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FINITE ELEMENT ANALYSIS OF STEADY FLOW OF VISCOUS FLUID USING STREAM FUNCTION

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1. INTRODUCTION

A number of finite element methods are available in the field of incompressible viscous fluid flow with nonlinear convection term. Huebner[1] has discussed a wide and useful review of the finite element anylyses. These investigations can be classified into three categories. The first category includes the finite element analyses which employ the values of the velocity and pressure at nodal points in the flow field as unknown variables. Oden and Wellford[2], Taylor and Hood[3],[4], Argyris and Mareczek[5], Gartling and Becker[6], Nickel, Tanner and Caswell[7], Kawahara, et al.[8]~[10] and Usuki and Kudo[11] presented finite element method using polynomial interpolation functions for velocity and pressure, respectively. The selections of interpolation functions have been discussed in 3), 4), 6), etc., free surface problem has been treated in 7) and thermally coupled problems are found in 10). These methods are wholly called velocity pressure method in this paper.

The second category is the group of finite element methods employing stream function and vorticity function. The basic equations of fluid flow can be described by the second order nonlinear simultaneous differential equation taking stream and vorticity functions as new unknown variables. The investigations by Tong[12], Baker[13]~[15], Cheng[16] and Bratanow, et. al.[17]~[19] belong to this second category. However, these methods seem to be rather unsuitable to describe vorticity boundary conditions. Taylor and Hood[3] presented a useful method to consider vorticity conditions. Eliminating vorticity function from the aforementioned basic differential equations, the governing equations are transformed into the fourth order nonlinear differential equation where stream function being the only unknown variable. This approach is called the stream function method and comes under the third category. Olson[20],[21] Lieber, et al.[22] presented finite element analyses, which are classified into this stream function method. Among these papers, boundary conditions for pressure and shearing stress are discussed as the natural boundary conditions of the basic variational equation. The formulation by Wu[23] is noteworthy as it discusses the treatment of the boundary conditions of infinite regions. In the investigation by Carlo and Piva[24], the mixed finite element method is used to solve thermal convection problem.

In this paper, a two dimensional finite element method will be presented based on the stream function method. Contrary to the conventional stream function method, the formulation in this paper will consider the conditions for surface force and momentum flux on the boundary conditions. To cover these two conditions, two different types of variational equations derived by multiplying both sides of the basic equation by weighting function and integrating over the whole field will be introduced. Introducing definition equation of stream function into these variational equations, the equations can be rewritten in terms of stream function. Employing the interpolation relation of stream function and the variational equations obtained above, finite element governing equation can be derived. For interpolation function, Olson[20],[21] has used the complete polynomials of the fifth degree considering the analogy of plate bending problems. Lieber, et al.[22] has employed the polynomials of the third degree and also referred to the polynomials of the second degree originally introduced by Morely[25]. In this paper, to save computational time and to preserve conforming condition, incomplete polynomials of the third degree will be applied. This interpolation function was intro-

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duced by Zienkiewicz[26] and discussed by Zlamal[27]. To solve discretized equations, Newton-Raphson method will be employed. Several sample problems and comparisons with the results obtained by finite difference method will be shown to illustrate the validity of the formulation.

2. BASIC EQUATIONS

Throughout this paper, equations are described by using inditial notation and usual summation convention with repeated indices. The analysis is restricted to two dimensional steady flow of incompressible viscous fluid. Spatial description with rectangular coordinate system x_i $(i=1, 2)$ is employed. Notation $(\ )_{,i}$ means partial differrentiation with respect to x_i. With the use of conservation of linear momentum, the equation of motion is expressed as:

$$\rho u_j u_{i,j} - \tau_{ij,j} = \rho \hat{f}_i \quad \cdots\cdots(2.1)$$

where u_i, and $\hat{f}_i$ mean velocity and body force, and ρ is density. In case of incompressible viscous fluid, the constitutive equation for stress τ_{ij} is written as follows employing Kronecker delta fuction δ_{ij}.

$$\tau_{ij} = -p\delta_{ij} + 2\mu d_{ij} \quad \cdots\cdots(2.2)$$

where p denotes pressure and μ is viscosity coefficient. Deformation rate d_{ij} is derived from velocity as:

$$d_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \cdots\cdots(2.3)$$

The equation of continuity of incompressible fluid is obtained in the following form.

$$u_{i,i} = 0 \quad \cdots\cdots(2.4)$$

Introducing equations (2.2) and (2.3) into equation (2.1) and using (2.4), the wellknown Navier-Stokes equation can be derived:

$$\rho u_j u_{i,j} + p_{,i} - \mu u_{i,jj} = \rho \hat{f}_i \quad \cdots\cdots(2.5)$$

Let ϕ be the stream function, i.e.,

$$u_i = \varepsilon_{ij} \phi_{,j} \quad \cdots\cdots(2.6)$$

where ε_{ij} means Edington's epsilon function. Substituting equation (2.6) into equation (2.5) and eliminating pressure p, the governing equation is written in the following form.

$$\nu \phi_{,kkjj} - \varepsilon_{ij} \phi_{,kki} \phi_{,j} = 0 \quad \cdots\cdots(2.7)$$

where ν is dynamic viscosity μ/ρ. In equation (2.7), body force $\rho \hat{f}_i$ has been neglected. The conventional finite element analyses[19)~21)] have used equation (2.7) as the basic equation. Instead, the present paper uses the variational equations described in section 3 as the basic equations.

Regarding boundary condition, the velocity is assumed to be prescribed on boundary S_1, i.e.,

$$u_i = \hat{u}_i \quad \text{on} \quad S_1, \quad \cdots\cdots(2.8)$$

surface force S_i to be on boundary S_2, i.e.,

$$S_i = \tau_{ij} n_j = \hat{S}_i \quad \text{on} \quad S_2 \quad \cdots\cdots(2.9)$$

and momentum flux Q_i to be on boundary S_3, i.e.,

$$Q_i = (-\rho u_i u_j + \tau_{ij}) n_j = \hat{Q}_i \quad \text{on} \quad S_3 \quad \cdots\cdots(2.10)$$

where superposed ^ means prescribed values on the boundary and n_j is the components of the unit normals to the boundary surface. The present paper discusses the following two cases. Case I assumes:

$$S_1 \cup S_2 = S \quad \cdots\cdots(2.11)$$

$$S_1 \cap S_2 = \phi \quad \cdots\cdots(2.12)$$

and case II assumes:

$$S_1 \cup S_3 = S \quad \cdots\cdots(2.13)$$

$$S_1 \cap S_3 = \phi \quad \cdots\cdots(2.14)$$

in which S means the whole boundary surface of the flow field to be analyzed and ϕ is the null set.

3. VARIATIONAL EQUATIONS

To apply the finite element method, variational equations are required based on the conventional discretization procedure called Galerkin method. Consider the variational equation in case I, i.e., corresponding equations (2.1), (2.2) and (2.3) with boundary conditions (2.8) and (2.9). Let u_i^* be the weighting function, the value of which is arbitrary except on boundary S_1, where it takes the value zero. Multiplying both sides of equation (2.1) by u_i^*, integrating over the whole volume V and using Green's theorem, then,

$$\int_V (\rho u_i^* u_j u_{i,j})\, dV + \int_V (u_{i,j}^* \tau_{ij})\, dV = \int_V (\rho u_i^* \hat{f}_i)\, dV + \int_{S_2} (u_i^* \tau_{ij} n_j)\, dS \quad \cdots(3.1)$$

Introducing equations (2.2) and (2.3) into equation (3.1) and rearranging it,

$$\int_V (\rho u_i^* u_j u_{i,j})\, dV - \int_V (u_{i,i}^* p)\, dV + \int_V \mu (u_{i,j}^* u_{i,j})\, dV + \int_V \mu (u_{i,j}^* u_{j,i})\, dV = \hat{\Gamma} \quad \cdots\cdots(3.2)$$

where

$$\hat{\Gamma} = \int_V (\rho u_i^* \hat{f}_i)\, dV + \int_{S_2} (u_i^* \hat{S}_i)\, dS \quad \cdots\cdots(3.3)$$

In the variational equation (3.2), the surface force condition on boundary S_2 is considered as the natural boundary condition. Using stream function equation (2.6), variational equation (3.2) can be reformulated in the following form.

$$\int_V \rho\varepsilon_{ik}\varepsilon_{jl}\varepsilon_{im}(\psi,_k{}^*\psi,_l\psi,_{mj})\,dV + \int_V \mu\varepsilon_{ik}\varepsilon_{il}(\psi,^*_{kj}\psi,_{lj})\,dV + \int_V \mu\varepsilon_{ik}\varepsilon_{jl}(\psi,^*_{kj}\psi,_{lj})\,dV = \hat{\Gamma} \quad \cdots\cdots(3.4)$$

This equation is called variational equation I in this paper.

On the other hand, variational equation for corresponding equations (2.1), (2.2) and (2.3) with boundary conditions (2.8) and (2.10) is obtained in the subsequent manner. This paper refers this as variational equation II. Applying Green's theorem to the first term of the left side of equation (3.1) and rearranging it, then:

$$-\int_V \rho(u^*_{i,j}u_ju_i)\,dV + \int_V (u^*_{i,j}\tau_{ij})\,dV = \int_V (u_i{}^*\rho\hat{f}_i)\,dV + \int_{S_3} (u_i{}^*(-\rho u_iu_j+\tau_{ij})n_j)\,dS \quad \cdots\cdots(3.5)$$

Introducing equations (2.2) and (2.3) into equation (3.5), the following equation can be obtained.

$$-\int_V \rho(u^*_{i,j}u_ju_i)\,dV - \int_V (u^*_{i,i}p)\,dV + \int_V \mu(u^*_{i,j}u_{i,j})\,dV + \int_V \mu(u^*_{i,j}u_{j,i})\,dV = \hat{\Sigma} \quad \cdots\cdots(3.6)$$

where

$$\hat{\Sigma} = \int_V (\rho u_i{}^*\hat{f}_i)\,dV + \int_{S_3} (u_i{}^*\hat{Q}_i)\,dS \quad \cdots\cdots(3.7)$$

Using stream function, alternate equation of equation (3.4) for case II can be derived as follows:

$$-\int_V \rho(\varepsilon_{ik}\varepsilon_{jl}\varepsilon_{im}\psi,^*_{kj}\psi,_l\psi,_m)\,dV + \int_V \mu\varepsilon_{ik}\varepsilon_{il}(\psi,^*_{kj}\psi,_{lj})\,dV + \int_V \mu\varepsilon_{ik}\varepsilon_{jl}(\psi,^*_{kj}\psi,_{li})\,dV = \hat{\Sigma} \quad \cdots\cdots(3.8)$$

4. FINITE ELEMENT ANALYSIS

Assume that the flow field to be analyzed is divided into small regions called finite elements, and that both trial and weighting functions for stream function are expressed by

$$\psi = \Phi_\alpha\psi_\alpha \quad \cdots\cdots(4.1)$$

$$\psi^* = \Phi_\alpha\psi^*_\alpha \quad \cdots\cdots(4.2)$$

where Φ_α denotes interpolation function. Nodal values of the stream function and corresponding weighting function are described as ψ_α and ψ^*_α. Introducing equations (4.1) and (4.2) into equation (3.4) or (3.8) and considering the arbitrariness of ψ^*_α lead to finite element governing equation,

$$A_{\alpha\beta\gamma}\psi_\beta\psi_\gamma + B_{\alpha\beta}\psi_\beta = \hat{\Omega}_\alpha \quad \cdots\cdots(4.3)$$

where coefficients $A_{\alpha\beta\gamma}$, $B_{\alpha\beta}$ and $\hat{\Omega}_\alpha$ are derived in the following forms when using (3.4), i.e., equation I.

$$A_{\alpha\beta\gamma} = \int_V \rho(\varepsilon_{ik}\varepsilon_{jl}\varepsilon_{im}\Phi_{\alpha,k}\Phi_{\beta,l}\Phi_{\gamma,mj})\,dV \quad \cdots\cdots(4.4)$$

$$B_{\alpha\beta} = \int_V \mu(\varepsilon_{ik}\varepsilon_{il}\Phi_{\alpha,kj}\Phi_{\beta,lj})\,dV + \int_V \mu(\varepsilon_{ik}\varepsilon_{jl}\Phi_{\alpha,kj}\Phi_{\beta,li})\,dV \quad \cdots\cdots(4.5)$$

$$\hat{\Omega}_\alpha = \int_V \rho(\varepsilon_{ij}\Phi_{\alpha,j}\hat{f}_i)\,dV + \int_{S_2} (\varepsilon_{ij}\Phi_{\alpha,j}\hat{S}_i)\,dS \quad \cdots\cdots(4.6)$$

In case of using equation (3.8), i.e., equation II, $A_{\alpha\beta\gamma}$ and $\hat{\Omega}_\alpha$ are expressed as follows.

$$A_{\alpha\beta\gamma} = -\int_V \rho(\varepsilon_{ik}\varepsilon_{jl}\varepsilon_{im}\Phi_{\alpha,kj}\Phi_{\beta,l}\Phi_{\gamma,m})\,dV \quad \cdots\cdots(4.7)$$

$$\hat{\Omega}_\alpha = \int_V \rho(\varepsilon_{ij}\Phi_{\alpha,j}\hat{f}_i)\,dV + \int_{S_3} (\varepsilon_{ij}\Phi_{\alpha,j}\hat{Q}_i)\,dS \quad \cdots\cdots(4.8)$$

Referring to Figure 1, it may be convenient to use $\xi-\eta$ coordinate system to compute the coefficients $A_{\alpha\beta\gamma}$ and $B_{\alpha\beta}$. To apply the conventional finite element superposition procedure for the whole flow field, local $\xi-\eta$ coordinate system of equa-

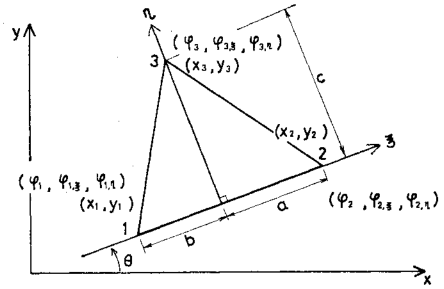

Fig. 1 Triangular finite element and coordinate system.

tion (4.3) is transformed into global $x-y$ coordinate system common to the whole finite elements in the flow field.

Expressing coordinate transformation in the form:

$$\phi_\beta = C_{\beta m}\phi_m \qquad \cdots\cdots(4.9)$$

equation (4.3) becomes:

$$A_{lmn}\phi_m\phi_n + B_{lm}\phi_m = \hat{\Omega}_l \qquad \cdots\cdots(4.10)$$

where

$$A_{lmn} = C_{\alpha l}A_{\alpha\beta\gamma}C_{\beta m}C_{\gamma n} \qquad \cdots\cdots(4.11)$$

$$B_{lm} = C_{\alpha l}B_{\alpha\beta}C_{\beta m} \qquad \cdots\cdots(4.12)$$

$$\hat{\Omega}_l = C_{\alpha l}\hat{\Omega}_\alpha \qquad \cdots\cdots(4.13)$$

Superposing equation (4.10) for the whole flow field yields nonlinear simultaneous equation system, which is called finite element governing equation. The finite element governing equation can be written in the same form as equation (4.10) and be denoted as:

$$F_\lambda = A_{\lambda\nu\mu}\phi_\nu\phi_\mu + B_{\lambda\nu}\phi_\nu - \hat{\Omega}_\lambda \equiv 0 \qquad \cdots\cdots(4.14)$$

where $A_{\lambda\nu\mu}$ and $B_{\lambda\nu}$ can be constructed using A_{lmn} and B_{lm} for each finite element. To solve equation (4.14), Newton-Raphson method is commonly used which seems to be one of the most efficient iteration schemes. The algorithm is as follows:

$$\phi_\nu^{(n+1)} = \phi_\nu^{(n)} - [K_{\lambda\nu}^{(n)}]^{-1}F_\lambda^{(n)} \qquad \cdots\cdots(4.15)$$

where

$$K_{\lambda\nu}^{(n)} = A_{\lambda\nu\mu}\phi_\mu^{(n)} + A_{\lambda\mu\nu}\phi_\mu^{(n)} + B_{\lambda\nu} \qquad \cdots(4.16)$$

To solve nonlinear simultaneous equation system by Newton-Raphson method, large scale non-symmetric linear simultaneous equation must be solved by repetition. For this purpose, unit partition method is employed in the numerical examples described in section 6.

5. INTERPOLATION FUNCTION

To save computational time and to preserve conforming condition, incomplete polynomials of the third degree are employed as the interpolation function for the numerical computation in this paper. The interpolation function has been originally introduced by Zienkiewicz[26]. It is assumed that stream function is expressed by the polynomials of the following form in each finite element,

$$\phi = a_1 + a_2\xi + a_3\eta + a_4\xi^2 + a_5\xi\eta + a_6\eta^2 + a_7\xi^3 + a_8\xi^2\eta + a_9\xi\eta^2 + a_{10}\eta^3 \qquad \cdots\cdots(5.1)$$

where $a_1 \sim a_{10}$ are unknown coefficients determined below, and ξ and η are local coordinate system shown in Fig. 1. Let the nodal unknown values be chosen as:

$$\phi_\alpha = [\phi_1, \phi_{1,\xi}, \phi_{1,\eta}, \phi_2, \phi_{2,\xi}, \phi_{2,\eta}, \phi_3, \phi_{3,\xi}, \phi_{3,\eta}, \phi_c] \qquad \cdots\cdots(5.2)$$

where ϕ_1, ϕ_2, ϕ_3 mean nodal values of the stream function at node 1, 2, 3, quantities $\phi_{1,\xi}$, $\phi_{1,\eta}$, $\phi_{2,\xi}$, $\phi_{2,\eta}$, $\phi_{3,\xi}$, $\phi_{3,\eta}$ are the values of their derivatives, which correspond to nodal velocities and ϕ_c is the value of stream function at centroid of the triangular finite element. Unknown coefficients are determined by the relation:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \end{pmatrix} = \begin{pmatrix} 1 & -b & & (-b)^2 & & & (-b)^3 & & & \\ & 1 & & 2(-b) & & & 3(-b)^2 & & & \\ & & 1 & & (-b) & & & (-b)^2 & & \\ 1 & a & & a^2 & & & a^3 & & & \\ & 1 & & 2a & & & 3a^3 & & & \\ & & 1 & & a & & & a^2 & & \\ 1 & & c & & & c^2 & & & & c^3 \\ & 1 & & & c & & & & c^2 & \\ & & 1 & & & 2c & & & & 3c^2 \\ 1 & f & e & f^2 & fe & e^2 & f^3 & f^2e & fe^2 & e^3 \end{pmatrix}^{-1} \begin{pmatrix} \phi_1 \\ \phi_{1,\xi} \\ \phi_{1,\eta} \\ \phi_2 \\ \phi_{2,\xi} \\ \phi_{2,\eta} \\ \phi_3 \\ \phi_{3,\xi} \\ \phi_{3,\eta} \\ \phi_c \end{pmatrix} \qquad \cdots\cdots(5.3)$$

where a, b and c in $f=(a-b)/3$ and $e=c/3$ are the coordinates of each nodal point of the finite element (see Figure 1). Denoting inverse matrix in the right side of equation (5.3) as $\mathfrak{A}_{p\alpha}$, equation (5.1) can be described as follows.

$$\phi = \sum_{\alpha=1}^{10}\sum_{p=1}^{10} \mathfrak{A}_{p\alpha}\xi^{m_p}\eta^{n_p}\cdot\phi_\alpha \qquad \cdots\cdots(5.4)$$

where

$$m_p = [0\ 1\ 0\ 2\ 1\ 0\ 3\ 2\ 1\ 0] \qquad \cdots\cdots(5.5)$$

$$n_p = [0\ 0\ 1\ 0\ 1\ 2\ 0\ 1\ 2\ 3] \qquad \cdots\cdots(5.6)$$

In practice, it is convenient to eliminate the value ϕ_G from the unknown values of the whole finite element field. For its elimination, the procedure of Zlamal[27] is employed:

$$\begin{aligned}\phi_G=&\frac{1}{3}(\phi_1+\phi_2+\phi_3)+\frac{1}{18}[(\xi_2+\xi_3-2\xi_1)\phi_{1,\xi}\\&+(\eta_2+\eta_3-2\eta_1)\phi_{1,\eta}+(\xi_1+\xi_3-2\xi_2)\phi_{2,\xi}\\&+(\eta_1+\eta_3-2\eta_2)\phi_{2,\eta}+(\xi_1+\xi_2-2\xi_3)\phi_{3,\xi}\\&+(\eta_1+\eta_2-2\eta_3)\phi_{3,\eta}]\end{aligned}\quad\cdots\cdots(5.7)$$

Equations (5.4) and (5.7) lead to interpolation function in the form:

$$\Phi_\alpha=\sum_{p=1}^{10}\mathfrak{A}_{p\alpha}\xi^{m_p}\eta^{n_p}\quad\cdots\cdots(5.8)$$

Using equation (5.8), equations (4.4) and (4.5) are reformulated as follows.

$$\begin{aligned}A_{\alpha\beta\gamma}=&\sum_{p=1}^{10}\sum_{q=1}^{10}\sum_{r=1}^{10}\mathfrak{A}_{p\alpha}\mathfrak{A}_{q\beta}\mathfrak{A}_{r\gamma}\\&\times\rho[\{n_pn_qn_rm_r-n_pm_qn_r(n_r-1)\\&\quad+n_pn_rn_qm_q-n_pm_rn_q(n_q-1)\}\\&\times F(m_p+m_q+m_r-1,\ n_p+n_q+n_r-3)/2\\&\quad+\{m_pn_qm_r(m_r-1)-m_pm_qm_rn_r\\&\quad+m_pn_rm_q(m_q-1)-m_pm_rm_qn_q\}\\&\times F(m_p+m_q+m_r-3,\ n_p+n_q+n_r-1)/2]\end{aligned}\quad\cdots\cdots(5.9)$$

$$\begin{aligned}B_{\alpha\beta}=&\sum_{p=1}^{10}\sum_{q=1}^{10}\mathfrak{A}_{p\alpha}\mathfrak{A}_{q\beta}\\&\times\mu[\{4m_pn_pm_qn_q-n_p(n_p-1)m_q(m_q-1)\\&\quad-m_p(m_p-1)n_q(n_q-1)\}\\&\times F(m_p+m_q-2,\ n_p+n_q-2)\\&\quad+\{n_p(n_p-1)n_q(n_q-1)\}\\&\times F(m_p+m_q,\ n_p+n_q-4)\\&\quad+\{m_p(m_p-1)m_q(m_q-1)\}\\&\times F(m_p+m_q-4,\ n_p+n_q)]\end{aligned}\quad\cdots\cdots(5.10)$$

where

$$\left.\begin{aligned}F(m,n)&=\frac{m!\,n!}{(m+n+2)!}c^{n+1}\{a^{m+1}-(-b)^{m+1}\}\\F(m,n)&=\int_V\xi^m\eta^n\,dV\end{aligned}\right\}\quad\cdots\cdots(5.11)$$

Equation (4.7) is also formulated in the form:

$$\begin{aligned}A_{\alpha\beta\gamma}=&-\sum_{p=1}^{10}\sum_{q=1}^{10}\sum_{r=1}^{10}\mathfrak{A}_{p\alpha}\mathfrak{A}_{q\beta}\mathfrak{A}_{r\gamma}\\&\times\rho[\{n_p(n_p-1)m_qn_r-n_pm_pn_qn_r\\&\quad+n_p(n_p-1)m_rn_q-n_pm_pn_rn_q\}\\&\times F(m_p+m_q+m_r-1,\ n_p+n_q+n_r-3)/2\\&\quad+\{m_pn_pm_qm_r-m_p(m_p-1)n_qm_r\\&\quad+m_pn_pm_rm_q-m_p(m_p-1)n_rm_q\}\\&\times F(m_p+m_q+m_r-3,\ n_p+n_q+n_r-1)/2]\end{aligned}$$

Transformation matrix in equation (4.9) is as follows.

$$\left.\begin{aligned}&C_{\alpha l}=\begin{bmatrix}1&&\\&\cos\theta&\sin\theta\\&-\sin\theta&\cos\theta\end{bmatrix}\\&\begin{Bmatrix}\phi(\xi,\eta)\\\phi_{,\xi}\\\phi_{,\eta}\end{Bmatrix}=[C]\begin{Bmatrix}\phi(x,y)\\\phi_{,x}\\\phi_{,y}\end{Bmatrix}\end{aligned}\right\}\quad\cdots\cdots(5.13)$$

where

$$\cos\theta=\frac{(x_2-x_1)}{r},\quad\sin\theta=\frac{(y_2-y_1)}{r}$$

$$r=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

6. NUMERICAL EXAMPLES

To illustrate the adaptability of the present finite element method, several numerical examples will be discussed in this section. Reynolds numbers in these examples are calculated by $R_e=du_0/\nu$,

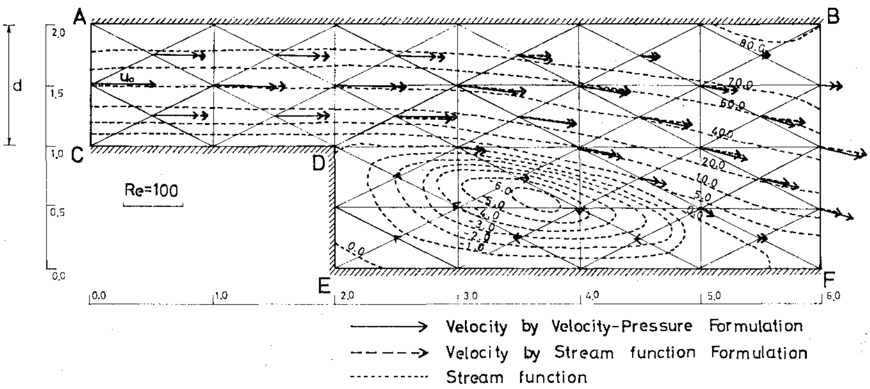

Fig. 2 Computed velocity and stream function at Reynolds number of 120.

where d and u_0 are the fundamental units of length and velocity, respectively. Their values are shown in the figures, and dynamic viscosity is μ/ρ. The hatched boundary indicates rigid wall on which both components of velocity are taken to be zero. Arrowed lines show the computed velocity. Stream lines are illustrated by contour lines. These stream lines are calculated by interpolating relation, i.e., equation (4.1), on the basis of the computed nodal stream function.

The first example is the computation of flow through a channel having sharp corners. Figure 2 is an illustration of the finite element idealization, computed velocity and computed stream function calculated by the present stream function method based on variational equation I. The computed velocity is compared with the velocity obtained by the velocity pressure method, i.e., the finite element method using velocity and pressure as the nodal unknown variables[8)~10)]. The boundary conditions used are as follows.

A-B: $\psi=0.6666u_0\,,\quad \psi_{,x}=0\,,\quad \psi_{,y}=0$

C-A: $\psi=\left(2y^2-\dfrac{4}{3}y^3\right)u_0\,,\quad \psi_{,x}=0\,,$

$\psi_{,y}=(4-4y^2)u_0$

C-D-E-F: $\psi=0\,,\quad \psi_{,x}=0\,,\quad \psi_{,y}=0$

B-F: $\hat{S}_x=0$

where x and y denote horizontal and vertical coordinates and $\hat{S}_x$ is horizontal surface force.

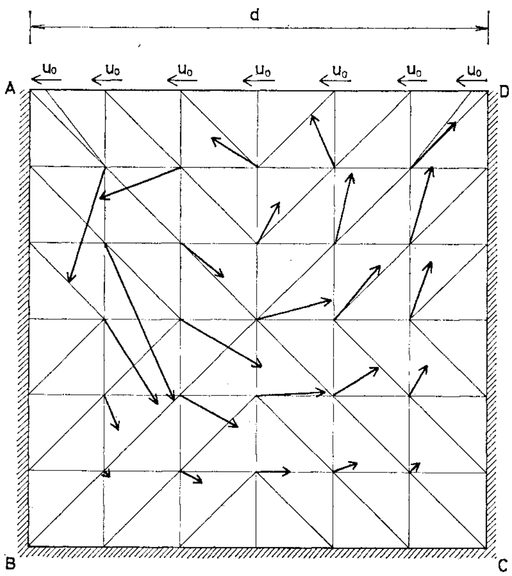

Fig. 3 Finite element idealization and computed velocity at Reynolds number of 100.

Dynamic viscosity is taken as 1.0. Solid arrowed lines denote velocity by the present method. Broken arrowed lines indicate velocity by the velocity pressure method. Dotted contour lines show the computed stream function. The computed results by both methods are well in agreement.

The second example is the computation of flow in a cavity. Figure 3 represents the finite element idealization and the computed velocity at Rey-

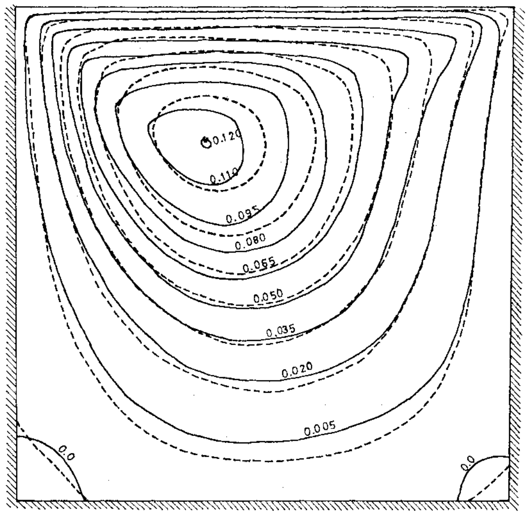

Fig. 4 Computed stream function at Reynolds number of 100.

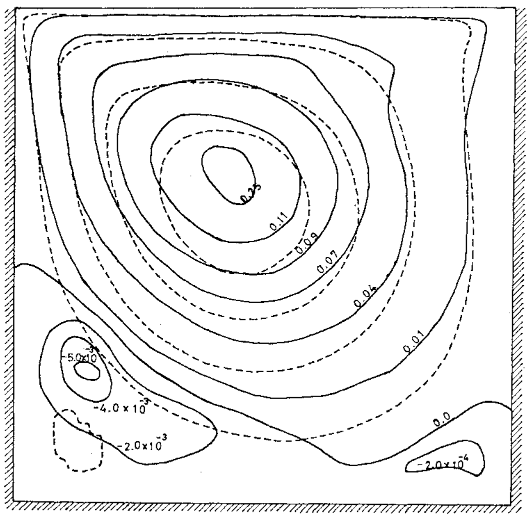

Fig. 5 Computed stream function at Reynolds number of 400.

nolds number of 100. Boundary conditions used in this computation are as follows.

A-B: $\psi=0$, $\psi_{,x}=0$, $\psi_{,y}=0$

B-C: $\psi=0$, $\psi_{,x}=0$, $\psi_{,y}=0$

C-D: $\psi=0$, $\psi_{,x}=0$, $\psi_{,y}=0$

D-A: $\psi=0$, $\psi_{,x}=0$, $\psi_{,y}=-u_0$

Variational equation I had been employed as the basic variational equation. Figure 4 shows the computed stream lines at Reynolds number of 100 (solid line) as compared with the results by the finite difference method of Bozeman and Dalton[28] (dotted line). Figure 5 is the computed stream lines at Reynolds number of 400 as compared with the results obtained by the finite element method of Marshall and Van Spiegel[29]. For the last two numerical results by the finite difference methods, 50×50 or 40×40 mesh size has been employed, respectively. In the computation by the present finite element method, only 52 nodal points and 156 degrees of freedom have been used to obtain the comparable results as shown in Fig. 3. Figure 6 is an illustration of the computed stream function at Reynolds number of 1,000. Three forms of circulated flow can

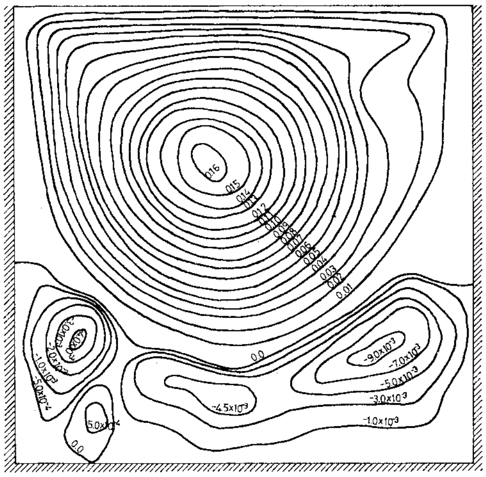

Fig. 6 Computed stream function at Reynolds number of 1000.

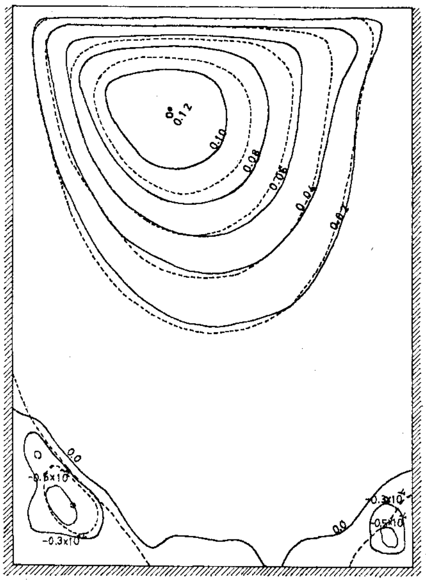

Fig. 7 Computed stream function at Reynolds number of 100 and aspect ratio 1.4.

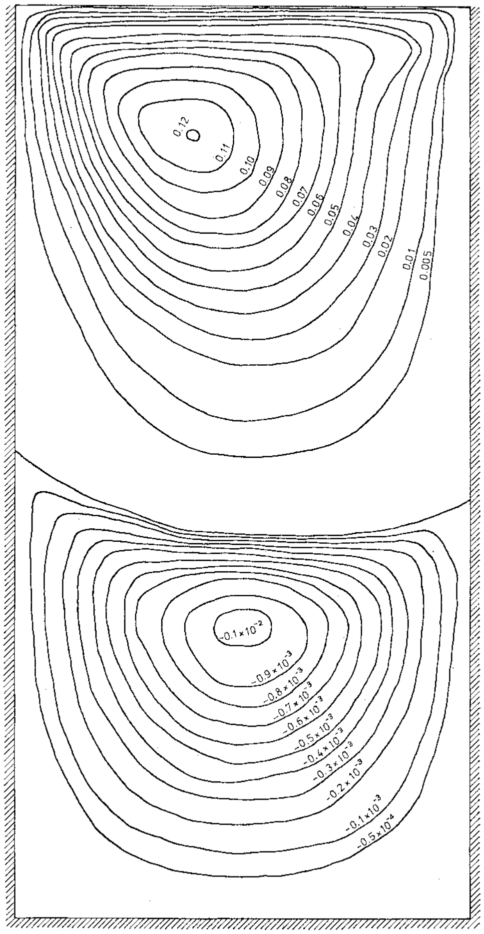

Fig. 8 Computed stream function at Reynolds number of 100 and aspect ratio 2.0.

be observed in the figure in addition to the main circulated flow. Figure 7 shows the computed stream function at Reynolds number of 100 in a cavity with aspect ratio of 1.4 as compared with the results by Bozeman and Dalton[28]. Figure 8 illustrates the computed stream function at Reynolds number of 100 in a cavity with aspect ratio of 2.0. In the figure, two circulated flows can be seen flowing in the opposite direction to each other. The position of the 0th stream function is calculated to have a common place in the various computed results for a cavity with different aspect ratio.

The last numerical example is the computation of flow over cylinders. Figures 9 and 10 show the finite element idealization and computed stream function at Reynolds number of 40 and 240, respectively. Total numbers of nodal points and elements used are 274 points and 428 elements. Boundary conditions employed are as follows:

A-B: $\psi=4u_0$, $\psi_{,x}=0$

C-A, H-B: $\psi=u_0 y$, $\psi_{,x}=0$, $\psi_{,y}=u_0$

C-D, E-F, G-H: $\psi=0$, $\psi_{,x}=0$

D-E, F-G: $\psi=0$, $\psi_{,x}=0$, $\psi_{,y}=0$

Completely different types of vortices can be recognized as circulated flows after the cylinders in Figures 9 and 10. As the basic formulation in this computation, variational equation I has been employed. Figure 11 is the comparison of computed stream function between by variational equations I and II at Reynolds number of 80. Boundary conditions employed here for flow over three cylinders are the same as in the example of Figure 9. Total numbers of nodal points and finite elements are 525 points and 872 elements. Figure 12 is the illustration of the finite element idealization and the computed momentum flux. The reactions of the momentum flux are observed as asymptotically decreasing for the second and third cylinders.

7. CONCLUSION

In this paper, a finite element method of two dimensional steady flow of incompressible viscous fluid has been presented. The finite element method is characterized by the following.

i) The method is based on variational equations in terms of stream function.
ii) The conditions of surface force and momentum flux on the boundary are considered.
iii) Comparable numerical results are obtained by using fewer total degrees of freedom than by the finite difference methods.
iv) Compared with the velocity pressure method, the present method is more stable in the sense of numerical computation.
v) Polynomials of the third degree are employed for the interpolating function of stream function.

In the previous papers[8)~10)], one of the authors has presented a finite element method using velocity and pressure as unknown variables. The coefficient matrix of the final governing equations in this finite element method includes zero terms among its main diagonal elements. This may be one of the main reasons for numerical instability especially in the computation of flow with extremely high Reynolds number. This is attributable to the fact that equation of continuity consists only of the function of velocity. Thus, if the formulation of velocity which automatically satisfies the equation of continuity would be employed in the analysis, a numerically stable finite element method could be obtained. For this purpose, it is concluded that it would be convenient to employ stream function as unknown variable.

In the present formulation, considering the conditions of surface force and momentum flux as natural boundary conditions, and upon formulating variational equation in terms of velocity and pressure, basic variational equation for stream function has been derived by substituting difinition equation of stream function into this velocity pressure variational equation. Two types of variational equation have been obtained corresponding to the conditions of surface force and momentum flux, respectively. It has been shown in the numerical examples that the conditions for the discharge to another fluid and for the momentum flux on the solid boundary are considered in the analysis by the present method. As has also been stated by Olson[20], comparison with the numerical results by the finite difference method has shown that the finite element method employing extremely fewer total number of unknown variables for computation yields accuracies comparable to the finite difference method.

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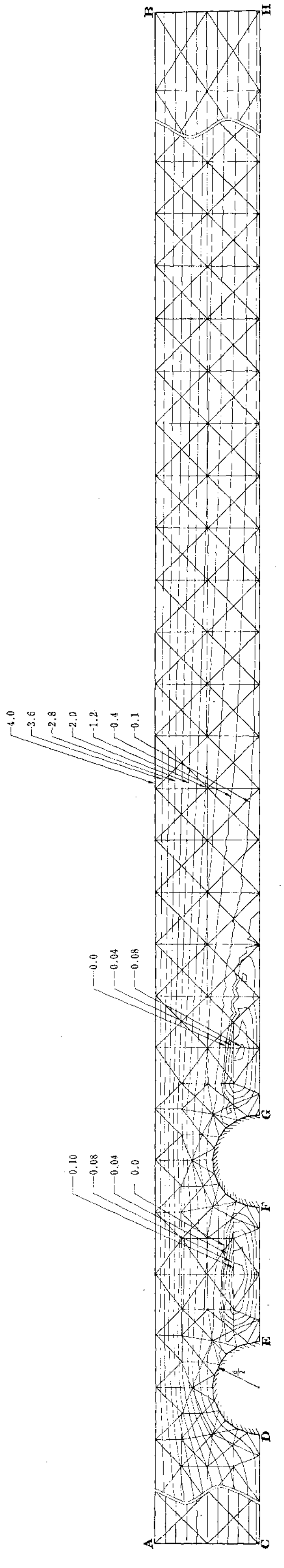


Fig. 9 Finite element idealization and computed stream function at Reynolds number of 40.

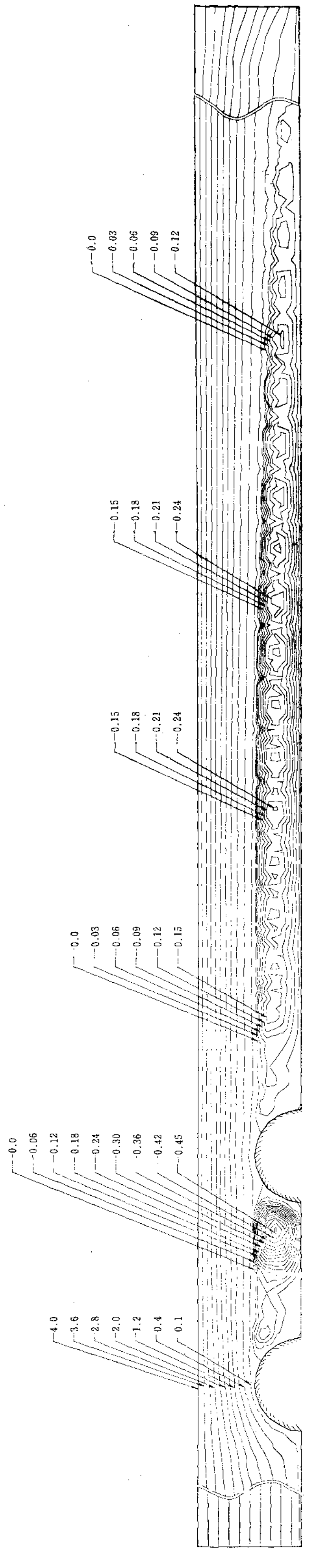
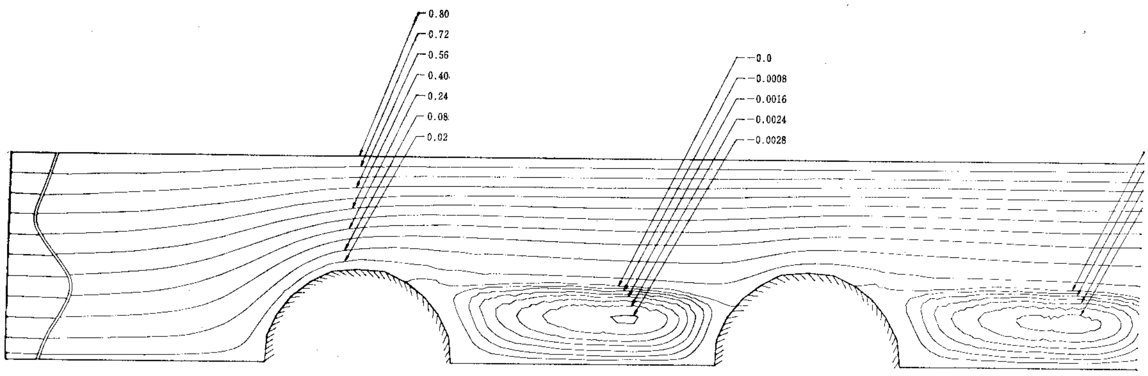
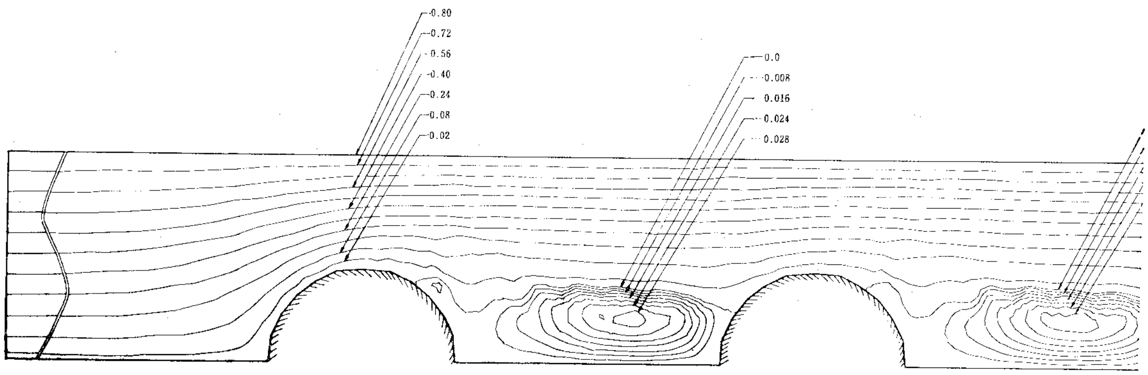


Fig. 10 Computed stream function at Reynolds number of 240.



by variational



by variational

Fig. 11 Computed stream function at

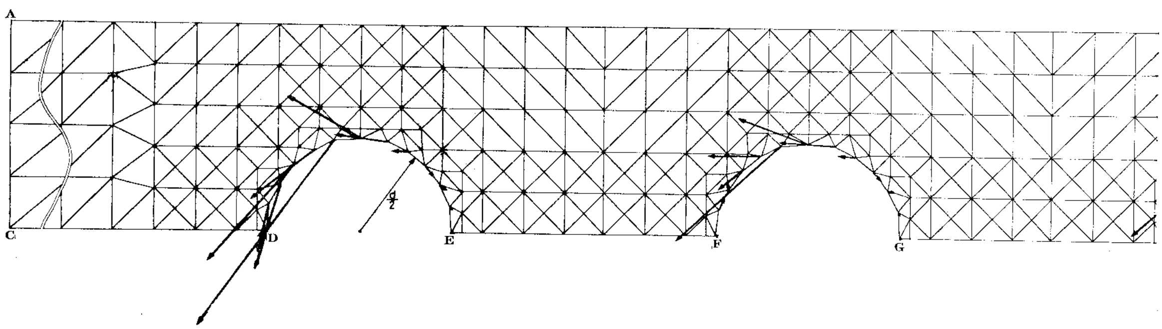
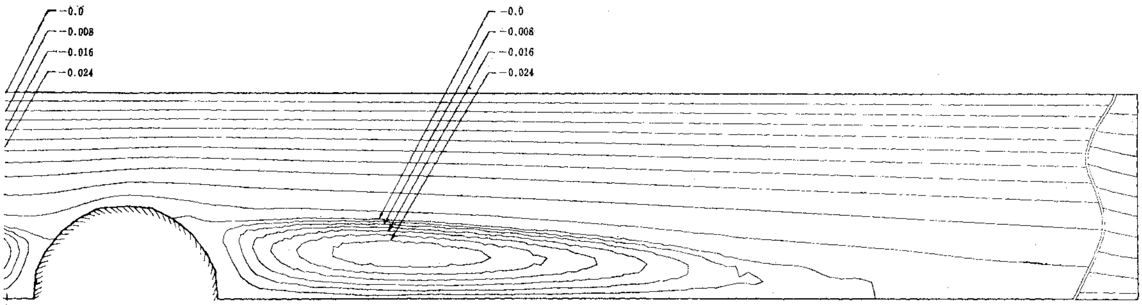
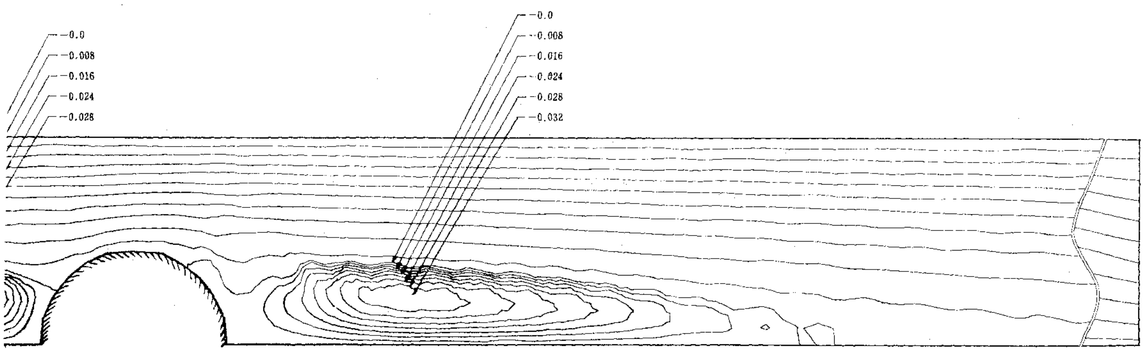


Fig. 12 Finite element idealization and computed

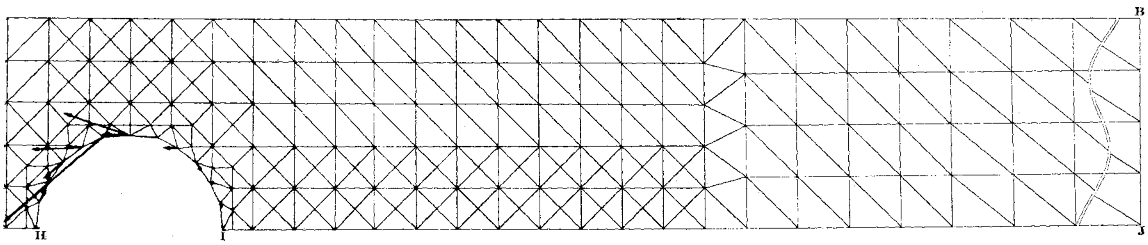


equation (1)



equation (2)

Reynolds number of 80.



momentum flux at Reynolds number of 80.

REFERENCES

1) Huebner, K. H.: The Finite Element Method for Engineering, John-Wiley & Sons, 1975.
2) Oden, J. T. and Wellford, L. C. Jr.: Analysis of Flow of Viscous Fluids by the Finite Element Method, AIAA J., Vol. 10, No. 12, pp. 1590～1599, 1972.
3) Taylor, C. and Hood, P.: A Numerical Solution of the Navier-Stokes Equations Using the Finite Element Technique, Computers Fluids, Vol. 1, No. 1, pp. 73～100, 1973.
4) Hood, P. and Taylor, C.: Navier-Stokes Equation Using Mixed Interpolation, Finite Element Method in Flow Problems, UAH press, pp. 121～132, 1974.
5) Argyris, H. and Mareczek, G.: Finite Element Analysis of Slow Incompressible Viscous Fluid Motion, Ingineur-Archiv., Vol. 43, No. 2, pp. 92～109, 1974.
6) Gartling, D. and Becker, E. B.: Computationally Efficient Finite Element Analysis of Viscous Flow Problems, Computation Methods in Nonlinear Mechanics, TICOM, pp. 603～614, 1974.
7) Nickel, R. E., Tanner, R. I. and Caswell, B.: The Solution of Viscous Incompressible Jet and Free Surface Flows Using Finite Element Methods, J. Fluid Mech., Vol. 65, Part 1, pp. 184～206, 1974.
8) Kawahara, M., Yoshimura, N., Nakagawa, K. and Ohsaka, H.: Steady Flow Analysis of Incompressible Viscous Fluid by the Finite Element Method, Theory and Practice in Finite Element Structural Analysis, University of Tokyo Press, pp. 557～572, 1973.
9) Kawahara, M., Yoshimura, N. and Nakagawa, K.: Analysis of Steady Incompressible Viscous Flow, Finite Element Methods in Flow Problems, UAH press, pp. 107～120, 1974.
10) Kawahara, M., Yoshimura, N., Nakagawa, K. and Ohsaka, H.: Steady and Unsteady Finite Element Analysis of Incompressible Viscous Fluid, Int. J. Num. Meth. Engng., (In Press).
11) Usuki, S. and Kudo, K.: The Local Potential Approach to Finite Element Method in Unsteady Viscous Incompressible Fluid Flow, Proc. J.S.C.E., Vol. 216, pp. 79～89, 1973.
12) Tong, P.: On the Solution of the Navier Stokes Equations in Two Dimensional and Axial Symmetric Problems, Finite Element Methods in Flow Probleus, UAH press, pp. 57～66, 1974.
13) Baker, A. J.: Finite Element Solution Algorithm for Incompressible Fluid Dynamics, Finite Element Method in Flow Problems, UAH press, pp. 51～55, 1974.
14) Baker, A. J.: Finite Element Solution Theory for Three-Dimensional Boundary Flows, Comp. Meth. Appl. Mech. Engng., Vol. 4, pp. 367～386, 1974.
15) Baker, A. J.: Finite Element Solution Algorithm for Viscous Incompressible Fluid Dynamics, Int. J. Num. Meth. Engng., Vol. 6, pp. 89～101, 1973.
16) Cheng, R. T.: Numerical Solution of the Naviel-Stokes Equations by the Finite Element Method, Phys. Fluid., Vol. 15, No. 12, pp. 2098～2105, 1972.
17) Bratanow, T. and Ecer, A.: Analysis of Moving Body Problems in Aerodynamics, Finite Element Method in Flow Problems, UAH press, pp. 225～241, 1974.
18) Bratanow, T., Ecer, A., Akutsu, H. and Spehert, T.: Nonlinearities in Analysis of Unsteady Flow Around Oscillating Wing, Computational Methods in Nonlinear Mechanics, TICOM, pp. 925～934, 1974.
19) Bratanow, T., Ecer, A. and Kobiske, M.: Finite Element Analysis of Unsteady Incompressible Flow Around an Oscillating Obstacle of Arbitrary Shape, AIAA J., Vol. 11, No. 11, pp. 1471～1477, Nov. 1973.
20) Olson, M. D.: Variational Finite Element Method for Two Dimensional Steady Viscous Flows, Joint McGill University-Engineering Institute of Canada Conference, 1972.
21) Olson, M. D.: Variational-Finite Element Method for Two Dimensional and Axisymmetric Navier-Stokes Equations, Finite Element Methods in Flow Problems, UAH press, pp. 103～106, 1974.
22) Lieber, P., Wen, K. S. and Attia, A. V.: Finite Element Method as an Aspect of the Principle of Maximum Uniformity: New Hydrodynamical Ramifications, Finite Element Methods in Flow Problems, UHA press, pp. 85～96, 1974.
23) Wu, J. C.: Integral Representation of Field Variables for the Finite Element Solution of Viscous Flow Problems, Finite Element Method in Engineering, The University of New South Wales, pp. 827～840, 1974.
24) Di Carlo, A. and Piva, R.: Finite Element Simulation of Thermally Induced Flow Fields, Computational Methods in Nonlinear Mechanics, TICOM, pp. 289～298, 1974.
25) Morley, L. S. D.: The Constant Moment

Plate-Bending Element, Jour. Strain Anal., Vol. 6, pp. 20~24, 1971.
26) Zienkiewicz, O. C.: Finite Element Method in Engineering Science, McGraw Hill, 1971.
27) Zlamal, M.: A Finite Element Procedure of Second Order Accuracy, Num. Math., Vol. 14, pp. 394~402, 1970.
28) Bozeman, J. D. and Dalton, C.: Numerical Study of Viscous Flow in a Cavity, Jour. Comp. Phys., Vol. 12, pp. 348~363, 1973.
29) Marshall, G. and Van Spiegel, E.: On the Numerical Treatment of the Navier-Stokes Equations for An Incompressible Fluid, Jour. Engng. Math., Vol. 7, No. 2, pp. 173~188, 1973.

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