is given by

$$f=f_s-F \quad \cdots\cdots(29)$$

Fig. 5 shows the manner how to determine f. In this figure, α has following form.

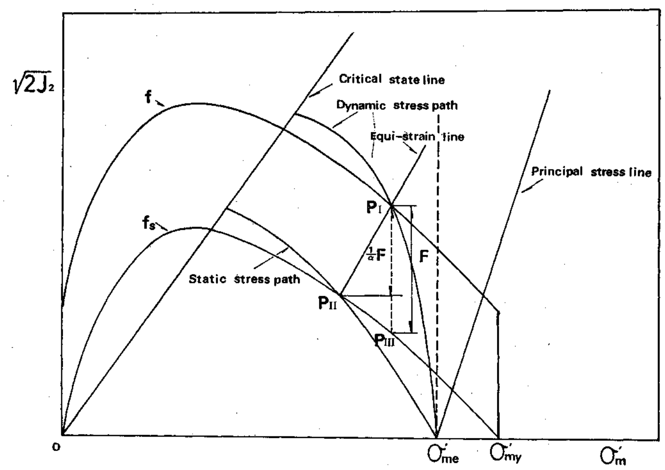

Fig. 5 Manner to determine the function f.

$$\alpha=\frac{\sqrt{2J_2}(P_{\rm I})-\sqrt{2J_2}(P_{\rm III})}{\sqrt{2J_2}(P_{\rm I})-\sqrt{2J_2}(P_{\rm II})} \qquad \cdots\cdots(30)$$

where

$$\sqrt{2J_2}(P_{\rm III})+M^*\sigma'_m(P_{\rm III})\ln\sigma'_m(P_{\rm III})/\sigma'_{my}=0 \qquad \cdots\cdots(31)$$

$$\sqrt{2J_2}(P_{\rm II})+M^*\sigma'_m(P_{\rm II})\ln\sigma'_m(P_{\rm II})/\sigma'_{my}=0 \qquad \cdots\cdots(32)$$

The point $P_{\rm I}$ has the inelastic strain equivalent to the point $P_{\rm II}$. From Eqs. (31) and (32), following relation is given

$$\sqrt{2J_2}(P_{\rm III})=\frac{\sigma'_m(P_{\rm III})}{\sigma'_m(P_{\rm II})}\sqrt{2J_2}(P_{\rm II})-M^*\sigma'_m(P_{\rm III})\ln\frac{\sigma'_m(P_{\rm III})}{\sigma'_{\rm II}(P_{\rm II})}=0 \qquad \cdots\cdots(33)$$

F is obtained from Eqs. (28), (30) and (33) as:

$$F=\sqrt{2J_2}(P_{\rm I})-\frac{\sigma'_m(P_{\rm I})}{\sigma'_m(P_{\rm II})}\sqrt{2J_2}(P_{\rm II})+M^*\sigma'_m(P_{\rm I})\ln\frac{\sigma'_m(P_{\rm I})}{\sigma'_m(P_{\rm II})} \qquad \cdots\cdots(34)$$

F is zero only when $\sigma'^d_m=\sigma'^s_m$ and $\sqrt{2J_2}^d=\sqrt{2J_2}^s$. So, F is satisfied with the condition that F is zero in equilibrium state. In view of Eq. (28),

$$f=\sqrt{2J_2}+M^*\sigma'_m\ln\frac{\sigma'_m}{\sigma'_{my}}-F \qquad \cdots\cdots(35)$$

$$\frac{\partial f}{\partial\sigma_{ij}}=\frac{s_{ij}}{\sqrt{2J_2}}+M^*\frac{\delta_{ij}}{3}\ln\frac{\sigma'_m}{\sigma'_{my}}+\frac{M^*}{3}\delta_{ij} \qquad \cdots\cdots(36)$$

Substituting Eq. (36) into Eq. (26), stress-strain relation is obtained as:

$$\dot\varepsilon_{ij}={}_1\beta_{ijkl}\dot\sigma_{kl}+\beta_2\left\{\left(M^*-\frac{\sqrt{2J_2}^s}{\sigma^s_m}\right)\frac{\delta_{ij}}{3}+\frac{s_{ij}}{\sqrt{2J_2}}+\frac{M^*}{3}(\ln\sigma'_m-\ln\sigma'^s_m)\delta_{ij}\right\} \quad \cdots(37)$$

In the case of conventional axi-symmetric triaxial compression,

$$\sqrt{2J_2}=\sqrt{\frac{2}{3}}(\sigma'_{11}-\sigma'_{33})=\sqrt{\frac{2}{3}}q\,, \qquad e_{11}=\varepsilon_{11}\,,$$

$$\sigma'_m=\frac{1}{3}\sigma'_{kk}\,, \qquad s_{ij}/\sqrt{2J_2}=\sqrt{\frac{2}{3}}\,,$$

$$\dot s_{11}=\frac{2}{3}(\sigma'_{11}-\sigma'_{33})$$

where σ'_{ij} denotes the effective stress tensor. From the Eq. (37), the stress-strain relation in undrained axi-symmetric triaxial compression is expressed by

$$\dot e_{11}=\gamma_1\dot s_{11}+\sqrt{\frac{2}{3}}\beta_2 \qquad \cdots\cdots(38)$$

As $\dot\varepsilon_{kk}=0$,

$$\gamma_2\sigma'_m+\beta_2\left(M^*-\frac{\sqrt{2J_2}^s}{\sigma'^s_m}+M^*\ln\frac{\sigma'_m}{\sigma'^s_m}\right)=0 \qquad \cdots\cdots(39)$$

where it is assumed that ${}_1\beta_{ijkl}\dot\sigma'_{kl}=\gamma_1\dot s_{ij}+\frac{1}{3}\dot\sigma'_m\gamma_2\delta_{ij}$.

4.2 Phenomenological Nature of Parameters

We shall discuss how the test results in undrained axi-symmetric triaxial state can be formulated by Eqs. (38) and (39). Fig. 6 shows the relation between the viscoplastic strain rate $\dot\varepsilon^{vp}_{ij}$ and $\sqrt{\frac{2}{3}}(\sqrt{2J_2}^d-\sqrt{2J_2}^s)/\sigma'_{me}$ in the undrained creep test in semi-log scale[32]. It is evident from Fig. 6 that $(q-q_s)/\sigma'_{me}$ increases in proportional

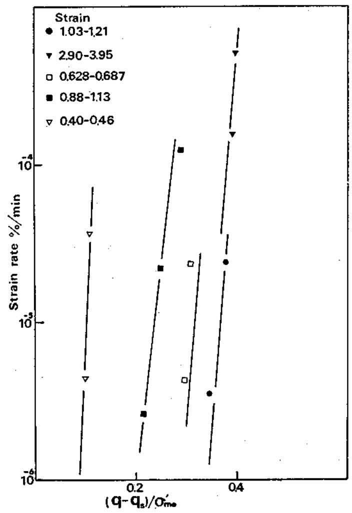

Fig. 6 Relationship between strain rate and $(q-q_s)/\sigma'_{me}$.

to the logarithm of strain rate if the amount of the strain is equal. Therefore, β_2 may be taken as

$$\beta_2=C_0(\varepsilon^{vp}_{11})\exp\left\{\frac{m(\varepsilon^{vp}_{11})}{\sigma'_{me}}(q-q_s)\right\} \qquad \cdots\cdots(40)$$

$$\ln\dot\varepsilon^{vp}_{11}=\ln\sqrt{\frac{2}{3}}C_0+\frac{m}{\sigma'_{me}}(q-q_s) \qquad \cdots\cdots(41)$$

Experimental equation in the relaxation test, proposed by Murayama *et al.*[38] is given by

$$q_r(\varepsilon_{11},t)-q_r(\varepsilon_{11},t_0)=-\beta(\varepsilon_{11})\log(t/t_0) \quad \cdots(42)$$

where q_r is the deviator stress at time t when the total strain is hold constant and $\beta(\varepsilon_{11})$ is the relaxation velocity, $dq_r/d\log(t)$. Substituting Eq. (42) into Eq. (38) and assuming $q_r(\varepsilon_{11},t_0)=q_s$, we get

$$-\frac{2}{3}\gamma_1\frac{\beta}{2.303}\frac{1}{t} + \sqrt{\frac{2}{3}}C_0\exp\left\{\frac{m}{\sigma'_{me}}\frac{\beta}{2.303}\ln(t/t_0)^{-1}\right\}=0 \quad \cdots\cdots(43)$$

The condition to satisfy Eq. (43) is obtained by

$$\frac{m(\varepsilon_{11}^{vp})}{\sigma'_{me}}\frac{\beta}{2.303}=1 \quad \cdots\cdots(44)$$

$$\frac{2}{3}\frac{\beta\gamma_1}{2.301}=C_1 t_0\,, \quad C_1=\sqrt{\frac{2}{3}}C_0 \quad \cdots\cdots(45)$$

From Eq. (44), β increases proportionally to the consolidation pressure. This results coinside with the test results. In the strain-rate controlled shear test indicated by Fig. 7 which shows the correlation between q and $\log\dot\varepsilon_{11}$, it is clearly noticed that the slope σ'_{me}/m increases as the strain increases and becomes constant. The intercept C_1 in Fig. 6 decreases as the strain increases, even if $(q-q_s)/\sigma'_{me}$ is constant. The rate of decrease becomes small as the strain increases.

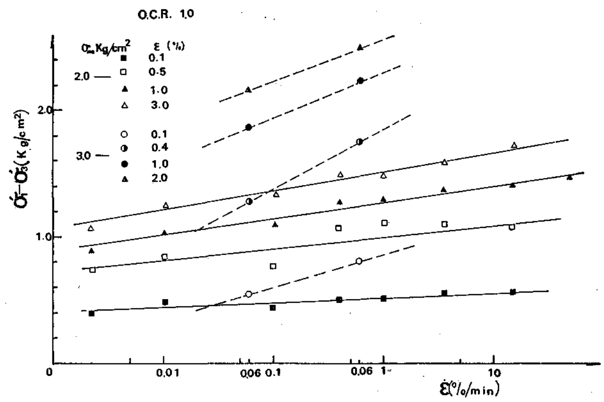

Fig. 7 Relationship between deviator stress and logarithm of strain rate.

Eq. (37) includes the following experimental equation developed by Yong and Japp[34] for dynamic loading constant strain-rate triaxial test if neglecting the elastic strain rate:

$$q(\varepsilon_{11},\dot\varepsilon_{11})-q(\varepsilon_{11},\dot\varepsilon^0_{11})=\alpha(\varepsilon_{11})\log(\dot\varepsilon_{11}/\dot\varepsilon^0_{11}) \quad \cdots\cdots(46)$$

Akai and Adachi[35] show that $\alpha(\varepsilon_{11})$ in Eq. (46) is approximately equal to $\beta(\varepsilon_{11})$ in Eq. (42) and Eq. (46) can express the results of undrained creep tests. Eq. (38) satisfies these results. Next Eq. (39) is discussed. Substituting Eq. (39) into Eq. (42),

$$\gamma_2\sigma'_m=-\beta_2(F)\left\{\left(M^*-\frac{\sqrt{2J_2}^{\,s}}{\sigma'^{s}_m}\right)+M^*\ln\sigma'_m/\sigma'^{s}_m\right\}$$

$$=C_2\exp\left\{\frac{m}{\sigma'_{me}}\frac{\beta}{2.303}\ln(t/t_0)^{-1}\right\}$$

$$=-C_2\frac{t_0}{t} \quad \cdots\cdots(46)$$

where $\frac{m}{\sigma'_{me}}\frac{\beta}{2.303}=1$ and

$$C_2=C_0\left\{\left(M^*-\frac{\sqrt{2J_2}^{\,s}}{\sigma'^{s}_{me}}\right)+M^*\ln\sigma'_m/\sigma'^{s}_m\right\}.$$

Therefore, if C_2/γ_2 only depends on strain, Eq. (47) can be integrated as:

$$\sigma'_m=-A\ln(t/t_0)+\sigma'_{m0} \quad \cdots\cdots(48)$$

where $A=\frac{1}{3}\frac{\beta}{2.303}$. Therefore, $C_2=\frac{\gamma_2 A}{t_0}$.

A depends on the test results that the stress path is parallel to the principal stress line in relaxation test. In view of Eqs. (48) and (42), $\dot\sigma'_m:\dot q=1:3$. Following the above discussion, C_1 is determined from Eq. (45).

$$C_1=\frac{1}{t_0}\frac{2\gamma_1\beta}{6.909}=\frac{2\gamma_1 C_2}{\gamma_2} \quad \cdots\cdots(49)$$

From the above discussion, the parameters which are necessary to use Eqs. (38) and (39) are γ_1, γ_2, β, σ'_{me}, M^*, λ, κ and e_0 (initial void ratio). By these parameters, the dynamic stress-strain relation and stress path for normally consolidated clay are obtained. From the test results of Akai *et al.*[35] and Murayama *et al.*[33], $\alpha(\varepsilon_{11})=0.20\ \text{kg/cm}^2$ and $\beta(\varepsilon_{11})=0.14$, $0.17\ \text{kg/cm}^2$ were obtained for the clay samples consolidated by $2\ \text{kg/cm}^2$. Therefore $m=23.0$, 32.7 from Eq. (44).

5. ONE-DIMENSIONAL STRESS WAVE PROPAGATION

5.1 Characteristics

By the method of characteristics, the original system of first order partial differential equations can be transformed into a system involving characteristic coordinates by which the differentiation becomes considerably simplified. According to Courant and Hilbert[36], the characteristic lines play a role as wave fronts. These are the lines across which solution of partial differential equations suffers discontinuities. We shall discuss the wave propagation in the material which is expressed by the following constitutive equation.

$$\frac{\partial\sigma}{\partial t}=f(\sigma,\varepsilon)\frac{\partial\varepsilon}{\partial t}+g(\sigma,\varepsilon) \quad \cdots\cdots(50)$$

where ε denotes the total strain and σ the axial stress. Thin rod is considered in order to neglect the lateral inertia. The coordinate of any particle is X in Lagrangian form and x in current coordinate. The displacement of particle u is defined by

$$u=X-x \quad \cdots\cdots(51)$$

The strain ε is given by

$$\varepsilon=1-\frac{\partial x}{\partial X} \quad \cdots\cdots(52)$$

ε is taken positive in compression. The particle velocity is given by $v=\partial x/\partial t$. Therefore,

$$-\frac{\partial \varepsilon}{\partial t}=\frac{\partial v}{\partial X} \quad \cdots\cdots(53)$$

We neglect the body force in the equation of motion.

$$-\frac{\partial \sigma}{\partial X}=\rho_0\frac{\partial v}{\partial t} \quad \cdots\cdots(54)$$

Along the curve $\phi=0$, the interior derivative of v, σ and ε is continuous. $\phi=0$ denotes the wave front.

$$\left.\begin{aligned} dv&=\frac{\partial v}{\partial X}dX+\frac{\partial v}{\partial t}dt\,,\\ d\sigma&=\frac{\partial \sigma}{\partial X}dX+\frac{\partial \sigma}{\partial t}dt\,,\\ d\varepsilon&=\frac{\partial \varepsilon}{\partial X}dX+\frac{\partial \varepsilon}{\partial t}dt \end{aligned}\right\} \quad \cdots\cdots(55)$$

As Eqs. (50), (53) and (54) form quasi-linear partial differential equations, the characteristics exist.

$$dX=0\,, \quad dX/dt=\pm\sqrt{\frac{f(\sigma,\varepsilon)}{\rho_0}} \quad \cdots\cdots(56)$$

Along these characteristics, there exist following differential relations:

Along $dX=0$, $\quad d\sigma=f(\sigma,\varepsilon)\,d\varepsilon+g(\sigma,\varepsilon)\,dt \quad \cdots\cdots(57)$

Along $dX/dt=C$, $\quad d\sigma=\rho_0 C\,dv-g(\sigma,\varepsilon)\,dt \quad \cdots\cdots(58)$

where $C=\sqrt{\dfrac{f(\sigma,\varepsilon)}{\rho_0}}$.

The most simplest method to solve the above ordinary differential equation, Eqs. (57) and (58), is numerical integration. Replacing Eq. (50) with Eq. (38), Eqs. (57) and (58) becomes

Along $dX=0$, $\quad d\varepsilon=\dfrac{1}{E}d\sigma+\beta_2(F)\sqrt{\dfrac{2}{3}} \quad \cdots\cdots(59)$

Along $dX/dt=\pm C$,

$$d\sigma=\rho_0 C\,dv-\beta_0(F)\sqrt{\frac{2}{3}}E \quad \cdots\cdots(60)$$

where $\sigma=\sigma'_{11}-\sigma'_{33}$ and $E=\dfrac{3}{2\gamma_1}$

5.2 Numerical Results and Discussion

The parameters used here are assumed to be constant, and we postulate that the behavior is purely elastic if $f_s<0$ in unloading condition. Eqs. (59) and (60) are integrated along the characteristics by Massau's method. Table 2 shows the parameters used in the calculations. Fig. 8 shows the variation in the wave shape during the wave propagation. The wave shape in the neighbourhood of the peak stress becomes round as the wave advances. The rise time becomes large as the wave shape collapses. From Fig. 8,

Table 2 Parameters used in numerical calculation.

Young's Modulus	$E=1.4\times10^{7}$ kg/m^2
Density	$\rho_0=1.963\times10^{2}$ kg/m^{2}sec^2
Slope of e-logσ'_m line of consolidation test	$\lambda=0.127$
Slope of e-logσ'_m line of swelling test	$\kappa=0.0214$
Value of $((\sigma_{11}\ \sigma_{33})/\sigma'_m)$ at critical state	$M^*=1.300$

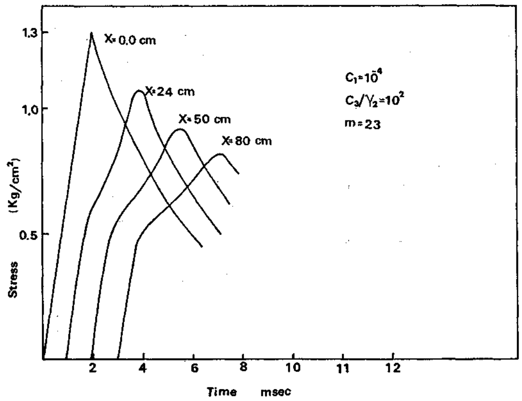

Fig. 8 Stress wave propagation (calculated results).

the attenuation of the peak stress is large in the first place of rod and becomes constant. Fig. 9 shows the stress-strain relations in wave propagation. The stress-strain relation is bilinear and the hysteretic type dissipation is predominant. The area of hysterises loop becomes small as wave propagates. The dynamic stress path is shown in Figs. 10 and 11 among which the former corresponds to Fig. 8. It is noticed from the stress path that the pore water pressure increases proportionally to the inelastic strain. Fig. 11 is the case that C_2 is larger than that in

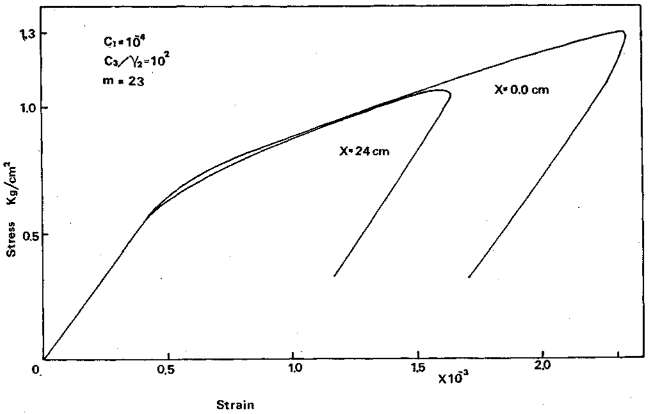

Fig. 9 Stress-strain relation (calculated results).

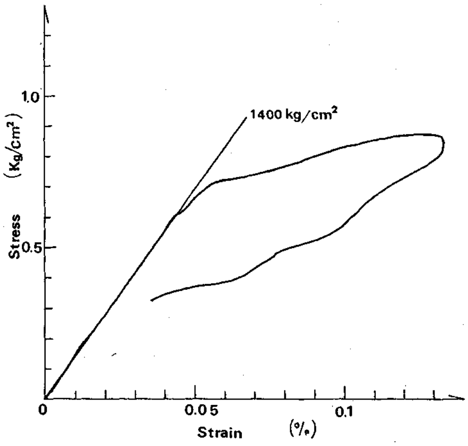

Fig. 12 Stress-strain relation during wave propagation (after Akai and Hori[6]).

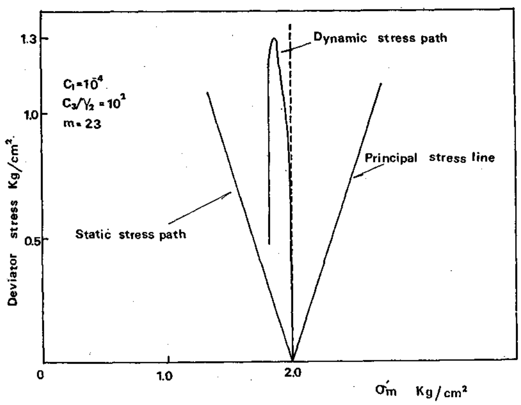

Fig. 10 Stress path (calculated result).

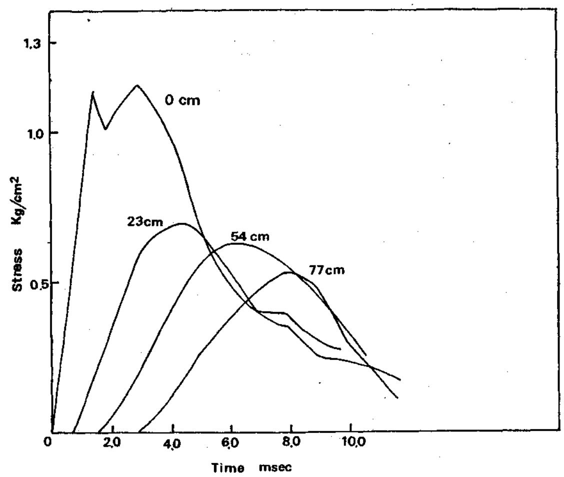

Fig. 13 Stress wave propagation (experimental result after Akai and Hori[6]).

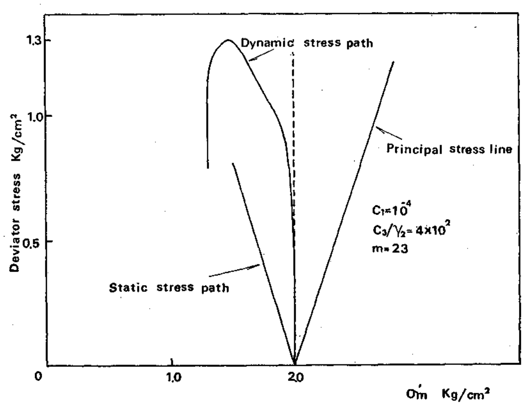

Fig. 11 Stress path (calculated result).

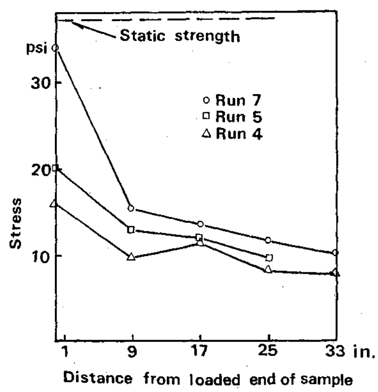

Fig. 14 Typical peak stress attenuation (after Vey & Strauss[2]).

Fig. 10. Fig. 12 shows the stress-strain relation obtained in the stress wave propagation test. The stress-strain relation is bi-linear type in loading part and coincides with the theoretical results. But in the unloading part the theoretical results differ from the experimental results. The reason may be considered that the elastic response of clay is not clearly known. Fig. 13 shows the experimental results of the variation in wave shape travelling through cohesive soil. The peak stress attenuates about 40 percent of the total in the first 23 cm in the specimen. The wave shape in the neighbourhood of peak stress becomes round and the rise time becomes large as the wave travels further distance. The above discussion shows that before-mentioned theoretical approach can approximotely explain the experimental results on the attenuation of the stress, the trend of rise time and the stress-strain relation. Figs. 14, 15 and 16 are the results of wave propagation test through compacted Kaolin clay, which was carried by Vey and Strauss[2]. Fig. 14 shows the stress-strain relations obtained in wave propagation tests, and it is similar to that of Fig. 9. The attenuation of the peak stress and the feature of the rise time are also similar to the numerical results as is shown in Fig. 8.

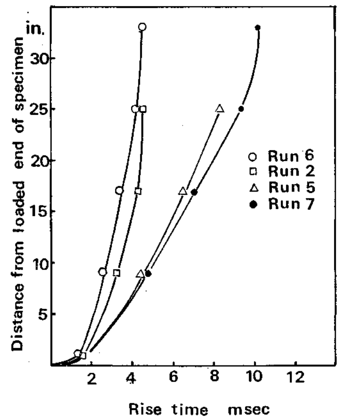

Fig. 15 Distance from loaded end of specimen versus rise time to peak stress (after Vey & Strauss[2]).

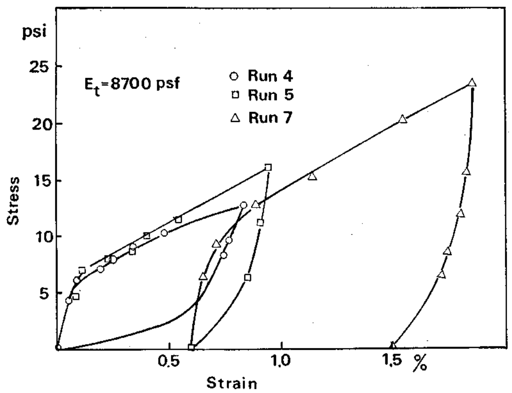

Fig. 16 Dynamic stress-strain curves (after Vey & Strauss[2]).

6. CONCLUSIONS

The authors propose a general constitutive relation for inelastic material based on the internal state variable theory and apply the proposed theory to express the behavior of clay. The dynamic stress-strain relation of clay, which is obtained from the theory, can explain the various test results: the undrained creep test, the stress relaxation test and the undrained strain rate-controlled shear test. And although, it can explain the unloading part, there remaines a distinction between the experiment and the theoretical results. In the future study, therefore, the elastic behavior of cohesive soil must be clarified to explain the dynamical behavior under unloading.

From a series of study the following main conclusions are obtained:

Conclusions in the experimental study.

(1) The peak stress attenuates by almost 30–50% at the neighbourhood of the surface of specimen and the attenuation rate slows down with distance.

(2) The attenuation rate of peak stress depends on p_0/p_c, in which p_0 denotes the input peak pressure and p_c is the consolidation pressure.

(3) The wave velocity c_r only depends upon the consolidated pressure p_c under the condition of constant void ratio.

(4) The rise time increases with time and then becomes constant.

Conclusions in the theoretical study.

(1) The general constitutive equation more suitable for clay than the elastic-viscoplastic body proposed by Perzyna is introduced based on the internal state variable theory to express the dynamic phenomena of continua, and it has been applied to a normally consolidated clay.

(2) Thus proposed constitutive relation can explain the dynamic behavior of a normally consolidated clay in various tests; the undrained creep test, the stress relaxation test, the un-

drained shear test and stress wave propagation test.

(3) The one-dimensional stress wave propagation through the proposed inelastic material is discussed by the method of characteristics. The numerical results by integrating the ordinary differential equation along characteristics coincide fairly well with the experimental results.

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APPENDIX (A)

(1) Preliminaries

Introducing a rectangular cartesian axial coordinate fixed in space, the coordinate of any particle is expressed by X_K in Lagrangian form and by x_k in current coordinate. A motion of the body is described by

$$x_k = x_k(X_K, t) \quad (k, K=1, 2, 3) \qquad \text{(A-1)}$$

where x_k reveals a single function of position X_K and time t. The line element of the substance is denoted by Eqs. (A-2) and (A-3).

$$dS^2 = dX_K\, dX_K \qquad \text{(A-2)}$$

$$ds^2 = dx_k\, dx_k = x_{k,K}\, x_{k,L}\, dX_K\, dX_L \qquad \text{(A-3)}$$

The strain tensor E_{KL} in Lagrangian form is defined by Eq. (A-4), using Kronecker's delta, δ_{kl}:

$$ds^2 - dS^2 = (x_{k,K}\, x_{k,L} - \delta_{KL})\, dX_K\, dX_L = 2E_{KL}\, dX_K\, dX_L \qquad \text{(A-4)}$$

Cauchy's stress tensor is expressed by second Kirchhoff stress tensor as follows:

$$t_{kl} = \frac{\rho}{\rho_0} x_{k,K}\, x_{l,L}\, T_{KL} \qquad \text{(A-5)}$$

where ρ is the current density and ρ_0 is the initial density. The deformation velocity tensor is defined by

$$d_{ij} = \dot{E}_{KL}\, X_{K,i}\, X_{L,j} \qquad \text{(A-6)}$$

(2) Thermodynamics of Inelastic Material

The energy balance equation for non-polar case in the local form is expressed by Eq. (A-7)

$$\rho\dot{\varepsilon} - t_{kl}\, d_{kl} = \rho r - \operatorname{div} q \qquad \text{(A-7)}$$

where ε denotes the internal energy density, q the heat flux vector and r the heat supply per unit mass and unit time. The Clausius-Duhem inequality may now be written in the material form by

$$\rho\theta\dot{\eta} - \rho r - \theta \operatorname{div}\left(\frac{q}{\theta}\right) \geq 0 \qquad \text{(A-8)}$$

where η is the entropy density and θ the absolute temperature. In view of Eq. (A-7), Eq. (A-8) becomes

$$\rho\dot{\eta} - (\rho\dot{\varepsilon} - t_{kl}\, d_{kl})\Big/\theta - \frac{1}{\theta^2} q_k\theta, k \geq 0 \qquad \text{(A-9)}$$

The free energy density ψ is introduced by

$$\psi = \varepsilon - \theta\eta \qquad \text{(A-10)}$$

Complementary energy density function ϕ is also introduced by

$$\phi = \frac{1}{\rho_0} E_{KL} T_{KL} - \psi \qquad \text{(A-11)}$$

Then

$$\dot{\phi} = \frac{1}{\rho_0}(\dot{E}_{KL} T_{KL} + E_{KL}\dot{T}_{KL}) - \dot{\psi} \qquad \text{(A-12)}$$

From Eqs. (A-10) and (A-12), Eq. (A-11) becomes

$$-\eta\dot{\theta} + \dot{\phi} - \frac{1}{\rho_0} E_{KL}\dot{T}_{KL} - \frac{1}{\rho_0\theta} q_k\theta, k \geq 0 \qquad \text{(A-13)}$$

From Eq. (A-13), internal dissipation inequality (Planck's inequality) is obtained by

$$\dot{\phi} - \frac{1}{\rho_0} E_{KL}\dot{T}_{KL} - \eta\dot{\theta} \geq 0 \qquad \text{(A-14)}$$

APPENDIX (B)

Stability of Solution

Coleman and Gurtin[22] defined the "attraction domain of constant temperature and strain" and discussed the stability of the solution of differential equation governing the internal state variables. Following the Coleman and Gurtin, the stability of the solution of Eq. (2) is discussed. It follows from Eq. (A-10) that

$$\dot{\psi}=\frac{\partial\psi}{\partial E_{KL}}\dot{E}_{KL}+\frac{\partial\psi}{\partial\theta}\dot{\theta}+\frac{\partial\psi}{\partial P_{KL}}\dot{P}_{KL}+\frac{\partial\psi}{\partial\kappa}\dot{\kappa} \quad \cdots\cdots(\text{A-15})$$

Substituting Eqs. (A-11) and (A-15) into Eq. (A-14),

$$\left(\eta+\frac{\partial\psi}{\partial\theta}\right)\dot{\theta}+\frac{\partial\psi}{\partial g}\dot{g}+\left(\frac{\partial\psi}{\partial E_{KL}}-\frac{1}{\rho_0}T_{KL}\right)\dot{E}_{KL}-\frac{\partial\psi}{\partial P_{KL}}\dot{P}_{KL}-\frac{\partial\psi}{\partial\kappa}\dot{\kappa}\geq 0 \quad \cdots\cdots(\text{A-16})$$

Since $\dot{\theta}$, $\dot{g}$ and $\dot{E}_{KL}$ may be given as arbitrary values,

$$\eta=-\frac{\partial\psi}{\partial\theta}, \quad \frac{\partial\psi}{\partial g}=0, \quad T_{KL}=\rho_0\frac{\partial\psi}{\partial E_{KL}} \quad \cdots\cdots(\text{A-17})$$

Therefore,

$$\dot{\phi}=\frac{1}{\rho_0}E_{KL}\dot{T}_{KL}+\eta\dot{\theta}-\frac{\partial\psi}{\partial P_{KL}}\dot{P}_{KL}-\frac{\partial\psi}{\partial\kappa}\dot{\kappa} \quad \cdots\cdots(\text{A-18})$$

On the other hand,

$$\dot{\phi}=\frac{\partial\phi}{\partial T_{KL}}\dot{T}_{KL}+\frac{\partial\phi}{\partial P_{KL}}\dot{P}_{KL}+\frac{\partial\phi}{\partial\kappa}\dot{\kappa}+\frac{\partial\phi}{\partial\theta}\dot{\theta} \quad \cdots\cdots(\text{A-19})$$

Comparing Eq. (A-18) with Eq. (A-19),

$$\frac{\partial\phi}{\partial P_{KL}}=-\frac{\partial\psi}{\partial P_{KL}}, \quad \frac{\partial\phi}{\partial\kappa}=-\frac{\partial\psi}{\partial\kappa} \quad \cdots\cdots(\text{A-20})$$

Consequently, following inequality is given.

$$-\frac{\partial\psi}{\partial P_{KL}}\dot{P}_{KL}-\frac{\partial\psi}{\partial\kappa}\dot{\kappa}\geq 0 \quad \cdots\cdots(\text{A-21})$$

$$\dot{\psi}\geq 0 \quad (E_{KL} \text{ and } \theta \text{ are constant})$$

When $\psi>0$, ψ serves Lyapunov function and the solution $(E_{KL}, P_{KL}, \theta, \kappa)$ is Lyapunov stable. If in addition $(E_{KL}, P_{KL}, \theta, \kappa)\rightarrow$fixed value as $t\rightarrow\infty$, the solution of Eq. (2) is called asymptotically stable in the sense of Lyapunov. In isothermal relaxation process, the solution of Eq. (2) is stable, but not necessarily stable in isothermal creep process.

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