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# CONVERGENCE OF FINITE ELEMENT LAX-WENDROFF METHOD FOR LINEAR HYPERBOLIC DIFFERENTIAL EQUATION

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## 1. INTRODUCTION

Recently, the finite element method has become one of the most commonly used methods for the computation of structural analysis by digital computers. A number of research workers have detected that the method is adaptable not only to structural analysis but also to other fields of engineering problems. This paper presents a form of finite element method applied to the analysis of linear hyperbolic differential equation. Other forms of finite element method applied to elliptic and parabolic equations are well studied and convergence is assured especially in the case of linear problems. On the other hand, the use of conventional finite element method to hyperbolic problems has sometimes failed to produce the well approximate solution. The computation often resulted in unstable solution. Therefore, it is noted that certain techniques should be employed to overcome instability. In the computation to solve hyperbolic equation by the finite difference method, the Lax-Wendroff method is the best known method which is widely used. The method has proven to be effective especially in the case of wave propagation problem. The method also assures stable solution even in nonlinear phenomena by numerical and theoretical considerations.

This paper will discuss the extended scheme of the Lax-Wendroff finite difference method, which is based on the techniques of finite element method. Thus, the scheme is called the finite element Lax-Wendroff method. Namely, expanding the unknown function into Taylor series of short time increment and employing the original equation, the algorithm of the finite element Lax-Wendroff method can be obtained. Convergence proofs of the method can be shown by using the concept of $L_2$ space.

The convergence studies of finite element methods applied to parabolic equation have been made by Douglas and Dupont [1970], Dupont [1972], Fix and Nassif [1972], Wheeler [1971, 1973], Rachford [1973], Meyer [1973], Wellford and Oden [1973], Thomée and Wendroff [1974] and others. These papers discuss the convergence properties of conventional procedure to solve parabolic equation. Because of the conveniences of using full explicit method, the idea of the lumped coefficient scheme is often employed in practical computation. The convergence of the lumped coefficient scheme was studied by Fujii [1971, 1972] in elasticity problem and by Kikuchi [1972] in wider problemes given by positive definite operators. This paper will prove two theorems, one is the convergence of consistent coefficient scheme and the other is the convergence of lumped coefficient scheme. The convergence properties are also studied by numerical computation.

## 2. PRELIMINARIES

Consider linear hyperbolic problem:

$$\frac{\partial u}{\partial t}+\sum_{i=1}^{p} b_i(x)u_{,i}=0 \qquad \cdots\cdots(2.1)$$

for $x\in\Omega\subset R^p$, $0<t\leqq T$, $1\leqq p\leqq 3$ where $\Omega$ is bounded domain with sufficiently smooth boundary $\partial\Omega$. Notation $(\ )_{,i}$ means the partial differentiation with respect to $x^i$, i.e., $i$th component of $x$. Here and henceforth summation convention is used for repeated indices unless otherwise specified. It is assumed that the solution $u\in C^0\cdot(\bar{\Omega}\times[0, T])$ satisfies:

$$u(x,t)=0 \quad x\in\partial\Omega \quad 0<t\leqq T \qquad \cdots\cdots(2.2)$$

$$u(x,0)=u^0(x) \quad x\in\Omega \qquad \cdots\cdots(2.3)$$

Let $C^n(\Omega)$ be the set of all functions being $n$ times differentiable on the domain $\Omega$. $L_2(\Omega)$ is the closure of $C^\infty(\Omega)$ function with norm:

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$$\|v\|^2_{L_2(\Omega)} \equiv \int_\Omega |v|^2 d\Omega \quad v \in L_2(\Omega) \cdots\cdots\cdots(2.4)$$

The inner product is defined as:

$$\langle u, v\rangle \equiv \int_\Omega u(x)v(x)d\Omega \quad u, v \in L_2(\Omega) \cdots(2.5)$$

The spaces $\mathcal{H}^1(\Omega)$ and $\mathcal{H}^2(\Omega)$ are the closures of $C^\infty$ functions with respect to the norm:

$$\|v\|^2_{H^1(\Omega)} \equiv \|v\|^2_{L_2(\Omega)} + \sum_{i=1}^{p} \|v_{,i}\|^2_{L_2(\Omega)} \quad v \in \mathcal{H}^1(\Omega) \cdots\cdots(2.6)$$

or the norm:

$$\|v\|^2_{H^2(\Omega)} \equiv \|v\|^2_{H^1(\Omega)} + \sum_{i=1}^{p}\sum_{j=1}^{p} \|v_{,ij}\|^2_{L_2(\Omega)} \quad v \in \mathcal{H}^2(\Omega) \cdots\cdots(2.7)$$

respectively. $\mathcal{H}^1_0(\Omega)$ is the subspace in $\mathcal{H}^1(\Omega)$ which vanishes identically in the neighborhood of $\partial\Omega$. The norm $\|\cdot\|_{H^1_0(\Omega)}$ is defined by

$$\|v\|^2_{H^1_0(\Omega)} \equiv \sum_{i=1}^{p} \|v_{,i}\|^2_{L_2(\Omega)} \quad v \in \mathcal{H}^1_0(\Omega) \cdots(2.8)$$

From eqs. (2.4) and (2.8), the wellknown inequality is obtained.

$$\|v\|_{L_2(\Omega)} \leqq C_\Omega \|v\|_{H^1_0(\Omega)} \quad \forall v \in \mathcal{H}^1_0(\Omega) \cdots(2.9)$$

where $C_\Omega$ is real constant.

The existence and uniqueness of the solution $u$ is assumed and, moreover, the following smothness conditions are postulated for functions $b_i$ and $u$.

Assumption:

i) $C_1 \leqq |b_i(x)| \leqq C_2$ for all $x \in \Omega$ and $1 \leqq i \leqq p$ where $C_1$ and $C_2$ are real constants.

ii) $u \in (\mathcal{H}^1_0(\Omega) \cap \mathcal{H}^2(\Omega)) \times C^3[0, T]$

iii) $u_{,t} \in \mathcal{H}^1(\Omega) \times C^2[0, T]$

iv) $u_{,tt} \in C^0(\Omega) \times C^1[0, T]$

Consider the weak form of eq. (2.1),

$$\left\langle \frac{\partial u}{\partial t}, v \right\rangle + \langle b_i u_{,i}, v\rangle = 0 \quad \text{for} \quad \forall v \in \mathcal{H}^1_0(\Omega),\ t \in (0, T] \cdots\cdots(2.10)$$

subject to

$$u(x, t) \in \mathcal{H}^1_0(\Omega), \quad \langle u, (\cdot, 0), v\rangle = \langle u^0, v\rangle \quad \text{for} \quad \forall v \in \mathcal{H}^1_0(\Omega) \cdots\cdots(2.11)$$

In the subsequent sections, $C_n$ $(n=1, 2, \cdots)$ is used as generic constants which are not necessarily the same each time they appear in this paper. The following lemma is derived from eqs. (2.10) and (2.11).

Lemma 1.

Let $u^n$ be $u(\cdot, n\Delta t)$ where $\Delta t = T/N$ and $N$ is an integer. Then, for all exact solution of (2.10) with eq. (2.11), the following equation is obtained.

$$\left\langle \frac{u^{n+1}-u^n}{\Delta t}, v \right\rangle + \langle b_i u^n_{,i}, v\rangle + \frac{\Delta t}{2}\langle b_i u^n_{,i}, b_j v_{,j}\rangle = \left\langle \frac{\varepsilon}{\Delta t}, v \right\rangle \quad \text{for} \quad \forall v \in \mathcal{H}^1_0(\Omega) \cdots\cdots(2.12)$$

and

$$|\varepsilon| < C_1(\Delta t)^3 \cdots\cdots(2.13)$$

Proof: For the exact solution of eq. (2.10), from assumption iv), it is obtained that

$$u^{n+1} = u^n + \Delta t\, u^n_{,t} + \frac{\Delta t^2}{2} u^n_{,tt} + \varepsilon(x, t) \cdots\cdots(2.14)$$

where $|\varepsilon| < C_1(\Delta t)^3$, $u_{,t} = \dfrac{\partial u}{\partial t}$ and $u_{,tt} = \dfrac{\partial^2 u}{\partial t^2}$.

Thus,

$$u^n_{,t} = \frac{u^{n+1}-u^n}{\Delta t} - \frac{\Delta t}{2} u^n_{,tt} - \frac{\varepsilon}{\Delta t} \cdots\cdots(2.15)$$

Substituting eq. (2.15) into eq. (2.10) and rearranging it, then,

$$\left\langle \frac{u^{n+1}-u^n}{\Delta t}, v \right\rangle - \frac{\Delta t}{2}\langle u^n_{,tt}, v\rangle + \langle b_i u^n_{,i}, v\rangle = \left\langle \frac{\varepsilon}{\Delta t}, v \right\rangle \cdots\cdots(2.16)$$

Using eqs. (2.1) and (2.2), $\langle u^n_{,tt}, v\rangle$ is reformulated as:

$$\begin{aligned}\langle u^n_{,tt}, v\rangle &= \left\langle -\frac{\partial}{\partial t}(b_i u^n_{,i}), v \right\rangle \\ &= \left\langle -b_i \left(\frac{\partial u^n}{\partial t}\right)_{,i}, v \right\rangle \\ &= \langle b_i(b_j u^n_{,j})_{,i}, v\rangle \\ &= -\langle b_i u^n_{,i}, b_j v_{,j}\rangle \cdots\cdots(2.17)\end{aligned}$$

Introducing eq. (2.17) into eq. (2.16), eq. (2.12) can be derived. Q. E. D.

## 3. FINITE ELEMENT LAX-WENDROFF SCHEME

To solve equation (2.1) approximately by the finite element method, $\Omega$ is decomposed into finite elements $\{\Omega^h_i\}^{M_h}_{i=1}$. For simplicity, boundary $\partial\Omega$ is assumed to consist of a finite number of simple closed polygons. Here $h$ is the mesh parameter of the finite element mesh (i.e., if $h_e = \text{dia}(\Omega^h_i)$, $h = \max(h_1, h_2, \cdots, h_{M_h})$). The refinement of $\Omega^h_i$ is assumed to be uniform, i.e., for each refinement of the mesh, let the radius of the largest sphere that can be inscribed in $\Omega^h_i$ be proportional to $h_e$. Let $S_M$ be the finite dimensional subspace of $\mathscr{H}_0^1(\Omega)$ consisting of functions which have the form:

$$W^n(x)=\sum_{\alpha=1}^{M}\Phi_\alpha(x)\cdot W_\alpha^n \quad x\in\Omega \quad \cdots\cdots\cdots(3.1)$$

where $\{\Phi_\alpha(x)\}_{\alpha=1}^{M}$ are bases of $S_M$. Let $P$ be the partition of $[0, T]$ composed of the set $\{t_0, t_1, \cdots, t_N\}$ where $0=t_0<t_1<\cdots<t_N=T$ with $t_{n+1}-t_n=\Delta t$. The function $f^n(x)$ is introduced to mean:

$$f^n(x)=f(x, t_n) \quad \cdots\cdots\cdots(3.2)$$

The function $W^n\in S_M$ is defined as the finite element Lax-Wendroff approximation in $S_M$ when $W^n$ is determined by the following form:

$$\langle W^{n+1}, V\rangle=\langle W^n, V\rangle-\Delta t\langle b_i W_{,i}^n, V\rangle -\frac{\Delta t^2}{2}\langle b_i W_{,i}^n, b_j V_{,j}\rangle \quad \cdots\cdots(3.3)$$

for all $V\in S_M$ and all $t_n\in P$, starting from the initial function $W^0\in S_M$ which is determined by:

$$\langle W^0, V\rangle=\langle u^0, V\rangle \quad \cdots\cdots\cdots(3.4)$$

for $\forall\, V\in S_M$. With the use of eqs. (3.1) and (3.3), discretized equation of the finite element method is obtained as follows.

$$\boldsymbol{A}\cdot\boldsymbol{W}^{n+1}=\boldsymbol{A}\cdot\boldsymbol{W}^n+\boldsymbol{B}\cdot\boldsymbol{W}^n+\boldsymbol{C}\cdot\boldsymbol{W}^n \quad \cdots(3.5)$$

where

$$\boldsymbol{A}=\langle\Phi_\alpha(x), \Phi_\beta(x)\rangle$$

$$\boldsymbol{B}=-\Delta t\langle b_i(x)\Phi_{\alpha,i}(x), \Phi_\beta(x)\rangle$$

$$\boldsymbol{C}=-\frac{\Delta t^2}{2}\langle b_i(x)\Phi_{\alpha,i}(x), b_j(x)\Phi_{\beta,j}(x)\rangle$$

Unknown vector $\boldsymbol{W}^n\in R^M$ has the form:

$$\boldsymbol{W}^n=(W_1^n, W_2^n, \cdots, W_M^n) \quad \cdots\cdots\cdots(3.6)$$

Assume initial value $u_0$ in the form

$$u^0=\sum_{\alpha=1}^{M}u_\alpha^0\cdot\Phi_\alpha(x) \quad \cdots\cdots\cdots(3.7)$$

then, from eqs. (3.4) and (3.7), the following is obtained

$$\boldsymbol{W}^0=\boldsymbol{u}^0 \quad \cdots\cdots\cdots(3.8)$$

where $\boldsymbol{u}^0\in R^M$ has the form:

$$\boldsymbol{u}^0=(u_1^0, u_2^0, \cdots, u_M^0) \quad \cdots\cdots\cdots(3.9)$$

In this paper, piecewise linear function is employed as $\Phi_\alpha(x)$, namely, $S_M$ is taken to be a subspace of $\mathscr{H}_0^1(\Omega)$ spanned by piecewise linear basis functions. Precise forms of matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are listed in the Appendix. From the viewpoint of practical computation, eq. (3.5) may not be convenient because eq. (3.5) is not a pure explicit scheme. In practical computation, it is suitable to diagonalize the coefficient matrix $\boldsymbol{A}$ of the left side of eq. (3.5) to obtain a pure explicit scheme in the form:

$$\bar{\boldsymbol{A}}\boldsymbol{W}^{n+1}=\boldsymbol{A}\boldsymbol{W}^n+\boldsymbol{B}\boldsymbol{W}^n+\boldsymbol{C}\boldsymbol{W}^n \quad \cdots\cdots\cdots(3.10)$$

where

$$\bar{\boldsymbol{A}}=\langle\bar{\Phi}_\alpha(x), \bar{\Phi}_\beta(x)\rangle$$

in which $\bar{\Phi}_\alpha(x)$ is the piecewise constant shape function. Let $T_M$ be the fiinite dimensional subspace of $\mathscr{H}_0^1(\Omega)$ consisting of functions which have the form:

$$\bar{W}^n(x)=\sum_{\alpha=1}^{M}\bar{\Phi}_\alpha(x)\cdot W_\alpha^n \quad x\in\Omega \quad \cdots\cdots\cdots(3.11)$$

Then, eq. (3.3) can be rewritten as follows.

$$\langle\bar{W}^{n+1}, \bar{V}\rangle=\langle W^n, V\rangle-\Delta t\langle b_i W_{,i}^n, V\rangle -\frac{\Delta t^2}{2}\langle b_i W_{,i}^n, b_j V_{,j}\rangle \quad \cdots\cdots(3.12)$$

for $\forall\, V\in S_M$ and $\bar{V}\in T_M$, Precise form of $\bar{A}$ is listed in the Appendix. Finite element scheme eq. (3.5) is called the consistent coefficient scheme and eq. (3.10) the lumped coefficient scheme respectively.

Lemma 2.

Let $u(x)$ be a function which has bounded continuous derivatives of the second order in $\bar{\Omega}$, i.e.,

$$|D^\alpha u(x)|\leqq M^{(2)} \quad |\alpha|=2 \quad \cdots\cdots\cdots(3.13)$$

and let $\tilde{W}(x)$ be the projection of $u(x)$ on $S_M$ then it holds

$$\|u-\tilde{W}\|_{L_2(\Omega)}\leqq C_1 h^2 M^{(2)} \quad \cdots\cdots\cdots(3.14)$$

$$\|u-\tilde{W}\|_{H_0^1(\Omega)}\leqq C_2 h M^{(2)} \quad \cdots\cdots\cdots(3.15)$$

Let $\bar{W}(x)$ be the projection of $u(x)$ on $T_M$, then,

$$\|u-\bar{W}\|_{L_2(\Omega)}\leqq C_3 h M^{(1)} \quad \cdots\cdots\cdots(3.16)$$

where

$$|D^\alpha u(x)|\leqq M^{(1)} \quad |\alpha|=1 \quad \cdots\cdots\cdots(3.17)$$

In these estimations, $C_1$, $C_2$ and $C_3$ are constants and independent of the triangulation.

Proof: This lemma is standard and special case obtained by Ciarlet and Wagshal [1971], Ciarlet and Raviart [1972] and Strang and Fix [1973]. For completeness, proof will be shown in a more elementary manner.

Referring to Figure 1, $u^*$ is a function in $\xi\eta\zeta$ coordinate system corresponding to $u$ as a function in $xyz$ coordinate system. Denoting $C_1\sim C_7$ as constants, the following is obtained:

$$\|u\|_{L_2(\Omega)}=\left(\int_\Omega|u|^2 d\Omega\right)^{1/2}=\left(\int_\Omega|u^*|^2\cdot|J|\,d\Omega\right)^{1/2} \leqq\left(\int_\Omega|u^*|^2 d\Omega\right)^{1/2}\cdot\left(\int_\Omega|J|\,d\Omega\right)^{1/2} \leqq C_1|J|^{1/2}\cdot\|u^*\|_{L_2(\Omega)} \quad \cdots\cdots\cdots(3.18)$$

where $J$ is a Jacobian from $xyz$ coordinate system

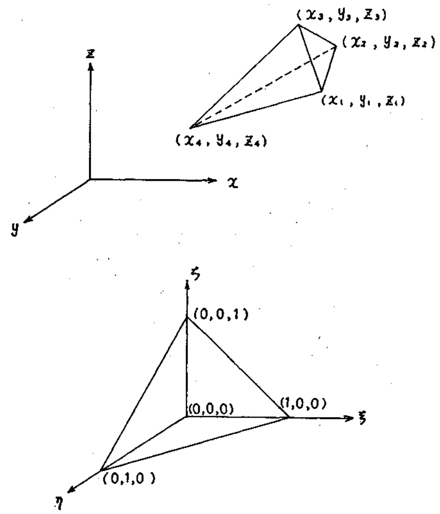


**Fig. 1** Coordinate System.

to $\xi\eta\zeta$ coordinate system. In the same manner, the equation:

$$\|u\|_{H_0^1(\Omega)}=\left(\int_\Omega |Du|^2\,d\Omega\right)^{1/2}$$

$$\leqq\left(\int_\Omega |Du^*|^2\cdot|J|\cdot\frac{1}{h^2}d\Omega\right)^{1/2}$$

$$\leqq\frac{1}{h}\cdot\left(\int_\Omega |Du^*|\,d\Omega\right)^{1/2}\cdot\left(\int_\Omega |J|\,d\Omega\right)^{1/2}$$

$$\leqq\frac{C_1}{h}\cdot|J|^{1/2}\cdot\|u^*\|_{H_0^1(\Omega)} \quad\cdots\cdots(3.19)$$

is obtained. Using eqs. (3.18) and (3.19), it gives:

$$\|u-\bar{W}\|_{L_2(\Omega)}\leqq C_1|J|^{1/2}\cdot\|u^*-\bar{W}^*\|_{L_2(\Omega)} \quad\cdots\cdots(3.20)$$

$$\|u-\bar{W}\|_{H_0^1(\Omega)}\leqq\frac{C_1}{h}|J|^{1/2}\cdot\|u^*-\bar{W}^*\|_{H_0^1(\Omega)} \quad\cdots\cdots(3.21)$$

Interpolant $\bar{W}^*$ has the form:

$$\bar{W}^*(\xi,\eta,\zeta)=a_1^*+a_2^*\xi+a_3^*\eta+a_4^*\zeta \quad\cdots(3.22)$$

where $a_1^*\sim a_4^*$ are constants to be determined. Exact solution can be expanded into Taylor series as:

$$u^*(\xi,\eta,\zeta)=a_1^*+a_2^*\xi+a_3^*\eta+a_4^*\zeta$$

$$+\frac{(\xi-\xi_0)^2}{2}\cdot\frac{\partial^2u^*}{\partial\xi^2}+\frac{(\eta-\eta_0)^2}{2}\cdot\frac{\partial^2u^*}{\partial\eta^2}$$

$$+\frac{(\zeta-\zeta_0)^2}{2}\cdot\frac{\partial^2u^*}{\partial\zeta^2}+(\zeta-\zeta_0)(\eta-\eta_0)\frac{\partial^2u^*}{\partial\xi\partial\eta}$$

$$+(\eta-\eta_0)(\zeta-\zeta_0)\frac{\partial^2u^*}{\partial\eta\partial\zeta}+(\zeta-\zeta_0)(\xi-\xi_0)\frac{\partial^2u^*}{\partial\zeta\partial\xi} \quad\cdots\cdots(3.23)$$

where $0\leqq\xi_0,\ \eta_0,\ \zeta_0\leqq1$

Defining the notation:

$$|Du^*|=\max\left(\left|\frac{\partial u^*}{\partial\xi}\right|,\ \left|\frac{\partial u^*}{\partial\eta}\right|,\ \left|\frac{\partial u^*}{\partial\zeta}\right|\right)$$

$$|Du|=\max\left(\left|\frac{\partial u}{\partial x}\right|,\ \left|\frac{\partial u}{\partial y}\right|,\ \left|\frac{\partial u}{\partial z}\right|\right)$$

$$|D^2u^*|=\max\left(\left|\frac{\partial^2u^*}{\partial\xi^2}\right|,\ \left|\frac{\partial^2u^*}{\partial\eta^2}\right|,\ \left|\frac{\partial^2u^*}{\partial\zeta^2}\right|,\ \left|\frac{\partial^2u^*}{\partial\xi\partial\eta}\right|,\ \left|\frac{\partial^2u^*}{\partial\eta\partial\zeta}\right|,\ \left|\frac{\partial^2u^*}{\partial\zeta\partial\xi}\right|\right)$$

$$|D^2u|=\max\left(\left|\frac{\partial^2u}{\partial x^2}\right|,\ \left|\frac{\partial^2u}{\partial y^2}\right|,\ \left|\frac{\partial^2u}{\partial z^2}\right|,\ \left|\frac{\partial^2u}{\partial x\partial y}\right|,\ \left|\frac{\partial^2u}{\partial y\partial z}\right|,\ \left|\frac{\partial^2u}{\partial z\partial x}\right|\right)$$

and using eqs. (3.22) and (3.23), the following expressions are derived.

$$|u^*-\bar{W}^*|\leqq C_2|D^2u^*| \quad\cdots\cdots(3.24)$$

$$|D(u^*-\bar{W}^*)|\leqq C_3|D^2u^*| \quad\cdots\cdots(3.25)$$

The inequality:

$$|D^2u^*|\leqq C_4h^2\cdot|D^2u| \quad\cdots\cdots(3.26)$$

yields the relation:

$$\left(\int_\Omega |D^2u^*|^2\,d\Omega\right)^{1/2}\leqq C_5h^2\left(\int_\Omega |D^2u|^2\,d\Omega\right)^{1/2}\cdot|J|^{-1/2} \quad\cdots(3.27)$$

and

$$\|u^*-\bar{W}^*\|_{L_2(\Omega)}\leqq C_2C_5h^2\left(\int_\Omega |D^2u|^2\,d\Omega\right)^{1/2}\cdot|J|^{-1/2}\cdots(3.28)$$

$$\|u^*-\bar{W}^*\|_{H_0^1(\Omega)}\leqq C_3C_5h^2\left(\int_\Omega |D^2u|^2\,d\Omega\right)^{1/2}\cdot|J|^{-1/2}\cdots(3.29)$$

Assuming the uniform refinement (i.e. $|J|\neq0$) and using eqs. (3.20) and (3.28), and eqs. (3.21) and (3.29) lead to the desired results eqs. (3.14) and (3.15) respectively. In almost the same manner, eq. (3.16) can be shown by using the following estimations instead of eqs. (3.24) and (3.27).

$$|u^*-\bar{W}^*|\leqq C_6|Du^*| \quad\cdots\cdots(3.30)$$

$$\left(\int_\Omega Du^*|^2\,d\Omega\right)^{1/2}\leqq C_7h\left(\int_\Omega |Du|^2\,d\Omega\right)^{1/2}\cdot|J|^{-1/2} \quad\cdots\cdots(3.31)$$

Q. E. D.

## 4. CONVERGENCE ESTIMATES FOR CONSISTENT COEFFICIENT SCHEME

The convergence estimates of the finite element

Lax-Wendroff solution with the consistent coefficient scheme can be obtained in the following Theorem 1 and proven by using Lemma 3~7.

Lemma 3.

$$0\leqq\langle b_i u_{,i},\ b_j u_{,j}\rangle\leqq\left(\sum_{i=1}^{p} b_i^2\right)\cdot\|u\|^2_{H_0^1(\Omega)}\cdots(4.1)$$

$$\langle b_i z_{,i},\ z\rangle=0 \quad \text{for} \quad z\in\mathscr{H}_0^1(\Omega)\cdots\cdots\cdots\cdots(4.2)$$

Proof: Eq. (4.1) is the simple consequence of the following matrix eigenvalue problem in case of $p=3$.

$$\left|\begin{bmatrix} b_1^2 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2^2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3^2\end{bmatrix}-\begin{bmatrix}\mu & & \\ & \mu & \\ & & \mu\end{bmatrix}\right|=0 \cdots\cdots\cdots\cdots(4.3)$$

For simple calculation, it is obvious that eigenvalues $\mu_1$, $\mu_2$, $\mu_3$ of (4.3) are:

$$\mu_1,\ \mu_2=0 \quad \text{and} \quad \mu_3=b_1^2+b_2^2+b_3^2 \cdots\cdots(4.4)$$

Thus, eq. (4.1) is derived. Eq. (4.2) is obtained by integrating the left side of eq. (4.2) by parts considering the boundary conditions, values of which are taken to be zero. Q. E. D.

Lemma 4.

There exists constant $\lambda$ such that

$$\|\phi\|_{H_0^1(\Omega)}\leqq\frac{\lambda}{h}\|\phi\|_{L_2(\Omega)}\cdots\cdots\cdots\cdots(4.5)$$

for $\phi\in S_M$.

Proof: Because $\phi\in S_M$, $\phi$ is the function expressed in the form:

$$\phi=\Phi_\alpha\varphi_\alpha\cdots\cdots\cdots\cdots(4.6)$$

where $\Phi_\alpha$ $(\alpha=1, 2, \cdots, M)$ is shape function and $\varphi_\alpha\in R^M$. Eq. (4.6) and in the sequal summation convention is used for indices $\alpha, \beta, \cdots$. Thus the following expressions are obtained.

$$\|\phi\|^2_{L_2(\Omega)}=\langle\Phi_\alpha,\ \Phi_\beta\rangle\varphi_\alpha\varphi_\beta\cdots\cdots\cdots\cdots(4.7)$$

$$\|\phi\|^2_{H_0^1(\Omega)}=\langle\Phi_{\alpha,i},\ \Phi_{\beta,i}\rangle\varphi_\alpha\varphi_\beta\cdots\cdots\cdots\cdots(4.8)$$

Consider Rayleigh quotient $R$:

$$R=\frac{\|\phi\|^2_{H_0^1(\Omega)}}{\|\phi\|^2_{L_2(\Omega)}}=\frac{\langle\Phi_{\alpha,i},\ \Phi_{\beta,i}\rangle\varphi_\alpha\varphi_\beta}{\langle\Phi_\alpha,\ \Phi_\beta\rangle\varphi_\alpha\varphi_\beta}\cdots\cdots(4.9)$$

Then, the following equation can be derived by using the results obtained by Irons and Treharnet [1971] and Fried [1973].

$$R\leqq\frac{\langle\Phi'_{\alpha,i},\ \Phi'_{\beta,i}\rangle\varphi'_\alpha\varphi'_\beta}{\langle\Phi'_\alpha,\ \Phi'_\beta\rangle\varphi'_\alpha\varphi'_\beta}\leqq\frac{\lambda^2}{h^2}\cdots\cdots\cdots\cdots(4.10)$$

where $\Phi'_\alpha$ $(\alpha=1, 2, 3)$ is the shape function restricted to a finite element $\Omega_i^h$. Constant $\lambda$ can be in fact calculated by using the matrices $A$ and $-\frac{2}{\Delta t^2}C$ listed in the Appendinx. Q. E. D.

From the algorithm eq. (3.3) and the exact solution eq. (2.12), the following lemma is obtained.

Lemma 5.

For $z^n\in S_M$ the following estimate is valid.

$$\begin{aligned}(\|z^{n+1}\|^2_{L_2(\Omega)}-\|z^n\|^2_{L_2(\Omega)})
&\leqq\Delta t|\langle b_i z^n_{,i},\ z^{n+1}\rangle|+\frac{\Delta t^2}{2}|\langle b_i z^n_{,i},\ b_j z^{n+1}_{,j}\rangle| \\
&+\Delta t\left|\left\langle\frac{\varepsilon}{\Delta t},\ z^{n+1}+z^n\right\rangle\right| \\
&+|\langle\tilde{z}^{n+1}+\tilde{z}^n,\ z^{n+1}+z^n\rangle| \\
&+\Delta t|\langle b_i\tilde{z}^n_{,i},\ z^{n+1}+z^n\rangle \\
&+\frac{\Delta t^2}{2}|\langle b_i\tilde{z}^n_{,i},\ b_j(z^{n+1}+z^n)_{,j}\rangle|\cdots\cdots(4.11)\end{aligned}$$

where

$$z^n=\tilde{W}^n-W^n\cdots\cdots\cdots\cdots(4.12)$$

$$\tilde{z}^n=u^n-\tilde{W}^n\cdots\cdots\cdots\cdots(4.13)$$

in which $u^n=u(\cdot,\ n\Delta t)$, $\tilde{W}^n$ is the projection of $u^n$ on $S_M$, and $W^n$ is the approximate solution by the finite element Lax-Wendroff method with consistent coefficient scheme eq. (3.5).

Proof: From eqs. (2.12) and (3.3), it holds that

$$\begin{aligned}&\frac{1}{\Delta t}\langle u^{n+1}-W^{n+1},\ V\rangle-\frac{1}{\Delta t}\langle u^n-W^n,\ V\rangle \\
&+\langle b_i(u^n_{,i}-W^n_{,i}),\ V\rangle \\
&+\frac{\Delta t}{2}\langle b_i(u^n_{,i}-W^n_{,i}),\ b_jV_{,j}\rangle \\
&=\left\langle\frac{\varepsilon}{\Delta t},\ V\right\rangle\cdots\cdots\cdots\cdots(4.14)\end{aligned}$$

for all $V\in S_M\subset\mathscr{H}_0^1(\Omega)$. Inserting $\tilde{W}^{n+1}$ and $\tilde{W}^n$, respectively, into eq. (4.14), one obtains:

$$\begin{aligned}&\frac{1}{\Delta t}\langle u^{n+1}-\tilde{W}^{n+1}+\tilde{W}^{n+1}-W^{n+1},\ V\rangle \\
&-\frac{1}{\Delta t}\langle u^n-\tilde{W}^n+\tilde{W}^n-W^n,\ V\rangle \\
&+\langle b_i(u^n_{,i}-\tilde{W}^n_{,i}+\tilde{W}^n_{,i}+W^n_{,i}),\ V\rangle \\
&+\frac{\Delta t}{2}\langle b_i(u^n_{,i}-\tilde{W}^n_{,i}+\tilde{W}^n_{,i}-W^n_{,i}),\ b_jV_{,j}\rangle \\
&=\left\langle\frac{\varepsilon}{\Delta t},\ V\right\rangle\cdots\cdots\cdots\cdots(4.15)\end{aligned}$$

Introducing eqs. (4.12) and (4.13) into eq. (4.15), eq. (4.15) is rewritten in the following form.

$$\frac{1}{\Delta t}\langle z^{n+1}-z^n,\ V\rangle+\langle b_i z^n_{,i},\ V\rangle$$
$$+\frac{\Delta t}{2}\langle b_i z^n_{,i},\ b_j V_{,j}\rangle$$
$$=\left\langle\frac{\varepsilon}{\Delta t},\ V\right\rangle-\frac{1}{\Delta t}\langle\tilde{z}^{n+1},\ V\rangle+\frac{1}{\Delta t}\langle\tilde{z}^n,\ V\rangle$$
$$-\langle b_i\tilde{z}^n_{,i},\ V\rangle-\frac{\Delta t}{2}\langle b_i\tilde{z}^n_{,i},\ b_j V_{,j}\rangle \qquad \cdots\cdots(4.16)$$

Putting $V=z^{n+1}+z^n$ in eq. (4.16) yields:

$$\frac{1}{\Delta t}(\|z^{+1}\|^2_{L_2(\Omega)}-\|z^n\|^2_{L_2(\Omega)})$$
$$+\langle b_i z^n_{,i},\ z^{n+1}\rangle+\langle b_i z^n_{,i},\ z^n\rangle$$
$$+\frac{\Delta t}{2}\langle b_i z^n_{,i},\ b_j z^{n+1}_{,j}\rangle+\frac{\Delta t}{2}\langle b_i z^n_{,i},\ b_j z^n_{,j}\rangle$$
$$=\left\langle\frac{\varepsilon}{\Delta t},\ z^{n+1}+z^n\right\rangle-\frac{1}{\Delta t}\langle\tilde{z}^{n+1},\ z^{n+1}+z^n\rangle$$
$$+\frac{1}{\Delta t}\langle\tilde{z}^n,\ z^{n+1}+z^n\rangle-\langle b_i\tilde{z}^n_{,i},\ z^{n+1}+z^n\rangle$$
$$-\frac{\Delta t}{2}\langle b_i\tilde{z}^n_{,i},\ b_j(z^{n+1}+z^n)_{,j}\rangle \quad\cdots\cdots(4.17)$$

Using the results in Lemma 3 and eq. (4.17), eq. (4.11) can be obtained. Q. E. D.

The estimated results of the right side of eq. (4.11) are listed in the following two Lemmas.

Lemma 6.

There exist constants $C_1$ and $C_2$ such that

$$\Delta t|\langle b_i z^n_{,i},\ z^{n+1}\rangle|\leqq\Delta t^2 C_1\left\|\frac{\partial u^n}{\partial t}\right\|_{H^1_0(\Omega)}$$
$$+\Delta t\, C_2\|z^{n+1}\|^2_{L_2(\Omega)} \qquad\cdots\cdots(4.18)$$

Proof: Using Lemma 3, and the wellknown inequality:

$$ab<\frac{1}{2}\left[\varepsilon a^2+\frac{b^2}{\varepsilon}\right] \quad \text{for } \varepsilon>0 \quad\cdots\cdots(4.19)$$

where $a$ and $b$ are real numbers, the following estimates are derived.

$$\Delta t|\langle b_i z^n_{,i},\ z^{n+1}\rangle|$$
$$=\Delta t|\langle b_i(z^n_{,i}-z^{n+1}_{,i}),\ z^{n+1}\rangle|$$
$$\leqq\Delta t^2 C_3\left\|\frac{1}{\Delta t}(z^n-z^{n+1})\right\|^2_{H^1_0(\Omega)}$$
$$+\Delta t C_4\|z^{n+1}\|^2_{L_2(\Omega)} \qquad\cdots\cdots(4.20)$$

Let $PC^1[O, T: S_M]$ be the space of functions which are continuous with piecewise continuous derivatives between the points of the partition $P$ of $[O, T]$, then $\overline{PC}^1[O, T: S_M]$ is the subspace of $PC^1[O, T: S_M]$ of functions which are piecewise linear. The space was originally introduced by Wellford and Oden [1973]. Let the functions $W(t)$ and $\tilde{W}(t)$ be respectively such that

$$W(t)\in\overline{PC}^1[O, T: S_M]$$
$$W(t_n)=W^n \quad \text{for } \forall\, t_n\in P$$

and

$$\tilde{W}(t)\in\overline{PC}^1[O, T: S_M]$$
$$\tilde{W}(t_n)=\tilde{W}^n \quad \text{for } \forall\, t_n\in P$$

Then, the right side of eq. (4.20) can be estimated in the following form by considering eq. (4.12).

$$=\Delta t^2 C_3\left\|\frac{\partial\tilde{W}(t)}{\partial t}-\frac{\partial W(t)}{\partial t}\right\|^2_{H^1_0(\Omega)}$$
$$+\Delta t C_4\|z^{n+1}\|^2_{L_2(\Omega)}$$
$$\leqq\Delta t^2 C_5\{1+f(\Delta t, h)\}\cdot\left\|\frac{\partial u^n}{\partial t}\right\|^2_{H^1_0(\Omega)}$$
$$+\Delta t C_4\|z^{n+1}\|^2_{L_2(\Omega)} \qquad\cdots\cdots(4.21)$$

The explicit expression for $f$ is not introduced because it can only result in the higher order terms in the error estimate. Considering this, eq. (4.18) is derived from eq. (4.21). Q. E. D.

Lemma 7.

Taking $\Delta t=\lambda\cdot h$ where $\lambda$ is the constant in Lemma 4, then, there exist constants $C_1\sim C_9$ such that

$$\frac{\Delta t^2}{2}\langle b_i z^n_{,i},\ b_j z^{n+1}_{,j}\rangle$$
$$\leqq\Delta t^2 C_1\|z^n\|^2_{H^1_0(\Omega)}+\Delta t^2 C_2\|z^{n+1}\|^2_{H^1_0(\Omega)}$$
$$\leqq C_1\lambda^4\|z^n\|^2_{L_2(\Omega)}+C_2\lambda^4\|z^{n+1}\|^2_{L_2(\Omega)} \qquad\cdots\cdots(4.22)$$

$$\Delta t\left|\left\langle\frac{\varepsilon}{\Delta t},\ z^{n+1}+z^n\right\rangle\right|$$
$$\leqq C_3\Delta t\frac{1}{\Delta t^2}\|\varepsilon\|^2_{L_2(\Omega)}$$
$$+\Delta t C_4(\|z^{n+1}\|^2_{L_2(\Omega)}+\|z^n\|^2_{L_2(\Omega)})\cdots(4.23)$$

$$|\langle\tilde{z}^{n+1}+\tilde{z}^n,\ z^{n+1}+z^n\rangle|$$
$$\leqq C_5(\|\tilde{z}^{n+1}\|^2_{L_2(\Omega)}+\|\tilde{z}^n\|^2_{L_2(\Omega)}$$
$$+\|z^n\|^2_{L_2(\Omega)}+\|z^{n+1}\|^2_{L_2(\Omega)})\cdots\cdots(4.24)$$

$$\Delta t|\langle b_i\tilde{z}^n_{,i},\ z^{n+1}+z^n\rangle|$$
$$\leqq C_6\Delta t\|\tilde{z}^n\|^2_{H^1_0(\Omega)}$$
$$+C_7\Delta t(\|z^{n+1}\|^2_{L_2(\Omega)}+\|z^n\|^2_{L_2(\Omega)})\cdots(4.25)$$

$$\frac{\Delta t^2}{2}|\langle b_i\tilde{z}^n_{,i},\ b_j(z^{n+1}+z^n)_{,j}\rangle|$$
$$\leqq C_8\Delta t^2\|\tilde{z}^n\|^2_{H^1_0(\Omega)}$$
$$+C_9\Delta t^2(\|z^{n+1}\|^2_{H^1_0(\Omega)}+\|z^n\|^2_{H^1_0(\Omega)})$$
$$\leqq C_8\Delta t^2\|\tilde{z}^n\|^2_{H^1_0(\Omega)}$$
$$+C_9\lambda^4(\|z^{n+1}\|^2_{L_2(\Omega)}+\|z^n\|^2_{L_2(\Omega)})\cdots(4.26)$$

Proof: These inequalities are the simple consequences of eq. (4.5) and eq. (4.19). Q. E. D.

Basing on the aforementioned Lemmas, the convergence of the finite element Lax-Wendroff solution with consistent coefficient scheme eq. (3.5) can be concluded in the following theorem.

Theorem 1.

Taking $\Delta t=\lambda h$ and assuming uniform refinement of $h$, the norm of the difference between $W^n$ determined by eq. (3.5) and $u^n=u(\cdot, t_n)$ can be estimated as follows.

$$\| W^n-u^n\|_{L_2(\Omega)}\leqq Ch \qquad \cdots\cdots(4.27)$$

where $C$ is a generic constant.

Proof: Inserting $\tilde{W}^n$ into the left hand side of eq. (4.27), the norm is estimated in the form:

$$\begin{aligned}&\| W^n-u^n\|_{L_2(\Omega)}\\ &\quad\leqq \| W^n-\tilde{W}^n\|_{L_2(\Omega)}+\|\tilde{W}^n-u^n\|_{L_2(\Omega)}\\ &\quad=\|\tilde{z}^n\|_{L_2(\Omega)}+\|z^n\|_{L_2(\Omega)} \qquad \cdots\cdots(4.28)\end{aligned}$$

Thus, it is enough to show that $\|\tilde{z}^n\|_{L_2(\Omega)}$ and $\|z^n\|_{L_2(\Omega)}$ are estimated by the terms of $h$. From Lemma 2,

$$\|\tilde{z}^n\|^2_{L_2(\Omega)}\leqq C_1h^4 \qquad \cdots\cdots(4.29)$$

$$\|\tilde{z}^{n+1}\|^2_{L_2(\Omega)}\leqq C_2h^4 \qquad \cdots\cdots(4.30)$$

$$\|\tilde{z}^n\|^2_{H_0^1(\Omega)}\leqq C_3h^2 \qquad \cdots\cdots(4.31)$$

and from Lemma 5, 6 and 7,

$$\begin{aligned}\|z^{n+1}\|^2_{L_2(\Omega)}&\leqq C_4\|z^n\|^2_{L_2(\Omega)}+C_5\Delta t^2\left\|\frac{\partial u^n}{\partial t}\right\|^2_{H_0^1(\Omega)}\\ &+C_6\|\varepsilon\|^2_{L_2(\Omega)}+C_7(\|\tilde{z}^{n+1}\|^2_{L_2(\Omega)}+\|\tilde{z}^n\|^2_{L_2(\Omega)})\\ &+C_8\Delta t\|\tilde{z}^n\|^2_{H_0^1(\Omega)}+C_9\Delta t^2\|\tilde{z}^n\|^2_{H_0^1(\Omega)}\\ &\qquad\cdots\cdots(4.32)\end{aligned}$$

are obtained. Introducing eqs. (4.29)~(4.31) into eq. (4.32) and rearranging the terms, one obtains:

$$\begin{aligned}\|z^{n+1}\|^2_{L_2(\Omega)}&\leqq C_4\|z^n\|^2_{L_2(\Omega)}+C_6\Delta t^6\\ &+C_{10}h^4+C_{11}\Delta t\, h^2+C_{12}\Delta t^2 \qquad \cdots\cdots(4.33)\end{aligned}$$

Considering $\Delta t=\lambda h$ then eq. (4.33) can be rewritten in the following form.

$$\|z^{n+1}\|^2_{L_2(\Omega)}\leqq C_4\|z^n\|^2_{L_2(\Omega)}+C_{13}h^2 \qquad \cdots\cdots(4.34)$$

Thus, recursive use of eq. (4.34) yields the important estimate as follows:

$$\|z^{n+1}\|^2_{L_2(\Omega)}\leqq C_{14}h \qquad \cdots\cdots(4.35)$$

Introducing eq. (4.35) and eq. (4.29) into eq. (4.28) and rearranging the terms, the conclusion eq. (4.27) can be obtained. Q. E. D.

## 5. CONVERGENCE ESTIMATES FOR LUMPED COEFFICIENT SCHEME

The convergence estimates of the finite element Lax-Wendroff solution with the lumped coefficients scheme are described in a similar manner as in Section 4. Functions $\tilde{U}\in S_M$ and $\bar{U}\in T_M$ are said to be associative if both functions have the same value at each nodal point. The following two Lemmas were originally introduced by Fujii [1971].

Lemma 8.

There exists constant $\gamma$ such that

$$\|\phi\|_{H_0^1(\Omega)}\leqq\frac{\gamma}{h}\|\bar{\phi}\|_{L_2(\Omega)} \qquad \cdots\cdots(5.1)$$

for $\phi\in S_M$ and $\bar{\phi}\in T_M$, where $\phi$ and $\bar{\phi}$ are associative.

Proof: Because $\phi\in S_M$ and $\bar{\phi}\in T_M$, these functions are expressed in the form:

$$\phi=\Phi_\alpha\varphi_\alpha \qquad \cdots\cdots(5.2)$$

$$\bar{\phi}=\bar{\Phi}_\beta\varphi_\beta \qquad \cdots\cdots(5.3)$$

where $\Phi_\alpha$ and $\bar{\Phi}_\alpha$ are shape functions on $S_M$ and $T_M$ respectively and $\varphi_\alpha\in R_M$ $(\alpha=1, 2, \cdots, M)$. Thus, the following expressions are obtained.

$$\|\bar{\phi}\|^2_{L_2(\Omega)}=\langle\bar{\Phi}_\alpha, \bar{\Phi}_\beta\rangle\varphi_\alpha\varphi_\beta \qquad \cdots\cdots(5.4)$$

$$\|\phi\|^2_{H_0^1(\Omega)}=\langle\Phi_{\alpha,i}, \Phi_{\beta,i}\rangle\varphi_\alpha\varphi_\beta \qquad \cdots\cdots(5.5)$$

Consider Rayleigh quotient $R$:

$$\begin{aligned}R&=\frac{\|\bar{\phi}\|^2_{H_0^1(\Omega)}}{\|\phi\|^2_{L_2(\Omega)}}=\frac{\langle\bar{\Phi}_{\alpha,i}, \bar{\Phi}_{\beta,i}\rangle\varphi_\alpha\varphi_\beta}{\langle\Phi_\alpha, \Phi_\beta\rangle\varphi_\alpha\varphi_\beta}\\ &\leqq\frac{\langle\bar{\Phi}'_{\alpha,i}, \bar{\Phi}'_{\beta,i}\rangle\varphi'_\alpha\varphi'_\beta}{\langle\Phi'_\alpha, \Phi'_\beta\rangle\varphi'_\alpha\varphi'_\beta}\leqq\frac{\gamma^2}{h^2} \qquad \cdots\cdots(5.6)\end{aligned}$$

where $\bar{\Phi}_\alpha$ and $\Phi_\alpha$ $(\alpha=1, 2, 3)$ are the shape functions restricted to a finite element $\Omega_i^h$. Constant $\gamma$ can be in fact determined by using matrices $\bar{A}$ and $-\frac{2}{\Delta t^2}C$ listed in the Appendix. Q. E. D.

Lemma 9.

Let $\tilde{U}\in S_M$ and $\bar{U}\in T_M$ and assume both functions are associative, then it holds that

$$C_1\|\tilde{U}\|_{L_2(\Omega)}\leqq\|\bar{U}\|_{L_2(\Omega)} \qquad \cdots\cdots(5.7)$$

where $C_1$ is a constant independent of the triangulation.

Proof: Becauge $\tilde{U}\in S_M$ and $\bar{U}\in T_M$ are associative, the following can be obtained.

$$\| \tilde{U}-\bar{U} \|^2_{L_2(\Omega)} \leqq C_2 \sum_{\Omega_e^h} h_e^2 \cdot \sum_{j=1}^{p} \| \tilde{U}_{,j} \|^2_{L_2(\Omega)}$$

$$\leqq C_3 h^2 \| \tilde{U} \|^2_{H_0^1(\Omega)} \quad \cdots\cdots\cdots\cdots (5.8)$$

Using Lemma 8, the right side of eq. (5.8) is estimated to be:

$$\| \tilde{U}-\bar{U} \|^2_{L_2(\Omega)} \leqq C_4 \| \bar{U} \|^2_{L_2(\Omega)} \quad \cdots\cdots\cdots\cdots (5.9)$$

Finally, the following estimate is derived.

$$\| \tilde{U} \|_{L_2(\Omega)} \leqq \| \bar{U} \|_{L_2(\Omega)} + \| \tilde{U}-\bar{U} \|_{L_2(\Omega)}$$

$$\leqq (1+\sqrt{C_4}) \| \bar{U} \|_{L_2(\Omega)} \quad \cdots\cdots\cdots (5.10)$$

Q. E. D.

Lemma 10.

For $\bar{z}^{n+1} \in T_M$ and $z^n \in S_M$, the following estimate is valid.

$$(\| \bar{z}^{n+1} \|^2_{L_2(\Omega)} - \| z^n \|^2_{L_2(\Omega)})$$

$$\leqq |\langle \bar{U}^{n+1}-u^{n+1}, \bar{z}^{n+1} \rangle| + |\langle u^{n+1}, \bar{z}^{n+1}-z^n \rangle|$$

$$+ |\langle \tilde{z}^n, z^n \rangle| + \Delta t |\langle b_i \tilde{z}^n_{,i}, z^n \rangle|$$

$$+ \frac{\Delta t^2}{2} |\langle b_i \tilde{z}^n_{,i}, b_j z^n_{,j} \rangle| + |\langle \varepsilon, z^n \rangle| \cdots (5.11)$$

where

$$\bar{z}^{n+1} = \bar{U}^{n+1} - \bar{W}^{n+1} \cdots\cdots\cdots\cdots (5.12)$$

$$z^n = \tilde{W}^n - W^n \cdots\cdots\cdots\cdots (5.13)$$

$$\tilde{z}^n = u^n - \tilde{W}^n \cdots\cdots\cdots\cdots (5.14)$$

in which $\bar{U}^{n+1}$ and $\tilde{W}^n$ are the projections of the exact solution $u$ and $T_M$ at $(n+1)\Delta t$ and on $S_M$ at $n\Delta t$, respectively, and $\bar{W}^{n+1}$ and $W^n$ are the solutions determined by the finite element Lax-Wendroff method with lumped coefficient scheme eq. (3.10).

Proof: Using eq. (2.12), the equality:

$$\langle \bar{U}^{n+1}, \bar{V} \rangle - \langle \bar{W}^{n+1}, V \rangle + \Delta t \langle b_i \tilde{W}^n_{,i}, V \rangle$$

$$+ \frac{\Delta t^2}{2} \langle b_i \tilde{W}^n_{,i}, b_j V_{,j} \rangle$$

$$= \langle \bar{U}^{n+1}, \bar{V} \rangle - \langle u^{n+1}, V \rangle$$

$$+ \langle u^n - \tilde{W}^n, V \rangle - \Delta t \langle b_i (u^n_{,i} - \tilde{W}^n_{,i}), V \rangle$$

$$- \frac{\Delta t^2}{2} \langle b_i (u^n_{,i} - \tilde{W}^n_{,i}), b_j V_{,j} \rangle + \left\langle \frac{\varepsilon}{\Delta t}, V \right\rangle$$

$$\cdots\cdots\cdots\cdots (5.15)$$

is obtained for $\bar{V} \in T_M$ and $V \in S_M$. Introducing the finite element solution eq. (3.12) into eq. (5.15), employing eqs. (5.12) and (5.14) and rearranging terms, the following equation is derived.

$$\langle \bar{z}^{n+1}, \bar{V} \rangle - \langle z^n, V \rangle + \Delta t \langle b_i z^n_{,i}, V \rangle$$

$$+ \frac{\Delta t^2}{2} \langle b_i z^n_{,i}, b_j V_{,j} \rangle$$

$$= \langle \bar{U}^{n+1}, \bar{V} \rangle - \langle u^{n+1}, V \rangle$$

$$+ \langle \tilde{z}^n, V \rangle - \Delta t \langle b_i \tilde{z}^n_{,i}, V \rangle$$

$$- \frac{\Delta t^2}{2} \langle b_i \tilde{z}^n_{,i}, b_j V_{,j} \rangle + \langle \varepsilon, V \rangle \cdots\cdots (5.16)$$

To put $\bar{V}$ and $V$ in eq. (5.16) as

$$\bar{V} = \bar{z}^{n+1} \cdots\cdots\cdots\cdots (5.17)$$

$$V = z^n \cdots\cdots\cdots\cdots (5.18)$$

leads to:

$$\| \bar{z}^{n+1} \|^2_{L_2(\Omega)} - \| z^n \|^2_{L_2(\Omega)} + \Delta t \langle b_i z^n_{,i}, z^n \rangle$$

$$+ \frac{\Delta t^2}{2} \langle b_i z^n_{,i}, b_j z^n_{,i} \rangle$$

$$= \langle \bar{U}^{n+1} - u^{n+1}, \bar{z}^{n+1} \rangle + \langle u^{n+1}, \bar{z}^{n+1} - z^n \rangle$$

$$+ \langle \tilde{z}^n, z^n \rangle - \Delta t \langle b_i \tilde{z}^n_{,i}, z^n \rangle$$

$$- \frac{\Delta t^2}{2} \langle b_i \tilde{z}^n_{,i}, b_j z^n_{,j} \rangle + \langle \varepsilon, z^n \rangle \cdots\cdots (5.19)$$

Thus, eq. (5.11) is obtained by estimating both sides of eq. (5.19) and using Lemma 3.

Q. E. D.

The estimated results of the right side of eq. (5.11) are listed in the following Lemma.

Lemma 11.

Taking $\Delta t = \lambda \cdot h$, where $\lambda$ is the constant in Lemma 4, then, there exist constant $C_1 \sim C_{11}$ such that

$$|\langle \bar{U}^{n+1} - u^n, \bar{z}^{n+1} \rangle| \leqq C_1 \| \bar{U}^{n+1} - u^{n+1} \|^2_{L_2(\Omega)}$$

$$+ C_2 \| \bar{z}^{n+1} \|^2_{L_2(\Omega)} \cdots\cdots\cdots\cdots (5.20)$$

$$|\langle u^{n+1}, \bar{z}^{n+1} - z^n \rangle|$$

$$\leqq C_3 \| u^{n+1} \|_{L_2(\Omega)} \cdot \| \bar{z}^{n+1} - z^n \|_{L_2(\Omega)}$$

$$\leqq C_4 \| \bar{z}^{n+1} \|^2_{L_2(\Omega)} + C_5 \| z^n \|^2_{L_2(\Omega)} \cdots\cdots (5.21)$$

$$|\langle \tilde{z}^n, z^n \rangle| \leqq C_6 \| \tilde{z}^n \|^2_{L_2(\Omega)} + C_7 \| z^n \|^2_{L_2(\Omega)}$$

$$\cdots\cdots\cdots\cdots (5.22)$$

$$\Delta t |\langle b_i \tilde{z}^n_{,i}, z^n \rangle|$$

$$\leqq \Delta t C_8 \| \tilde{z}^n \|^2_{H_0^1(\Omega)} + \Delta t C_9 \| z^n \|^2_{L_2(\Omega)} \cdots (5.23)$$

$$\frac{\Delta t^2}{2} |\langle b_i \tilde{z}^n_{,i}, b_j z^n_{,j} \rangle|$$

$$\leqq C_{10} \Delta t^2 \| \tilde{z}^n \|^2_{H_0^1(\Omega)} + C_{11} \Delta t^2 \| z^n \|^2_{H_0^1(\Omega)}$$

$$\leqq C_{12} \Delta t^2 \| \tilde{z}^n \|^2_{H_0^1(\Omega)} + C_{13} \lambda^2 \| z^n \|^2_{L_2(\Omega)}$$

$$\cdots\cdots\cdots\cdots (5.24)$$

Proof: These inequalities are the simple consequences of eq. (4.5) and eq. (4.19). Q. E. D.

On the basis of the aforementioned Lemmas, the convergence of the finite element Lax-Wendroff solution with lumped coefficient scheme eq. (3.10) can be concluded in the following theorem.

Theorem 2.

Taking $\Delta t=\lambda h$ and assuming uniform refinement of $h$, the norm of the difference between $W^n$ determined by eq. (3.10) and $u^n=u(\cdot, t_n)$ can be estimated as follows.

$$\| W^n-u^n\|_{L_2(\Omega)}\leqq Ch \quad \cdots\cdots(5.25)$$

where $C$ is a generic constant.

Proof: Inserting $\tilde{W}^n$ into the left hand side of eq. (5.25), the norm is estimated in the form.

$$\begin{aligned}\| W^n-u^n\|_{L_2(\Omega)} &\leqq \| W^n-\tilde{W}^n\|_{L_2(\Omega)}+\|\tilde{W}^n-u^n\|_{L_2(\Omega)}\\ &=\|\tilde{z}^n\|_{L_2(\Omega)}+\|z^n\|_{L_2(\Omega)} \quad \cdots\cdots(5.26)\end{aligned}$$

Thus, it is enough to show that $\|z^n\|_{L_2(\Omega)}$ is estimated by the terms of $h$. From Lemma 11, the right side of eq. (5.11) can be estimated in the following form employing new constants $C_1\sim C_7$.

$$\begin{aligned}&(\|\bar{z}^{n+1}\|^2_{L_2(\Omega)}-\|z^n\|^2_{L_2(\Omega)})\\ &\leqq C_1\|\bar{U}^{n+1}-u^{n+1}\|^2_{L_2(\Omega)}+C_2\|\bar{z}^{n+1}\|^2_{L_2(\Omega)}\\ &\quad +C_3\|z^n\|^2_{L_2(\Omega)}+C_4\|\bar{z}^n\|^2_{L_2(\Omega)}\\ &\quad +\Delta t C_5\|\bar{z}^n\|^2_{H_0^1(\Omega)}+\Delta t^2 C_6\|\bar{z}^n\|^2_{L_2(\Omega)}\\ &\quad +C_7\Delta t^6 \quad \cdots\cdots(5.27)\end{aligned}$$

The use of eqs. (4.29), (4.31) and (3.16) yields:

$$\|\bar{z}^{n+1}\|^2_{L_2(\Omega)}\leqq C_8\|z^n\|^2_{L_2(\Omega)}+C_9h^2 \quad \cdots\cdots(5.28)$$

Because $\bar{U}^{n+1}$ and $\tilde{W}^{n+1}$ are associative, $\bar{z}^{n+1}$ and $z^{n+1}$ are associative. Thus, eqs. (5.7) and (5.28) lead to:

$$\|z^{n+1}\|_{L_2(\Omega)}\leqq C_{10}h \quad \cdots\cdots(5.29)$$

The conclusion eq. (5.25) can be derived by introducing eqs. (5.29) and (4.29) into eq. (5.26).

Q. E. D.

## 6. NUMERICAL STUDIES

To illustrate the adaptability of the finite element Lax-Wendroff method, numerical calculations were made for a simple well-known problem. It is the analysis of the behavior of one dimensional wave propagation such that

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=0$$

$$u(0, t)=0.5$$

$$u(l, t)=u(-l, t)=0$$

where $u(x, t)$ is the unknown function and domain is taken as $[-l, l]$ with $l=20$. Numerical examples in this paper are restricted to the one dimensional problem since convergence properties can be clearly shown by using one dimensional examples. It is a simple and straitforward task to extend one dimensional algorithm to two or three dimensional algorithm. The computation was carried out by using the algorithm of the finite element Lax-Wendroff method with the lumped coefficient scheme for 4, 10, 20, 40 and 80 elements divisions and the results plotted in the figures by solid lines. Dotted lines in the figures show the exact solution. Figure 2 is an illustration of the numerical results, taking space mesh $h$ hold constant and changing time mesh $\Delta t$ as $\Delta t=h$, 0.5$h$ and 0.1$h$. The figure shows the numerical results at $t=10$. The computed

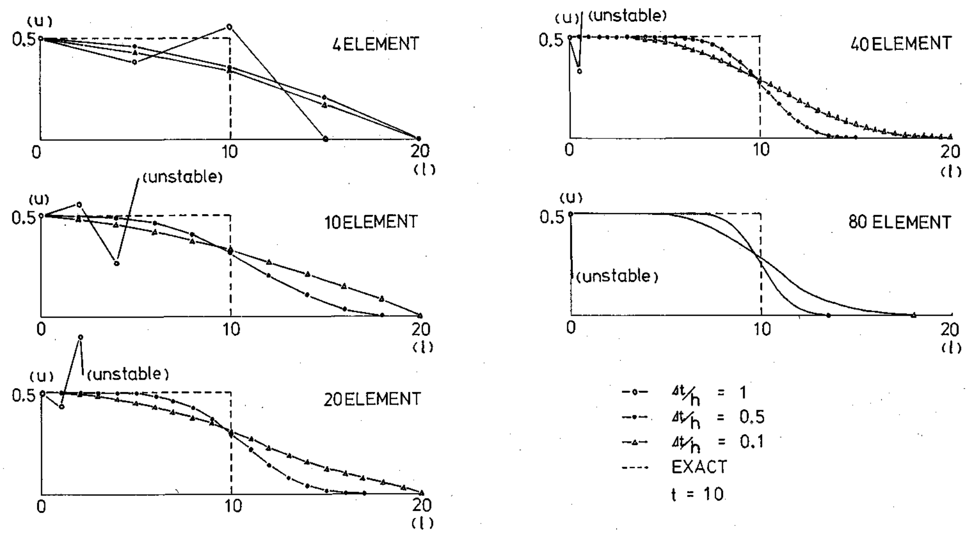


**Fig. 2** Numerical Resuults Mesh Hold.

norm $\|u-W\|_{L_2(\Omega)}$, i.e., the norm of the difference between the exact solution and the finite element approximate solution is illustrated in Figure 3. From the figure 3, the difference between the exact solution and the finite element approximate solution takes the minimum value at $\Delta t=0.8h$. Therefore, it can be concluded that the optimum value $\gamma$ in Theorem 2 is nearly 0.8.

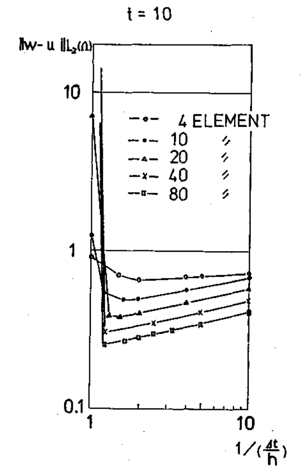


**Fig. 3** Difference Norm of Numerical Results in Fig. 2.

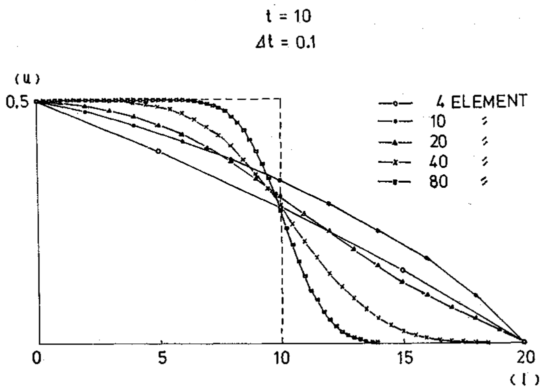


**Fig. 4** Numerical Results Taking Time Mesh Hold Constant.

Figures 4 and 5 show convergence properties as space mesh $h$ tends to approach zero while holding time mesh $\Delta t=0.1$. The computed results at $t=10$ are plotted as shown in figures 4 and 5. Figures 6 and 7 show the convergence properties as space mesh $h$ tends to approach zero while holding $\Delta t=\gamma\cdot h$, where $\gamma$ is taken as a constant. The computed results at $t=42$ and 84 are plotted as given in figures 6 and 7. The norm of the difference between the exact and approximate solution is illustrated in figure 7. From this figure, the order of convergence is observed to correspond to the order of $h$. These numerical result prove the conclusion obtained theoretically in Theorem 2.

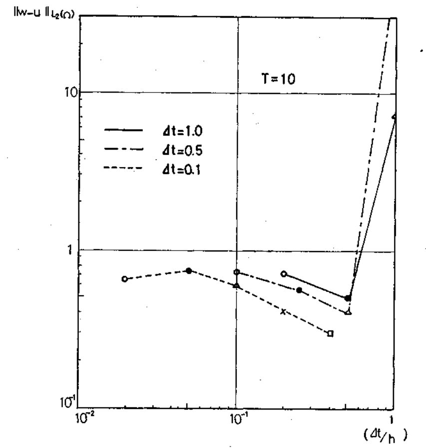


**Fig. 5** The Difference Norm of Numerical Results in Fig. 4.

## 7. CONCLUSIONS

This paper has presented two schemes of the numerical method, their convergence proofs and certain numerical illustrations. One of the schemes is called the finite element Lax-Wendroff method with the consistent coefficient scheme and the other the finite element Lax-Wendroff method with the lumped coefficient scheme. Convergence proofs of both schemes were obtained theoretically in Theorem 1 and Theorem 2. Numerical illustrations were also discussed for the case of one dimensional wave propagation problem. The methods employed the mesh points involving both time and space. The basic idea for discretization of time was originated from Lax-Wendroff difference method and for discretization of space from finite element method.

The numerical algorithm of the lumped coefficient scheme is a pure explicit scheme and, therefore, need not to solve algebraic simultaneous equation system. As far as the author's numerical examples are concerned, their computation has brought stable solutions. By both numerical and theoretical considerations, the order of convergence has shown to be the order of mesh parameter. For the choice of mesh spacing, it is concluded that both time and space mesh should be selected to be of equivalent

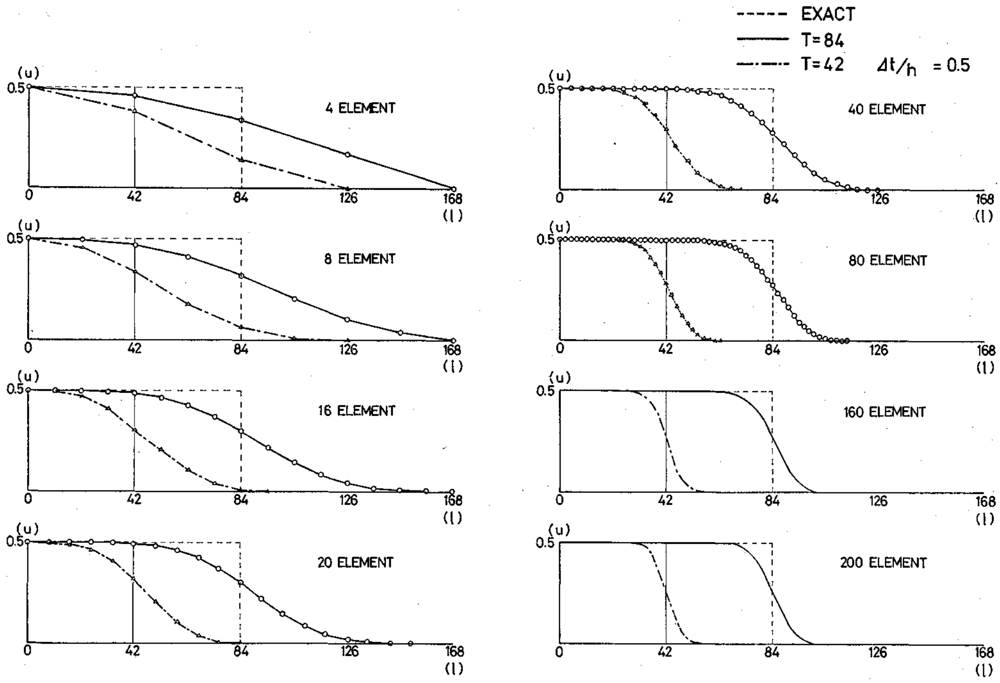


**Fig. 6** Numerical Results Taking $\Delta t/h=0.5$ Hold Constant.

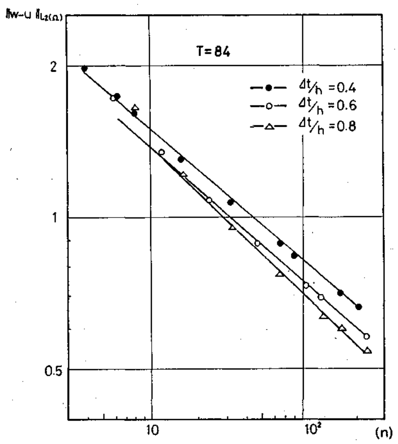


**Fig. 7** Difference Norm of Numerical Results in Fig. 6.

order.

The numerical examples in this paper were restricted to one dimensional problems because of simplicity. However, the adaptability of the conclusions in Theorem 1 and 2 is not limited to one dimensional problems, but also applicable to two and three dimensional problems. This paper has discussed only the case of linear problems. However, when taking the time mesh to be reasonably small and assuming the behavior of the phenomena to have linear property during the time increment, the nonlinear problems can be considered to be of piecewise linear problems. Thus, the conclusions in this paper are extended to apply to certain nonlinear problems such as tidal flow, flow in rivers and lakes and in many other fields of engineering problems.

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## APPENDIX

The matrices $\boldsymbol{A}$, $\bar{\boldsymbol{A}}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are constructed by superposing the elementwise matrices $A$, $\bar{A}$, $B$ and $C$ over the whole field. In case of three dimensional problem, the elementwise matrices are expressed in the following form, where $V$ is the volume of an element and

$$L_j=(z_k y_l-z_l y_k)+(z_l y_m-z_m y_l)+(z_m y_k-z_k y_m)$$

$$M_j=(z_k x_l-z_l x_k)+(z_l x_m-z_m x_l)+(z_m x_k-z_k x_m)$$

$$N_j=(y_k x_l-y_l x_k)+(y_l x_m-y_m x_l)+(y_m x_k-y_k x_m)$$

in wnich $(x_k, y_k, z_k)$ $k=1, 2, 3, 4$ is the coordinate of the nodal point $k$ [figure 1].

$$A=\frac{V}{20}\begin{pmatrix}2&1&1&1\\1&2&1&1\\1&1&2&1\\1&1&1&2\end{pmatrix},\qquad \bar{A}=\frac{V}{4}\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}$$

$$B=-\frac{\Delta t}{24}\begin{pmatrix}b_1L_1+b_2M_1+b_3N_1 & b_1L_2+b_2M_2+b_3N_2 & b_1L_3+b_2M_3+b_3N_3 & b_1L_4+b_2M_4+b_3N_4\\ b_1L_1+b_2M_1+b_3N_1 & b_1L_2+b_2M_2+b_3N_2 & b_1L_3+b_2M_3+b_3N_3 & b_1L_4+b_2M_4+b_3N_4\\ b_1L_1+b_2M_1+b_3N_1 & b_1L_2+b_2M_2+b_3N_2 & b_1L_3+b_2M_3+b_3N_3 & b_1L_4+b_2M_4+b_3N_4\\ b_1L_1+b_2M_1+b_3N_1 & b_1L_2+b_2M_2+b_3N_2 & b_1L_3+b_2M_3+b_3N_3 & b_1L_4+b_2M_4+b_3N_4\end{pmatrix}$$

$$
C=-\frac{\Delta t^2}{72V}\begin{bmatrix}
\begin{array}{l} b_1^2L_1^2+b_2^2M_1^2+b_3^2N_1^2 \\ \quad +2b_1b_2L_1M_1+2b_2b_3M_1N_1+2b_3b_1N_1L_1\ , \end{array} &
\begin{array}{l} b_1^2L_1L_2+b_2^2M_1M_2+b_3^2N_1N_2 \\ \quad +b_1b_2(L_1M_2+M_1L_2)+b_2b_3(M_1N_2+N_1M_2) \\ \quad +b_3b_1(N_1L_2+L_1N_2)\ , \end{array} &
\begin{array}{l} b_1^2L_1L_3+b_2^2M_1M_3+b_3^2N_1N_3 \\ \quad +b_1b_2(L_1M_3+M_1L_3)+b_2b_3(M_1N_3+N_1M_3) \\ \quad +b_3b_1(N_1L_3+N_3L_1)\ , \end{array} &
\begin{array}{l} b_1^2L_1L_4+b_2^2M_1M_4+b_3^2N_1N_4 \\ \quad +b_1b_2(L_1M_4+L_4M_1)+b_2b_3(M_1N_4+N_1M_4) \\ \quad +b_3b_1(N_1L_4+N_4L_1)\ , \end{array} \\
 &
\begin{array}{l} b_1^2L_1^2+b_2^2L_2^2+b_3^2L_3^2 \\ \quad +2b_1b_2L_2M_2+2b_2b_3M_2N_2+2b_3b_1N_2L_2 \end{array} &
\begin{array}{l} b_1^2L_2L_3+b_2^2M_2M_3+b_3^2N_2N_3 \\ \quad +b_1b_2(L_2M_3+M_2L_3)+b_2b_3(M_2N_3+N_2M_3) \\ \quad +b_3b_1(N_2L_3+N_3L_2)\ , \end{array} &
\begin{array}{l} b_1^2L_2L_4+b_2^2M_2M_4+b_3^2N_2N_4 \\ \quad +b_1b_2(L_2M_4+L_4M_2)+b_2b_3(M_2N_4+N_2M_4) \\ \quad +b_3b_1(N_2L_4+N_4L_2)\ , \end{array} \\
 & & 
\begin{array}{l} b_1^2L_3^2+b_2^2M_3^2+b_3^2N_3^2 \\ \quad +2b_1b_2L_3M_3+2b_2b_3M_3N_3+2b_3b_1N_3L_3 \end{array} &
\begin{array}{l} b_1^2L_3L_4+b_2^2M_3M_4+b_3^2N_3N_4 \\ \quad +b_1b_2(L_3M_4+L_4M_3)+b_2b_3(M_3N_4+N_3M_4) \\ \quad +b_3b_1(N_3L_4+N_4L_3)\ , \end{array} \\
 & \text{SYM.} & &
\begin{array}{l} b_1^2L_4^2+b_2^2M_4^2+b_3^2N_4^2 \\ \quad +2b_1b_2L_4M_4+2b_2b_3M_4N_4+2b_3b_1N_4L_4 \end{array}
\end{bmatrix}
$$

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