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ELASTIC STABILITY OF CENTRALLY LOADED THIN-WALLED MEMBERS WITH OPEN SECTIONS

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SYNOPSIS

The present study is concerned with theoretical and numerical investigation of elastic stability of a thin-walled straight column which is composed of flat plates and subjected to uniform axial compression that acts at simply supported ends. An approach that differs slightly from existing methods in computing critical loads is proposed on the basis of Kirchhoff's hypothesis for the out-of-plane deformation of the plate and Euler-Bernoulli's hypothesis for its in-plane deformation.

In the theoretical part, a unified analysis in determining critical stresses of arbitrary plate assemblies is given, in which all possible interactions between column and local buckling are taken into account. Numerical results on torsionally weak columns with channel-sections show that the consideration of all possible interactions results in significant changes in column buckling stresses.

1. INTRODUCTION

When thin-walled elastic columns composed of plates are subjected to uniform axial compression, either column or local buckling may take place depending on their lengths and their cross-sectional dimensions[1].

In the theories[2)~4)] of column buckling, it is assumed that cross-sectional shape is preserved. As a result, three possible modes exist; (1) flexural buckling; (2) torsional buckling; (3) torsion-flexural buckling[5)~11)].

While in the usual treatments[12)~17)] of local buckling, the unloaded edges of a buckling plate are considered to be either completely restrained against translation and elastically restrained against rotation by the adjacent plates, or being completely free against translation and rotation. In reality it is not the single plate which becomes unstable, but the whole assembly of plates reaches a state of instability characterized by the beginning of distortion of the cross-section[3)].

The effects of profile deformation on overall buckling have been considered by many investigators[18)~25)]. In connection with the interaction between column and local buckling, Bijlaard[1)] pointed out that the real buckling stress is smaller than either column or local buckling stress. According to his conclusions, the interaction is important only for the case of columns with T-sections and angle sections, in which torsion-flexural buckling mode governs.

Pflüger[26)] investigated the interaction on a column with a channel section by reducing the section to an equivalent structural model consisting of three plates, and computed critical loads by means of the Rayleigh-Ritz method. However, the assumed shape functions in his calculation excluded the possibility of transverse bending of the flange as a plate.

Under a similar structural model, Ghobarah and Tso[27)] determined critical loads with the aid of transfer matrix method. But their approximate analysis did not include the possibility of torsion-flexural buckling.

Wittrick and Williams[28)] solved the same problem by treating the section as three flat plates connected together, in which torsion-flexural type of modes is taken into account. One puzzling aspect of their results is that the modes are always either symmetrical or antisymmetrical; whereas the real buckling mode can not necessarily decompose into the two distinct modes.

The purpose of this paper is to present a unified approach to elastic stability of a thin-walled straight column subjected to uniform axial

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compression, in which all possible interactions between modes are considered. The systematic equilibrium method based on Kirchhoff's hypothesis and Euler-Bernoulli's hypothesis[4], is applicable to a buckling analysis for plate assemblies with arbitrary cross-sections. The deformation of every component plate is assumed to vary sinusoidally in the longitudinal direction. However, the present analysis does not assume beforehand the buckled pattern of cross-sections; thus it leads to explicit exact expressions for coupled buckling.

Although the present analysis is similar to that of folded plate theories[29],[30], it differs from Bijlaard's theory in that the concept of a "spring constant of restraining plate" is not used, and also differs from the approximate methods such as the finite strip methods[31],[32] in that the exact buckling displacement functions are introduced in place of approximate ones. The analysis used herein is exact, apart from the assumptions inherent in the small deformation theory, in the sense that all possible interactions between modes are considered.

A numerical approach is to arrange linear homogeneous equations governing the buckling stability, and to find critical stress by trial with the condition that the determinant of coefficient matrix becomes zero. The elements of the matrix are transcendental functions of the longitudinal half wave length. However, the computational procedure is not a hard task for a high speed computer, as long as the column is made up of a few plates. It has the advantage of using small matrices, thus requiring little time for computer execution.

To illustrate application of the present analysis, the coupled buckling stress for I-sectioned columns and channel-sectioned columns is compared to column buckling stress and local buckling stress, in which the former is calculated from the assumption that the cross-section does not distort, and the latter is computed from the assumption that the corners of the section remain straight lines. Comparisons are also made between the results of this paper and those of existing literature[3],[27],[28].

2. THEORETICAL DEVELOPMENT

Consider a straight column of open cross-section composed of m thin flat plates, as shown in Fig. 1. Each plate is assumed to be of uniform thickness and the cross-section of the column is supposed to be constant.

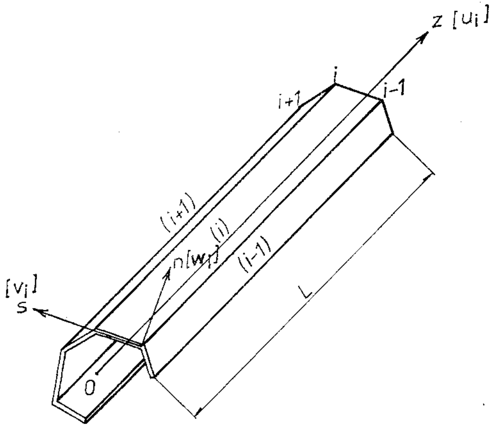


Fig. 1 Structural Model for Thin-walled Column.

The local coordinate system (z, s, n) with origin on the middle surface of each plate is used to describe the location of a point, where the direction of z-axis coincides with that of the longitudinal axis of the column, s is taken along the middle line and n normal to it. The orientation of the system follows the right hand rule. The displacements of a point lying on the middle surface of the i-th plate in the z, s, and n directions are denoted by u_i, v_i, and w_i, respectively.

The assumptions are summarized as follows:

1) Material is homogeneous, isotropic and linearly elastic.
2) Deformation in the buckled state is supposed to be infinitesimally small; and no postbuckling phenomena are considered.
3) Kirchhoff's hypothesis[4] is applicable to the out-of-plane deformation of each plate.
4) Euler-Bernoulli's hypothesis[4] is applicable to the in-plane deformation of each plate.
5) Joints of plates are rigidly connected along the edges.

(1) Equilibrium Equations

a) The Out-of-plane Equilibrium of Plates

Consider a thin flat plate which is loaded at both ends by the uniformly distributed compressive load $\delta_i\sigma^{(0)}$ where δ_i is the thickness of the i-th plate and $\sigma^{(0)}$ is the normal stress in the pre-critical state.

From Kirchhoff's hypothesis, the basic differrential equations for the buckling deflections $w_i(z, s)$ are written as[3]

$$D_i\left(\frac{\partial^4 w_i}{\partial z^4}+2\frac{\partial^4 w_i}{\partial z^2\partial s^2}+\frac{\partial^4 w_i}{\partial s^4}\right)+\delta_i\sigma^{(0)}\frac{\partial^2 w_i}{\partial z^2}=0\,,$$

$$(i=1, 2, 3, \cdots, m)\,, \quad \cdots\cdots\cdots\cdots(1)$$

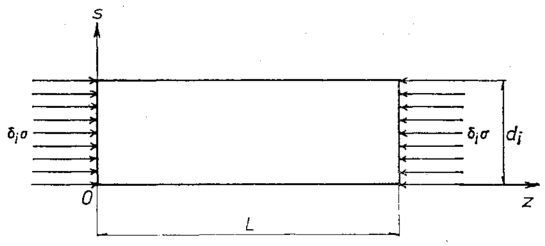


Fig. 2 The i-th Plate subjected to Uniform Axial Compression.

in which $D_i = E\delta_i^3/12(1-\mu^2)$ is the flexural rigidity of the i-th plate, E being the modulus of elasticity and μ being Poisson's ratio.

b) The In-plane Equilibrium of Plates

Fig. 3 shows the positive directions of distributed loads p_i, transverse forces $S_{i,i}$, $S_{i,i-1}$, distributed forces $t_{i,i}$, $t_{i,i-1}$, shearing forces V_i, normal forces N_i and moments M_i in the buckled state.

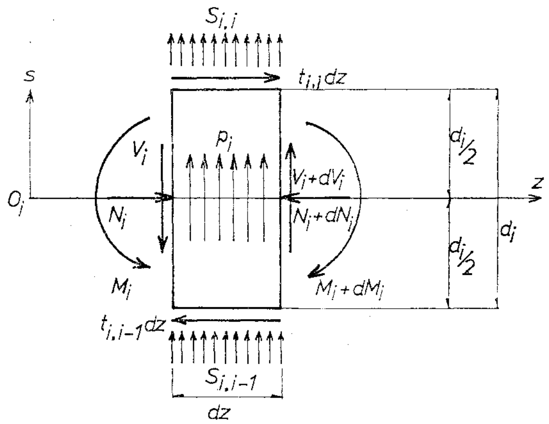


Fig. 3 Force Diagram for an Elemental Strip of the i-th Plate.

The equilibrium equations in the s direction are given by

$$\frac{dV_i}{dz} = -S_{i,i} - S_{i,i-1} - p_i\,, \quad (i=1, 2, 3, \cdots, m)\,, \quad \cdots\cdots\cdots(2)$$

in which, for the loading condition of a distributed load in the buckled state[3],

$$p_i = -\sigma^{(0)} A_i \frac{d^2 v_i}{dz^2}\,, \quad (i=1, 2, 3, \cdots, m)\,, \quad \cdots\cdots\cdots(3)$$

where A_i is the cross-sectional area of the i-th plate. The equilibrium equations in the z direction are written as

$$\frac{dN_i}{dz} = t_{i,i} - t_{i,i-1}\,, \quad (i=1, 2, 3, \cdots, m)\,. \quad \cdots\cdots\cdots(4)$$

The equilibrium equations of distributed forces along the edges are of the form

$$t_{i+1,i} = t_{i,i}\,, \quad (i=1, 2, 3, \cdots, m-1)\,, \quad \cdots\cdots\cdots(5a)$$

$$t_{1,0} = t_{m,m} = 0\,, \quad \text{(at free edges)}\,. \quad \cdots\cdots(5b)$$

Other equilibrium equations are obtained from taking moments of all the forces acting on the element, i.e.,

$$\frac{dM_i}{dz} = V_i - (t_{i,i} + t_{i,i-1})\frac{d_i}{2}\,, \quad (i=1, 2, 3, \cdots, m)\,. \quad \cdots\cdots\cdots(6)$$

According to Euler-Bernoulli's hypothesis[4], the relations between moments and curvatures are

$$M_i = -E_r I_i \frac{d^2 v_i}{dz^2}\,, \quad (i=1, 2, 3, \cdots, m)\,, \quad \cdots\cdots\cdots(7)$$

where I_i is the moment of inertia of the i-th plate, and the reduced modulus of elasticity $E_r = E/(1-\mu^2)$ is used.

The curvatures d^2v_i/dz^2 and the additional mean strains ε_i of plates in the buckled state are not independent of each other, because where two plates join, the strains in both plates must be equal. Then, the compatibility conditions become

$$\varepsilon_i + v_i''\frac{d_i}{2} = \varepsilon_{i+1} - v_{i+1}''\frac{d_{i+1}}{2}\,, \quad (i=1, 2, \cdots, m-1)\,, \quad \cdots\cdots\cdots(8)$$

in which a prime ($'$) denotes differentiation with respect to z, and ε_i are positive in contraction.

The additional normal forces in the buckled state are written as

$$N_i = E_r A_i \varepsilon_i\,, \quad (i=1, 2, 3, \cdots, m)\,, \quad \cdots(9)$$

which must satisfy the following condition:

$$\sum_{i=1}^{m} N_i = 0\,. \quad \cdots\cdots\cdots(10)$$

Hence, in view of Eqs. (2), (3), (6), and (7), the in-plane equilibrium equations are expressed in terms of the transverse forces $S_{i,i}$, $S_{i,i-1}$, the distributed forces $t_{i,i}$, $t_{i,i-1}$, and the displacements v_i, i.e.,

$$S_{i,i} + S_{i,i-1} = E_r I_i v_i'''' + \sigma^{(0)} A_i v_i'' - (t'_{i,i} + t'_{i,i-1})\frac{d_i}{2}\,, \quad (i=1, 2, 3, \cdots, m)\,. \quad \cdots\cdots\cdots(11)$$

Note that Eqs. (4), (5), (8), (9), and (10) are used to express $t_{i,i}$ and $t_{i,i-1}$ in terms of v_i. Finally, the expressions $(S_{i,i} + S_{i,i-1})$ are related to the buckling displacements v_i.

(2) Boundary Conditions at the Edges of Plates

a) Geometrical Boundary Conditions

Consider two adjacent plates (i) and ($i+1$) which meet along the i-th edge. In general, another plate may also join at this edge as shown in Fig. 4 by the dashed line.

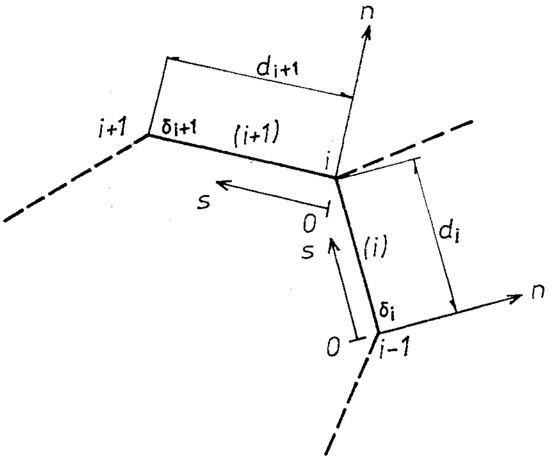


Fig. 4 Geometry and Layout of Cross-section.

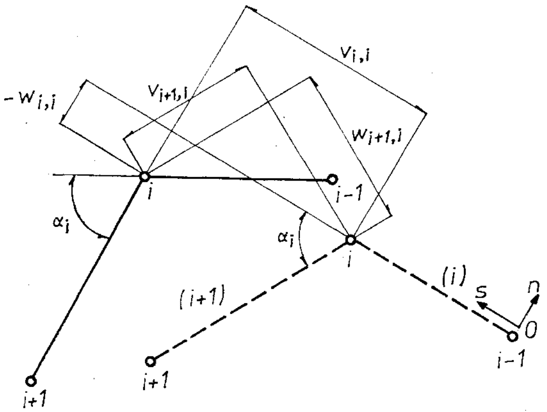


Fig. 5 Continuity of Displacements at the i-th Edge.

The continuity equations of displacements at the i-th edge are represented by

$$v_{i+1,i}=v_{i,i}\cos\alpha_i-w_{i,i}\sin\alpha_i\ ,\qquad (i=1,2,3,\cdots,m)\ ,\quad\cdots\cdots(12a)$$

$$w_{i+1,i}=v_{i,i}\sin\alpha_i+w_{i,i}\cos\alpha_i\ ,\qquad (i=1,2,3,\cdots,m)\ ,\quad\cdots\cdots(12b)$$

where α_i is the angle between the i-th plate and the ($i+1$)-th plate.

Since the two plates are rigidly connected, they rotate through the same angle at the i-th edge. The continuity equations are of the form

$$\frac{\partial w_{i+1,i}}{\partial s}=\frac{\partial w_{i,i}}{\partial s}\ ,\qquad (i=1,2,3,\cdots,m-1)\ .\quad\cdots\cdots(13)$$

In accordance with the assumption of infinitesimal deformation, the change of distance between the two edges of the i-th plate in the buckled state is negligible, so that

$$v_{i,i}=v_{i,i-1}\ ,\qquad (i=1,2,3,\cdots,m)\ .\quad\cdots\cdots(14)$$

b) Statical Boundary Conditions

With reference to Fig. 6, the equilibrium equations of moments are expressed by

$$M_{i+1,i}+M_{i,i}=0\ ,\qquad (i=1,2,3,\cdots,m-1)\ ,\quad\cdots\cdots(15)$$

where $M_{i,i}$ and $M_{i+1,i}$ are the bending moment of the i-th plate at the i-th edge and that of the ($i+1$)-th plate at the i-th edge, respectively. Here anticlockwise moments are considered positive.

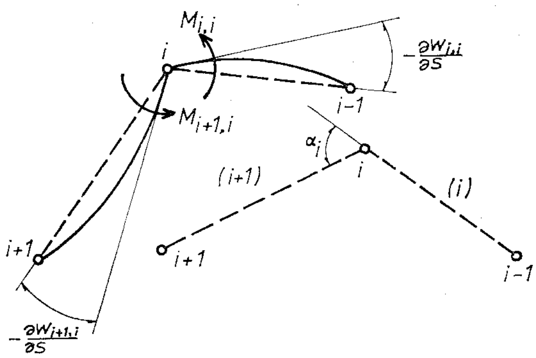


Fig. 6 Rotations and Moments at the i-th Edge.

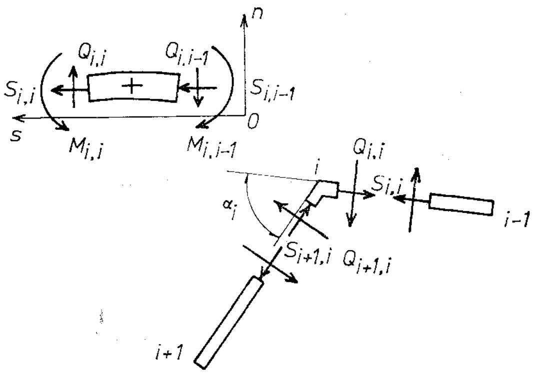


Fig. 7 Equilibrium of Forces at the i-th Edge.

Eqs. (15) take the simple form at free edges, i.e.,

$$M_{1,0}=M_{m,m}=0\ . \quad \cdots\cdots(16)$$

The equilibrium equations of forces at the i-th edge are represented by

$$Q_{i+1,i}=S_{i,i}\sin\alpha_i+Q_{i,i}\cos\alpha_i\ , \quad \cdots\cdots(17a)$$

$$(i=1, 2, 3, \cdots, m-1)\ ,$$

$$S_{i+1,i}=Q_{i,i}\sin\alpha_i-S_{i,i}\cos\alpha_i\ , \quad \cdots\cdots(17b)$$

where $Q_{i,i}$ and $Q_{i+1,i}$ are the transverse shear of the i-th plate at the i-th edge and that of the $(i+1)$-th plate at the i-th edge, respectively. Eqs. (17) take the simple form at the free edges, i.e.,

$$Q_{1,0}=0\ , \quad S_{1,0}=0\ ; \quad Q_{m,m}=0\ , \quad S_{m,m}=0\ . \quad \cdots\cdots(18)$$

(3) Boundary Conditions at both Ends of Column

When the ends $z=0$ and $z=L$ are simply supported, the boundary conditions may be written as follows:[4]

a) As a plate: $w_i(0, s)=w_i(L, s)=0\ ,$

$w_i''(0, s)=w_i''(L, s)=0\ ,$

$(i=1, 2, 3, \cdots, m)\ , \cdots(19a)$

b) As a beam: $v_i(0, s)=v_i(L, s)=0\ ,$

$v_i''(0, s)=v_i''(L, s)=0\ ,$

$(i=1, 2, 3, \cdots, m)\ . \cdots(19b)$

(4) General Solution and Stability Criterion

The buckling displacements v_i and w_i are assumed in terms of the following product functions which satisfy the boundary conditions (19a) and (19b)

$$v_i(z, s)=V_i(s)\sin\frac{k\pi}{L}z=V_i\sin\lambda z\ , \quad \cdots\cdots(20a)$$

$$(i=1, 2, 3, \cdots, m)\ ,$$

$$w_i(z, s)=W_i(s)\sin\frac{k\pi}{L}z=W_i\sin\lambda z\ , \quad \cdots(20b)$$

where $V_i(s)$ and $W_i(s)$ are unknown functions of s, and k is the number of half waves. Since two adjacent plates are rigidly connected, $\lambda=k\pi/L$ has to be the same for all plates[1]. Substitution of Eqs. (20b) into Eqs. (1) and canceling $\sin\lambda z$ lead to

$$\ddddot{W}_i-2\lambda^2\ddot{W}_i-\left(\frac{\sigma^{(0)}\delta_i}{D_i}\lambda^2-\lambda^4\right)W_i=0\ ,$$

$$(i=1, 2, 3, \cdots, m)\ , \quad \cdots\cdots(21)$$

where a dot ($\cdot$) denotes differentiation with respect to s. The general solutions for Eqs. (20) can be put in the following form:

$$W_i(s)=C_{i1}\cosh\gamma_i s+C_{i2}\sinh\gamma_i s+C_{i3}\cos\beta_i s+C_{i4}\sin\beta_i s\ , \quad (i=1, 2, 3, \cdots, m)\ , \quad \cdots\cdots(22)$$

where γ_i and β_i are

$$\left.\begin{aligned}\gamma_i&=\sqrt{\lambda^2+\sqrt{\frac{\sigma^{(0)}\delta_i}{D_i}}\lambda}\ ,\\ \beta_i&=\sqrt{-\lambda^2+\sqrt{\frac{\sigma^{(0)}\delta_i}{D_i}}\lambda}\ .\end{aligned}\right\} \quad \cdots\cdots(23)$$

Here C_{i1} to C_{i4} are constants.

In view of the in-plane equilibrium equations and the boundary conditions at edges, all the equations from (2) to (18) may reduce to $4m$ homogeneous equations with respect to the buckling displacements w_i. Hence the $4m$ equations for C_{i1}, C_{i2}, C_{i3}, C_{i4} can be obtained from using Eqs. (21).

The buckled form of column becomes possible only if the $4m$ equations for C_{i1}, C_{i2}, C_{i3}, C_{i4} yield solutions different from zero, which requires that the determinant of the coefficient matrix of these equations must vanish.

3. APPLICATION OF PROPOSED METHOD

(1) I-section

Consider first the I-sectioned column as shown in Fig. 8. The boundary conditions may be given by the following equations where the superscript $^{(0)}$ of normal stress is omitted for the sake of simplicity.

a) Continuity of displacements:

i) $w_1(z, 0)=w_2(z, 0)=-v_3(z, d_2/2)\ ,$

ii) $w_4(z, 0)=w_5(z, 0)=-v_3(z, -d_2/2)\ ,$

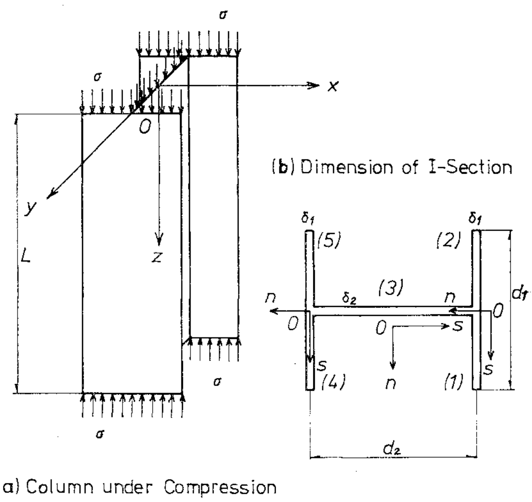


Fig. 8 Layout of Column with I-section.

iii) $w_3(z, d_2/2)=v_1(z,0)=v_2(z, 0)$,

iv) $w_3(z, -d_2/2)=v_4(z, 0)=v_5(z, 0)$,

v) $v_3(z, d_2/2)=v_3(z, -d_2/2)$.

b) Continuity of rotational angles:

i) $\dot{w}_1(z, 0)=\dot{w}_2(z, 0)=\dot{w}_3(z, d_2/2)$,

ii) $\dot{w}_4(z, 0)=\dot{w}_5(z, 0)=\dot{w}_3(z, -d_2/2)$.

c) Equilibrium of moments:

i) $M_1(z, 0)-M_2(z, 0)-M_3(z, d_2/2)=0$,

ii) $M_4(z, 0)-M_5(z, 0)+M_3(z, -d_2/2)=0$.

d) Moments at the free edges:

$$M_1(z, d_1/2)=M_2(z, -d_1/2)=M_4(z, d_1/2)=M_5(z, -d_1/2)=0 .$$

e) Transverse shears at the free edges:

$$Q_1(z, d_1/2)=Q_2(z, -d_1/2)=Q_4(z, d_1/2)=Q_5(z, -d_1/2)=0 .$$

f) Equilibrium of transverse forces:

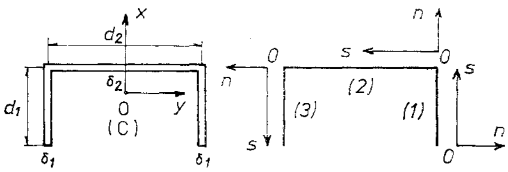
i) $$E_r(I_1+I_2)v_1''''(z, 0)=Q_2(z, d_2/2)-E_r(A_1+A_2)v_1''''(z, 0)d_1^2/16-\sigma(A_1+A_2)v_1''(z, 0) ,$$

ii) $$E_rI_3v_3''''(z, 0)=Q_1(z, 0)-Q_2(z, 0)+Q_4(z, 0)-Q_5(z, 0)-E_r(A_1+A_2+A_4+A_5)v_3''''(z, 0)d_2^2/4-\sigma A_3v_3''(z, 0) ,$$

iii) $$E_r(I_4+I_5)v_4''''(z, 0)=-Q_3(z, -d_2/2)-E_r(A_4+A_5)v_4''''(z, 0)d_1^2/16-\sigma(A_4+A_5)v_4''(z, 0) .$$

(2) Channel-section

The final results of calculations for the section shown in Fig. 9 are,



(a) Dimension of Channel Section (b) Local Coordinate System

Fig. 9 Sketch of Channel-section showing Notations.

1) $W_1(d_1)+W_3(0)=0$,

2) $\dot{W}_1(d_1)-\dot{W}_2(0)=0$,

3) $\dot{W}_2(d_2)-\dot{W}_3(0)=0$,

4) $$D_1[\ddot{W}_1(d_1)-\mu\lambda^2W_1(d_1)]-D_2[\ddot{W}_2(0)-\mu\lambda^2W_2(0)]=0 ,$$

5) $$D_2[\ddot{W}_2(d_2)-\mu\lambda^2W_2(d_2)]-D_3[\ddot{W}_3(0)-\mu\lambda^2W_3(0)]=0 ,$$

6) $D_1[\ddot{W}_1(0)-\mu\lambda^2W_1(0)]=0$,

7) $D_3[\ddot{W}_3(d_1)-\mu\lambda^2W_3(d_1)]=0$,

8) $D_1[\dddot{W}_1(0)-(2-\mu)\lambda^2\dot{W}_1(0)]=0$,

9) $D_3[\dddot{W}_3(d_1)-(2-\mu)\lambda^2\dot{W}_3(d_1)]=0$,

10) $$\left[E_rI_1\lambda^4+E_rA_1\frac{A_1+A_2}{2A_1+A_2}\frac{d_1^2}{4}\lambda^4-\sigma A_1\lambda^2\right]W_2(0)-E_rA_1\frac{d_1d_2}{4}\lambda^4W_1(d_1)-E_rA_1\frac{A_1}{2A_1+A_2}\frac{d_1^2}{4}\lambda^4W_2(d_2)+D_2[\dddot{W}_2(0)-(2-\mu)\lambda^2\dot{W}_2(0)]=0 ,$$

11) $$\left[E_rI_2\lambda^4+2E_rA_1\frac{d_2^2}{4}\lambda^4-\sigma A_2\lambda^2\right]W_1(d_1)-E_rA_1\frac{d_1d_2}{4}\lambda^4W_2(0)+E_rA_1\frac{d_1d_2}{4}\lambda^4W_2(d_2)-D_3[\dddot{W}_3(0)-(2-\mu)\lambda^2\dot{W}_3(0)]-D_1[\dddot{W}_1(d_1)-(2-\mu)\lambda^2\dot{W}_1(d_1)]=0 ,$$

12) $$\left[E_rI_1\lambda^4+E_rA_1\frac{A_1+A_2}{2A_1+A_2}\frac{d_1^2}{4}\lambda^4-\sigma A_1\lambda^2\right]W_2(d_2)-E_rA_1\frac{A_1}{2A_1+A_2}\frac{d_1^2}{4}\lambda^4W_2(0)+E_rA_1\frac{d_1d_2}{4}\lambda^4W_1(d_1)-D_2[\dddot{W}_2(d_2)-(2-\mu)\lambda^2\dot{W}_2(d_2)]=0 .$$

4. NUMERICAL EVALUATION

(1) I-section

Figs. 10 (a), (b) show the results of calculation of coupled buckling stresses σ_{cr} for the section shown in Fig. 8. Here the critical stresses σ_{cr} are plotted against the ratio a/d_2 of the half wave length a to the web width d_2, in which $a=L/k$ $(k=1, 2, 3, \cdots)$. The flexural buckling stresses σ_{Fy} and σ_{Fx} are, respectively, defined by[4)]

$$\sigma_{Fy}=\frac{E_rI_y\pi^2}{Aa^2} ; \quad \sigma_{Fx}=\frac{E_rI_x\pi^2}{Aa^2} , \cdots\cdots\cdots(24)$$

where I_y and I_x are the moments of inertia about the weak and strong axes, respectively, and A is the total cross-sectional area. The torsional buckling stress σ_T is defined by[3)]

$$\sigma_T=\frac{E_rI_\omega\pi^2/a^2+GJ_d}{I_p} , \cdots\cdots\cdots\cdots\cdots\cdots\cdots(25)$$

where E_rI_ω is the warping rigidity, GJ_d being St. Venant's torsional rigidity and I_p being the polar moment of inertia with respect to the shear center. The local buckling stress σ_L is calcu-

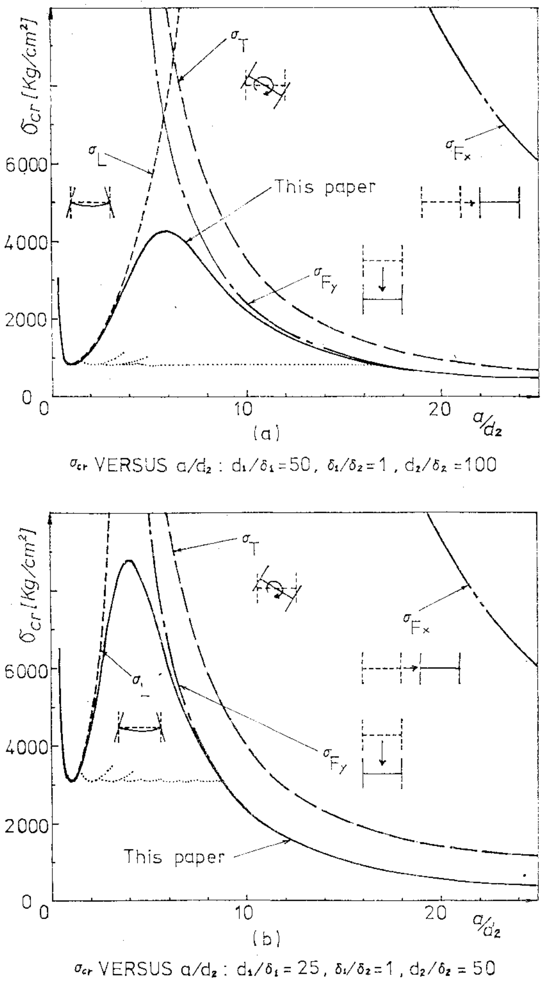


Fig. 10 (a), (b) Relations between Critical Stress σ_{cr} and Half Wave Length—Web Width Ratio a/d_2 ($E=2.1\times10^6$ kg/cm²; $\mu=0.3$).

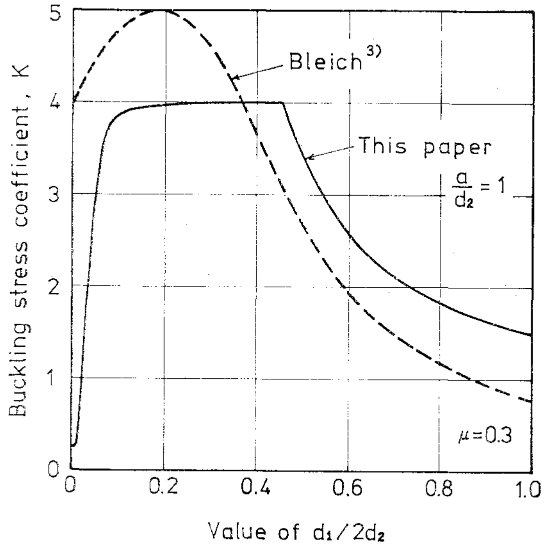


Fig. 11 $K-d_1/2d_2$ Relationship for $d_2/\delta_2=50$; $\delta_1/\delta_2=1$.

lated from the assumption that there is no outward deflection at the junction of plates. It will be observed that the effects of interaction between column and local buckling are pronounced only for the case of columns with intermediate half wave lengths. However, considering the coupled buckling stresses σ_{cr} of columns with one or more half waves, as indicated by the fine dotted lines in Figs. 10, yields the conclusion that the interaction between column and local buckling is practically negligible for columns with I-sections. This agrees with Bijlaard's results[1].

To illustrate the features of local buckling, the results of computation are compared with those of Bleich[3] and presented in Fig. 11, where the buckling stress coefficient K is plotted against the ratio $d_1/2d_2$. The coefficient K is determined from the formula[3]

$$\sigma_{cr}=\frac{K\pi^2E}{12(1-\mu^2)}\left(\frac{\delta_2}{d_2}\right)^2. \quad \cdots\cdots\cdots\cdots(26)$$

The significant discrepancy will be found between the exact results of this paper and the approximate ones of Bleich[3]. For a small ratio $d_1/2d_2$, the I-section is considered as a web plate with stiffeners at both unloaded edges. As a particular case, when $d_1/2d_2$ equals to zero, the section reduces to a web plate only so that the value K of the present analysis amounts to that of a plate having free longitudinal edges. This case corresponds to the flexural buckling (Euler buckling) of the plate. For a large ratio $d_1/2d_2$, the value K of conventional local buckling analysis[3] gives conservative value. It is to be inferred from the fact that the conventional approximate analyses[3),13] are based on the local buckling analysis of a single plate with simulated boundary conditions and an assumed half wave length a. For example, Bleich[3] introduced a "coefficient of restraint" at the unloaded edges, and supposed the half wave length $a=\infty$ for the buckling of outstanding flanges.

(2) Channel-section

The results of computation of coupled buckling stresses σ_{cr} are given as a set of curves in Figs. 12 (a), (b), (c). Here the torsion-flexural buckling stress σ_{TF_x} is defined by[5]

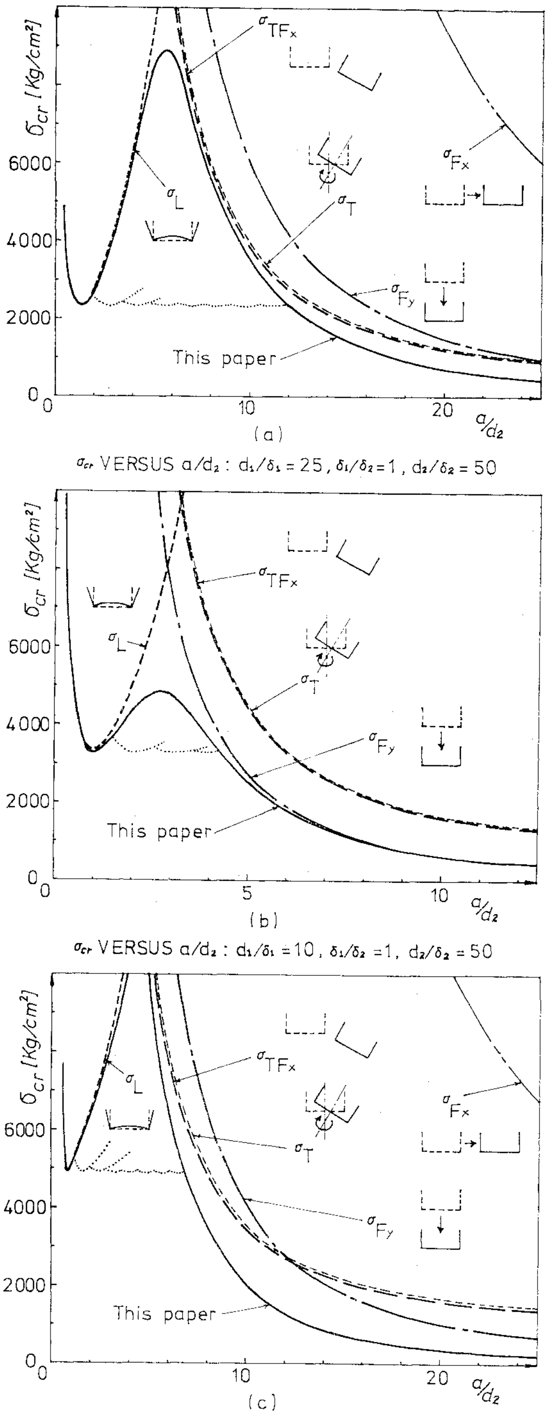
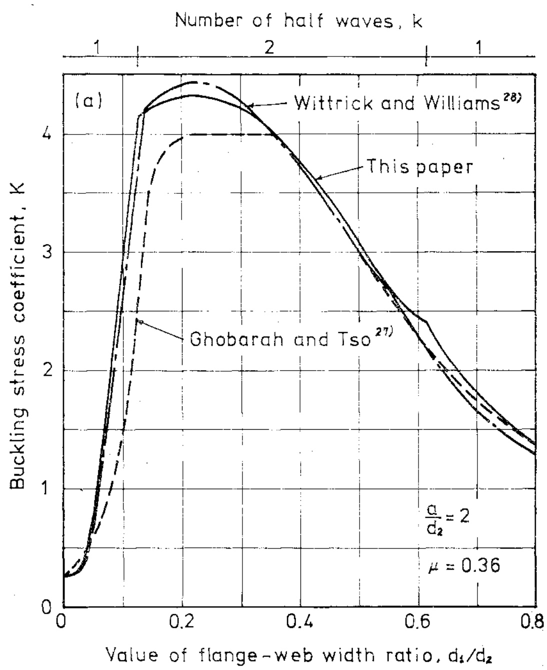


Fig. 12 (a), (b), (c) Relations between Critical Stress σ_{cr} and Half Wave Length—Web Width Ratio a/d_2 ($E=2.1\times10^6$ kg/cm^2; $\mu=0.3$).

$$\sigma_{TF_x} = \frac{(\sigma_{F_x}+\sigma_T)-\sqrt{(\sigma_{F_x}-\sigma_T)^2+4\sigma_{F_x}\sigma_T a_x^2 A/I_p}}{2(1-a_x^2 A/I_p)}. \quad \cdots\cdots(27)$$

where a_x is the distance from the centroid to the shear center. It will be observed that, if one of the three buckling stresses (σ_{F_y}, σ_{TF_x}, σ_L) is much smaller than the others, the coupled buckling stress σ_{cr} is nearly equal to it. On the other hand, if two or three of the three (σ_{F_y}, σ_{TF_x}, σ_L) are close to one another, the difference between the coupled buckling stress σ_{cr} and the smallest one of the three becomes large. It must be remarked, however, that the consideration of all possible interactions between modes results in significant reduction in column buckling stresses in which torsion-flexural buckling mode governs. On the other hand, the interaction between column and local buckling is pronounced for the case of flexural buckling mode.

The values of buckling stress coefficient K are plotted against the ratio d_1/d_2 in Figs. 13 (a), (b), (c), for three values of a/d_2. For comparison, the corresponding curves are taken from Ref. 27) and Ref. 28). Note that the number of half waves k is shown at the top of Figs. 13. Discrepancy between the exact results of this paper and the approximate ones of Ghobarah and Tso[27] is due to the fact that their analysis excludes the possibility of torsion-flexural buckling. How-



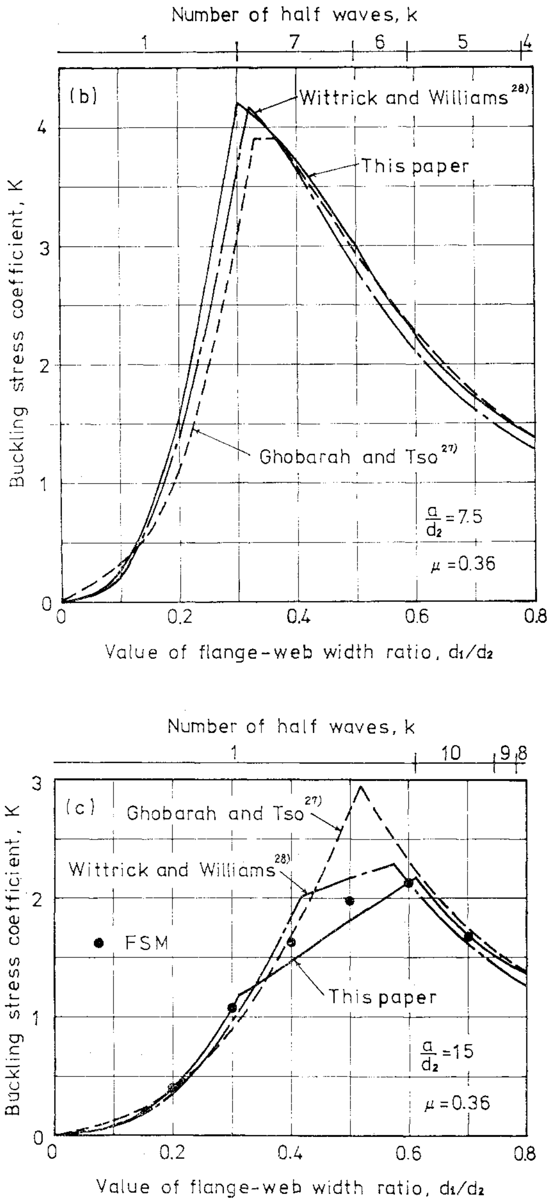


Fig. 13 (a), (b), (c) $K-d_1/d_2$ Relationship for $d_2/\delta_2=50$; $\delta_1/\delta_2=1$.

ever, the results of Wittrick and Williams[28] show excellent agreement with those of the present analysis except for the case of $a/d_2=15$. The difference in the case of $a/d_2=15$ arises from the fact that the buckling modes of Wittrick and Williams are always symmetrical or antisymmetrical, while the buckling modes of the present evaluation do not decompose into the two distinct modes. This lack of symmetry yields the conclusion that there exists an interaction between the torsion-flexural buckling σ_{TF_x} and the flexural buckling σ_{F_y}, as long as all possible interactions between column and local buckling are considered in the analysis. The validity of the present analysis is also confirmed by comparison with the approximate values of the finite strip method (FSM)[31], as shown in Fig. 13 (c). Note that Wittrick and Williams used nonlinear theory of plane elasticity for the in-plane deformation, however, the present analysis is based on the linear beam theory.

5. CONCLUSIONS

The following conclusions may be drawn from this study.

For theoretical study:

1) The present analysis based on Kirchhoff's hypothesis and Euler-Bernoulli's hypothesis involves all possible interactions between column and local buckling.
2) The unified approach is applicable to a buckling analysis of arbitrary plate assemblies under compression.
3) The systematic derivation of basic equations for a plate assembly is helpful in pointing out wider theoretical application of the theory of folded plates.

For numerical study:

4) Comparison between the analysis and the numerical evaluation reveals that the unified analysis is adequate to predict coupled buckling stresses.
5) In the buckling of I-sectioned columns, all possible interactions between column and local buckling are negligible. Therefore, only the smallest buckling stress is needed for design purposes.
6) For the case of columns with channel-sections, the consideration of all possible interactions between modes results in significant reduction in column buckling stress, in which torsion-flexural buckling mode governs. Thus sufficient stiffeners are required to increase the torsional rigidity.
7) Although the present stability matrix contains transcendental functions, the computational procedure requires little time to achieve satisfactory accuracy. The procedure is especially effective and advantageous for a column consisting of a few plates.

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